A CLASS OF LINEAR DIFFERENTIAL EVASION GAMES

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In this paper, L.S.Pontrjagin's and E.F.Mischenko's method of evasion [1] - [2] is generalized for linear differential games with bounded resource of control.

Consider the statement of the problem.

A. Let the behaviour of a vector z in n -dimensional Euclidean space R^n be described by the linear vector differential equation

$$\dot{z} = Az - Bu + Cv + a \quad , \tag{1}$$

where A is constant quadratic matrix of order n , $u \in R^k$ and $v \in R^e$ are control vectors, B and C are constant matrices of corresponding dimension , $a \in R^n$ is a given vector. The vector $u(t)$ is chosen by the pursuer, and the vector $v(t)$ is chosen by the evader. The vector subspace M of a dimension less than or equal to $n-2$ is given in R^n .

The pursuit is assumed to be completed in the first time t when the point $z(t)$ reaches the set M . The controls $u = u(t)$ and $v = v(t)$ are measurable functions. Assume that the following conditions are valid:

$$v(t) \in Q \quad , \quad u(t) \in P , \tag{2}$$

$$\int_0^\infty |v(s)| ds \leq \nu \quad , \quad \int_0^\infty |u(s)| ds \leq \mu , \tag{3}$$

$$\int_0^\infty (v(s), v(s)) ds \leq \sigma^2 , \quad \int_0^\infty (u(s), u(s)) ds \leq \rho^2 , \tag{4}$$

where $P \subset R^k$ and $Q \subset R^e$ are convex compact sets, (b, b) is scalar product of vector b, $\text{Int}\, Q \ni 0$, $P \ni 0$, $\nu > 0$, $\mu \geq 0$, $\sigma > 0$, $\rho \geq 0$.

The conditions (2)-(4) may be considered from physical point of view as restriction on value, on impulse and on energy of control respectively.

We consider the game (1) from the evader's point of view. It is assumed that evader knows equation of game (1), the type of restrictions on control, the set M, vectors $z(s)$ and $u(s)$ for any t from the interval $\max(0, t-h) \leq s \leq t$, where h is a constant parameter.

The aim of evader is not to attack the set M. We suppose that the control of pursuier is arbitrarily permissible.

We shall give some sufficient conditions that the evader can guarantee the satisfying of condition $z(t) \overline{\in} M$ for all $t \geq 0$ for any initial condition $z_0 \overline{\in} M$.

B. We denote the orthogonal complement of M in the space R^n by L. Further, let W be an arbitrary (so far) vector subspace of the space L. We denote by π the operation of orthogonal projection from R^n onto W. To every point $z \in R^n$ we assign two nonnegative numbers: $z \to (\xi, \eta)$, where ξ is the distance from z to M, and η is the distance from ξ to L.

C. <u>Conditions of Evasion.</u> There exists a two-dimensional vector subspace W of the space L for which the following conditions hold:

1. There does not exist any fixed one-dimensional vector subspace W^1 of W such that the inclusion

$$\pi e^{\tau A} C Q \subset W^1$$

holds for all small positive values of τ .

2. There exists a constant $\lambda > 1$ such that

$$\lambda \pi e^{\tau A} BP \subset \pi e^{\tau A} CQ$$

for all sufficiently small positive values of τ .

3. $\sigma > \varkappa \rho$, $\nu > \varkappa \mu$, $\varkappa$ is non- negative constant which depends on the game (1)(we'll define constant $\varkappa$ in part E).

It turns out that the following theorem holds.

<u>Theorem of Evasion.</u> If in the game (1) with restrictions (2)--(4) the conditions of evasion are satisfied, then, for any initial value of z_0 which does not belong to M , the evasion game can be played in such a way that the point $z(t)$ will never reach the space M $(0 \leq t < \infty)$ and, moreover, the following estimate of the distance from $z(t)$ to M holds.

