THE PURSUIT GAME WITH THE INFORMATION LACK OF THE EVADING PLAYER

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We consider zero-sum differential game with perscribed duration T . The kinematic equations have the form

$$P \; : \quad \dot{x} = f(x, u) \quad , \qquad u \in \mathcal{U} \subset \text{Comp}\, R^{k} ,$$
$$E \; : \quad \dot{y} = g(y, v) \quad , \qquad v \in V \subset \text{Comp}\, R^{e} ,$$

where $x \in R^{n}$, $y \in R^{n}$, $x(0) = x_0$, $y(0) = y_0$. We suppose that for every starting positions x_0 , y_0 and every pair of measurable open-loop controls $u(t)$, $v(t)$ there exists a unique solution of the system (1) with the initial conditions x_0 , y_0 .

The state of information is defined in the following way. The player E (evader) at each instant $t \in [0, T]$ knows the time t , the initial position of P (pursuier), x_0 , and his own position $y(t)$. When $t \in [0, \ell]$, P at each moment t knows his position $x(t)$, the time t and the initial state of the player E at $t = 0$, y_0 ; when $t \in [\ell, T]$, P at each moment t knows his position $x(t)$, the time t and the position of player E at moment $t - \ell$, $y(t-\ell)$., $\cancel{t}\ell > 0)$.

The payoff of the player E is defined as

$$M(x(T), y(T)) ,$$

where $M(x, y)$ is a given continuous function on $R^{n} \times R^{n}$. The game is supposed to be zero-sum.

Pure strategies. Under pure strategy of the player in the considered game we shall understand the so called piecewise control strategies (PCS). Under the (PCS) of the player E, $v(\cdot)$, we shall understand the pair $\{\tau, b\}$, where τ is a finite decomposition of the time interval $[0, T]$, $0 = t_1 \leq t_2 \leq \cdots \leq t_s = T$ and b-mapping which assigns to every state of information at the moment $t_k \in \tau$, t_k , x_0 , $y(t_k)$, a measurable open-loop control $v(t)$ defined on the time interval $[t_k, t_{k+1})$. Under the (PCS) of the player P, $u(\cdot)$, we shall understand the pair $\{\sigma, a\}$ where σ is a finite decomposition of the time interval $[0, T]$, $0 = t'_1 \leq t'_2 \leq \cdots \leq t'_q = T$ and a-mapping which assigns to every state of information at the moments $0 \leq t'_k \leq \ell$ $(t'_k \in \sigma)$, t'_k, y_0, $x(t'_k)$, a measurable open-loop control $u(t)$ defined on the time interval $[t'_k, t'_{k+1})$; and at moments $\ell < t'_k \leq T$ $(t'_k \in \sigma)$, it assigns to the t'_k, $y(t'_k - \ell)$, $x(t'_k)$ a measurable open-loop control $u(t)$ defined on the time interval $[t'_k, t'_{k+1})$.

Every pair of (PCS) $u(\cdot)$, $v(\cdot)$ and the initial conditions x_0, y_0 uniquely determine the trajectories $x(t), y(t)$ as solutions of (1), $t \in [0, T]$, and the payoff $M(x(T), y(T))$.

Thus we can define the pay-off function, as functional of pure strategies (PCS) in the following way

$$K(x_0, y_0; u(\cdot), v(\cdot)) = M(x(T), y(T)), \tag{3}$$

where $x(t)$, $y(t)$ are the trajectories corresponding to the strategy pair $u(\cdot)$, $v(\cdot)$ and initial conditions x_0, y_0.

The game under consideration is one with incomplete information for both players, as we well know from the general game theory, usually

$$\begin{aligned} &\operatorname{Sup}_{v(\cdot)} \operatorname{Inf}_{u(\cdot)} K(x_0, y_0; u(\cdot), v(\cdot)) \neq \\ &\neq \operatorname{Inf}_{u(\cdot)} \operatorname{Sup}_{v(\cdot)} K(x_0, y_0; u(\cdot), v(\cdot)). \end{aligned} \tag{4}$$

In case (4) holds it is difficult to speak about the solution of the game in any sense. So we have to follow the von Neumann's approach (see [1]) and introduce the mixed strategies in hope of finding the saddle point in an enlarged class.

