CONVEXITY PROPERTIES IN STRUCTURAL OPTIMIZATION

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INTRODUCTION

Optimization by a digital computer of a given structural design necessarily implies the reduction of a continuum to a finite number of degrees of freedom, be it rather large. In what follows it is understood that this discretization is achieved by a finite element method, although most of the properties to be exhibited are shared by finite difference procedures. The main variables discribing the response of the structure to its environment are either

- a finite dimensional vector of generalized displacements, noted q
- a finite dimensional vector of generalized deformations, noted e
- a finite dimensional vector of generalized stresses, noted s.

The action of the environment is limited here to the specification of sets of generalized loads, a given set being noted as a finite dimensional vector g.
The sources of such loads are multiple; they may be of gravitational, aerodynamical or thermal origin.
The optimization itself consists in the determination of finite sets of design variables for which the following hierarchy may be conveniently adopted [1,2] :

1. Transverse dimensional design variables.
   They are most easily described in terms of the discretized model of the structure. If we conceive the structure as made of a set of interconnected bars and plates, the local cross-sectional area of a bar, or thickness of a plate are design variables of this type. It is understood that the specification of such variables in a finite number of locations is accompanied by the specification of interpolation functions allowing the transverse dimensions to be known everywhere.
   It must be noted that alterations of the transverse design variables in thin-walled structures does not in principle modify either the external geometry of the structure, nor the topology of the interconnexions of its component parts.
2. Configuration variables.
   Some of them may still keep the external geometry invariant, while altering the length of bars and plates and modifying the relative angles between component parts. More generally they can also modify the external shape and the permissibility of this depends on the function the structure has to fulfill. Clearly there is more freedom in this respect for a bridge design, while there is very little for an aircraft wing, where the external shape is largely dictated by aerodynamic considerations. Configuration variables are those that do not belong to the first group but that still keep the topology of component interconnexions invariant.

3. Material properties.
   While variables of the two preceding groups have continuous variations between upper and lower bounds, the choice of material properties for each component is of discrete type. For this reason the optimization of the choice of materials is largely one of direct engineering judgment, possibly a problem of direct comparison between few designs involving different options. An exception must however be made for composite materials such as fiber and matrix where fiber orientation is a continuous variable very similar to a configuration variable.
4. Topological variables.
   Again differences in topology in the interconnexions cannot be mapped as a continuous change of variables. Any particular choice is mostly based on previous expérience and engineering intuition, although purely technological considerations are usually also involved.

Our conclusions about design variables is that little can be done presently in the matter of a useful mathematical formalism concerning the two last groups, except perhaps for very simple component parts. Moreover the changes in configuration variables have essentially non linear repercussions on the response of the structure, while the changes in transverse dimensions lead to simple properties of linearity or convexity. For this reason most of the efforts towards computerized optimization of structures is presently concerned, as in this paper, with the first group of variables only.

The optimality criterion itself may be very complex when aiming at a significant estimation of cost. For this reason, optimization in civil engineering where cost of materials, manufacture, manpower, delays, stock and investment are essential ingredients is totally different from optimization in aerospace as envisaged in this paper. The consideration of weight is so predominant in this last case, that it usually supersedes all other factors and leaves a very simple functional to be minimized, one that is both linear and homogeneous in the design variables of the first group. Moreover the cost of aerospace structures being high and the consequences of a bad design extremely heavy, the investments in scientific computation of the structural response and the search for optimality are more easily accepted.

We must now describe the types of constraints imposed on either the design variables themselves or on the structural response.

The transverse dimensional design variables are usually bounded from below and from above for reasons of manufacture and handling or for safegard against haphazard environmental actions that would unreasonably complicate the mathematical description of the loading cases. If c denotes the set of design variables we have thus for each component

$$0 < \underline{c}_i \leq c \leq \bar{c}_i$$

The result of a continuous approach to design variables may conflict with the use of a standardized scale of gauge thicknesses, in which case the gauge closest to the

value obtained will generally be tried for the final answer.
The structural response itself receives at least the two following constraints :

1. For a specified set of external loads the elastic limit of the materials involved may not be exceeded, or a limit well below the elastic limit is set to obtain a lower bound to the safe number of loading cycles in fatigue.
2. For a specified set of external loads there may be no loss of or even bifurcation of the stability of equilibrium.