D. If, in the game (1), the conditions of evasion C are satisfied, then there exist sequences of positive constants T_n, ε_n and C_n, and a positive integer k , which depends only on the game and does not depend either on its initial condition or on its progress, such that the evasion game can be played in such a way that

1. for $\xi_0 > \varepsilon_1$

$$\xi(t) > \frac{C_n \varepsilon_n^k}{(1+\eta(t))^k} ,$$

$$T_0 = 0 , \; T_{n-1} \leq t \leq T_n , \; n = 1, 2, \ldots ,$$

2. for $\xi_0 \leq \varepsilon_1$

$$\xi(t) > \frac{C_1 \xi_0^k}{(1+\eta(t))^k} , \; 0 \leq t \leq T_1 ,$$

$$\xi(t) > \frac{C_n \varepsilon_n^k}{(1+\eta(t))^k} , \; T_{n-1} \leq t \leq T_n ,$$

$$n = 2, 3, \ldots .$$

E. In this part we shall give the formula for definition of the constant $\varkappa$ and shall show the main distinction of the given method of evasion from L.S.Pontrjagin's method.

Condition of evasion 2 guarantees the following representation of the matrices (for sufficiently small τ) (see [1]):

$\pi e^{\tau A} C = g_\tau \cdot \varphi_\tau \cdot H$ and $\pi e^{\tau A} B = g_\tau \cdot h_\tau$, where g_τ is $n \times \ell$ analytical matrix, the rank of which is equal to the rank of matrix $\pi e^{\tau A} C$ for $\tau \neq 0$, H is $\ell \times \ell$ orthogonal matrix, φ_τ and h_τ are analytical matrices, moreover φ_0 is the matrix of orthogonal projection. The matrix h_τ is defined from these conditions non-uniquely, but the condition of evasion 2 guarantees the upper boundedness of the norm of matrix h_0. Let C be infinum of the norm of all such matrices. Constant $\varkappa$ is defined by condition

$$\varkappa = C \cdot \sqrt{1 + \left(\max_{w \in \partial \varphi_0(\frac{1}{\lambda} Q)} \ \min_{\substack{v \in Q \\ \varphi_0(v) = w}} \frac{|w - v|}{|w|} \right)^2} ,$$

where $\partial \varphi_0(\frac{1}{\lambda} Q)$ is the boundary of set $\varphi_0(\frac{1}{\lambda} Q)$. We shall give later examples of calculation of constant $\varkappa$ in some given cases.

We shall construct the evasion in game (1) with conditions (2)-(4) on decreasing time interval θ_n , using L.S.Pontrjagin's evasion control (see [1]) $v_n(t) = v_n^1(t) + v_n^2(t)$, where the control $v_n^2(t)$ is used for neutralization of the second player (see §5 [1]) and the control $v_n^1(t)$, $|v_n^1(t)| \leq \rho_n$ is used for doing pass maneuver. We choose the sequences θ_n and ρ_n ($\theta_n = \mathrm{const} \cdot \frac{1}{n}$, $\rho_n = \mathrm{const} \cdot \frac{1}{n}$) in such a way that evasion control be admissible.

F. Example 1. Control example from [2] . In a Euclidean space R^2 , we consider the motions of two points x and y , where x is pursuer and y is the evading object . The process of pursuit is finished when $x=y$. The motions of the points x and y are given by the equations

$$\ddot{x}+\alpha\dot{x}=u, \tag{5}$$

$$\ddot{y}+\beta y=v, \tag{6}$$

where α, β are positive constants, and u and v are control vectors from R^2 satisfying the conditions (2), (3), (4), where P and Q are compact convex subsets of R^2 with $\dim Q=2$, $\operatorname{Int} Q \ni 0$, $P \ni 0$.

It is easy to show, that $h_0=E$, $\varkappa=1$ for game (1) and condition of evasion will be satisfied if there exists such a number $\lambda>1$ that $\lambda P \overset{*}{\subset} Q$ and $\sigma>\rho$, $\nu>\mu$.

Example 2 (given in [3]):

$$\ddot{x}_1=u_1, \quad \dot{y}_1=v_1,$$

$$\dddot{x}_2=u_2, \quad \ddot{y}_2=v_2,$$

where $x_i, y_i, u_i, v_i \in R^1$, $i=1,2$, $x=(x_1,x_2)$ is the pursuer and $y=(y_1,y_2)$ is the evading object, u and v are control vectors satisfying the conditions (2), (3), (4). The process of pursuit is finished when $x_1=y_1$, $x_2=y_2$. It is easy to show that $h_0=0$, $\varkappa=0$ for this game, and the condition of evasion will be satisfied if $\operatorname{Int} Q \ni 0$, $\sigma>0$, $\nu>0$.

R e f e r e n c e s

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