Mixed behaviour piecewise control strategies (MB PCS). Under the (MB PCS) of the player E, $\nu(\cdot)$, we shall understand the pair $\{\tau, d\}$, where τ is a finite decomposition of the time interval $[0, T]$, $0 = t_1 \le t_2 \le \cdots \le t_s = T$, and b is a mapping which assigns to every state of information at moments $t_k \in \tau$, t_k, x_0, $y(t_k)$, a probability measure ν concentrated on a finite set of open-loop measurable controls $v(t)$, $t \in [t_k, t_{k+1})$. Under the (MB PCS) of the player P, $\mu(\cdot)$, we shall understand the pair $\{\sigma, c\}$, where σ is a finite decomposition of the time interval $[0, T]$, $0 = t_1' \le t_2' \le \cdots \le t_q' = T$, and c is a mapping which assigns to every state of information at the moments $0 \le t_k' \le \ell$ $(t_k' \in \sigma)$, t_k', y_0, $x(t_k')$ a probability measure μ concentrated on a finite set of open-loop measurable controls $u(t)$, $t \in [t_k', t_{k+1}')$; and at the moments $\ell \le t_k' \le T$ $(t_k' \in \sigma)$ it assigns to the t_k', $y(t_k' - \ell)$, $x(t_k')$ a probability measure μ concentrated on a finite set of open-loop measurable controls $u(t)$, $t \in [t_k', t_{k+1}')$.

Every strategy pair of (MB PCS) $\mu(\cdot)$, $\nu(\cdot)$ defines random trajectories $x(t)$, $y(t)$ from the initial position x_0, y_0. Thus the payoff $M(x(T), y(T))$ becomes random reachable and we have to consider its mathematical expectation. The latter is uniquely determined by the initial conditions x_0, y_0 and (MB PCS) strategy pair $\mu(\cdot)$, $\nu(\cdot)$. We shall write it as a functional of $\mu(\cdot)$, $\nu(\cdot)$

$$E(x_0, y_0; \mu(\cdot), \nu(\cdot)) = \mathrm{Exp}\, M(x(T), y(T)), \qquad (5)$$

when the expectation is taken by the probability measure over the trajectories $x(t), y(t)$ $(x(0) = x_0, y(0) = y_0)$ corresponding to the (MB PCS) strategy pair $\mu(\cdot)$, $\nu(\cdot)$.

We shall derive later some sufficient condition under which the equation (4) holds in the class of (MB PCS) strategies.

An auxiliary zero-sum game Γ_y. Let $C_P^t(x)$, $C_E^t(y)$ be reachable sets of positions for the players P and E from the starting positions x, y by the moment t. We shall consider a simultaneous game Γ_y, $y \in C_E^{T-\ell}(y_0)$ over the sets of strategies $C_P^T(x_0)$, $C_E^{\ell}(y)$. The game proceedes as follows. The players P and E choose simultaneously and independently of each other the points $\xi \in C_P^T(x_0)$ and corresponding ly $\eta \in C_E^{e}(y)$. The payoff of player E is defined as $M(\xi,\eta)$. If we suppose the compactness of the sets $C_P^{T}(x_0)$ and $C_E^{t}(y)$, the game Γ_y for every $y \in C_E^{T-e}(y_0)$ (sec [1]) has the saddle point in mixed strategies, that means in the class of probability measures over the sets $C_P^{t}(x_0)$, $C_E^{e}(y)$ (the payoff $M(\xi,\eta)$ is assumed to be continuous).

We shall pose the following conditions on the class of games $\Gamma_y,\ y \in C_E^{T-e}(y_0)$.

1. For every $\varepsilon > 0$, there exists such N , that in the game Γ_y , P has an ε -optimal mixed strategy μ_ε , which prescribes equal probabilities $1/N$, to N points $\xi_i(y) \in C_P^{T}(x_0)$ and the number of the points N does not depend on y , when $y \in C_E^{T-e}(y_0)$.

2. Let $y(t)$ be any motion of E on the time interval $t_1 \leq t \leq t_2$, then there exists such N nonintersecting trajectories $\xi_i[y(t-e)] = \xi_i(t)$, that $\xi_i(y(t-e)) \in C_P^{T}(x_0)$, where every $\xi_i[y(t-e)]$ is a spectrum point of the strategy μ_ε , which is ε -optimal in the game $\Gamma_{y(t-e)}$.

Let $V(y)$ be the value of the game Γ_y and

$$V(\bar{y}) = \max_{y \in C_E^{T-e}(y_0)} V(y) .$$

Now we can describe the construction of the ε -optimal (MB PCS) for both players in the previous game.

Theorem. Let the sets $C_P^{T}(x_0)$, $C_E^{e}(y)$, be compact for each $y \in C_E^{T-e}(y_0)$ and the conditions 1, 2 be satisfied. Suppose that for every $\varepsilon_1 > 0$ player P can **guarantee** ε_1 -capture at the moment T with any of the points ξ_i moving along the trajectories $\xi_i[y(t-e)] = \xi_i(t)$, when E moves along $y(t)$.

Then the value of the game is equal to $V(\bar{y})$ (see (6)) . The ε -optimal (MB PCS) for E includes the open-loop control transition to $\bar{y}$ on the time interval $t \in [0, T-e]$, and further transfer to any point $y(T) \in C_E^{\ell}(\bar{y})$, which occurs after the realisation of the random device according to the $\varepsilon/3$ - optimal mixed strategy of the player E in the game $\Gamma_{\bar{y}}$ at the moment $t = T-\ell$.

The ε -optimal (MB PCS) for P randomly chooses at $t=0$ with the probability $1/N$ any of the points ξ_i contained in the spectrum of his $\varepsilon/3$ optimal strategy in the game Γ_{y_0} and prescribes the pursuit of this point to guarantee the $\varepsilon/3$ capture with it at the terminal moment T .

When $M(\xi, \eta) = \rho(\xi, \eta)$, where ρ is an euclidean distance the value of the game, strategies mentioned in the theorem have an interesting geometric interpretation. The value $V(y)$ of Γ_y is equal to the radius $R(y)$ of the minimal sphere $S(y)$ which contains the set $C_E^{\ell}(y)$. The value of the previous game $V(\bar{y})$ is equal to the maximal radius $R(\bar{y}) = \max_{y \in C_E^{T-\ell}(y_0)} R(y)$.
The optimal (NB PCS) of player P is pure and consists in the pursuing of the centre $O(y)$ of the minimal sphere $S(y)$.

The optimal strategy of the player E is (MB PCS). On the time interval $[0, T-\ell)$ he moves to the point $\bar{y}$, for wich the radius $R(y)$ of minimal sphere containing the set $C_E^{\ell}(y)$ reaches its maximal value. Let us consider now the auxilary game $\Gamma_{\bar{y}}$. The payoff function in this game is convex, so the maximising player has an optimal mixed strategy which prescribes positive probabilities to no more than $n+1$ points of the set $C_E^{\ell}(\bar{y})$, where n is a dimension of the space R^n . One can prove that these points lie on the boundary of the minimal sphere containing $C_E^{\ell}(\bar{y})$. We denote them $\eta_1, \cdots, \eta_{n+1}$. Let $O(\bar{y})$ be the center of this minimal sphere, then there exist such

$$\lambda_i \ , \ \lambda_i \geqslant 0 \ , \quad i = 1, \dots, n+1 \ ,$$
$$\sum_{i=1}^{n+1} \lambda_i = 1 \ ,$$

that

$$\sum_{i=1}^{n+1} \lambda_i \eta_i = O(\bar{y})$$

At the moment $t = T - \ell$, E chooses with the probability λ_i, $i = 1, \ldots, n+1$, the direction to one of the points η_i, $i = 1, \ldots, n+1$ and on the interval $(T - \ell, T]$ moves to reach it at moment T.

REFERENCES

1 . Karlin S. Mathematical methods and theory of games, programming and economics, Pergamon Press, 1953.