In many cases haphazard exceptional loading cases are specified for which bifurcation of the equilibrium is allowed, provided the structure continues to resist elastically with a redistributed state of stress. Loads may be envisaged under which the elastic limits are exceeded and the structure becomes permanently damaged, provided there is no catastrophic collapse leading to loss of lives.
While structures optimized under constraints of type 1 and 2 can be tested against such geometrically or materially non linear phenomena, it does not seem reasonable at present to include them in the optimization procedure itself.
The following constraints are also technically significant :

3. Some linear combination of the displacements must satisfy a given equality or inequality under a given set of loads.
   In this category we find the prescription of limitation of a global rigidity characteristic of the structure, such as the torsional rigidity of a wing under tip torque or of an automobile chassis.
   Another example is the requirement that the trailing edge of an aircraft spoiler, straight in the retracted position, should remain straight when fully opened in the air stream [3].
4. Specified values or bounds are set to the low frequency vibration spectrum of the structure.

STRUCTURAL RELATIONS

The relations between the structural response variables and the loads can conveniently be decomposed and presented in matrix form as follows [4].
There are purely kinematical relations linking generalized displacements and strains; they imply compatibility of the strains,

$$e = S^T q \tag{1}$$

and a dual relationship involves the equilibrium between loads and stresses

$$g = Ss \tag{2}$$

The global kinematical matrix S depends solely on the topology of element interconnexions and is <u>independent</u> of the values of the dimensional design variables and of material properties.
The conjugate character of displacements and loads and of stresses and strain appears

clearly in the virtual work theorem

$$q^T g = q^T S s = (S^T q)^T s = e^T s \tag{3}$$

Assuming the material properties to be linear elastic, we add the constitutive equations

$$s = Je \qquad J \text{ positive definite.} \tag{4}$$

From this we can derive the global stiffness relation between loads and displacements

$$g = K\,q \tag{5}$$

$$K = SJS^T = K^T \qquad \text{the global stiffness matrix.}$$

K is certainly non negative, it is not restrictive, even if we have to suppress some rigid body modes by adding artificial kinematical boundary conditions, to assume it also positive definite. The elements of J, hence also those of K, are linear homogeneous functions of the design parameters

$$K = \sum_i c_i \frac{\partial K}{\partial c_i} \qquad J = \sum_i c_i \frac{\partial J}{\partial c_i} \qquad c_i > 0 \tag{6}$$

The matrices of partial derivatives depend only on material properties.

WEIGHT FUNCTIONAL and CONSTRAINTS

The weight functional is obviously a positive linear form in the design parameter

$$\omega = \sum_i p_i c_i \qquad p_i > 0 \tag{7}$$

the coefficients p_i depending on the material properties. It has the lower bound

$$\underline{\omega} = \sum_i p_i \underline{c_i} \tag{8}$$

Consider now the constraints stemming from upper bounds to the stressing of the material. In an isotropic continuum the Hüber-Hencky-Von Mises bound on the elements τ_{ij} of the local stress tensor

$$(\tau_{11}-\tau_{22})^3 + (\tau_{22}-\tau_{33})^2 + (\tau_{33}-\tau_{11})^2 + 6(\tau_{12}^2+\tau_{23}^2+\tau_{31}^2) \leq 2\sigma_e^2$$

(where σ_e is the elastic limit under uniaxial stress) is very convenient to use. It is better adapted to our purpose, when expressed in terms of the strain tensor ε_{ij}

$$(\varepsilon_{11}-\varepsilon_{22})^2 + (\varepsilon_{22}-\varepsilon_{33})^2 + (\varepsilon_{33}-\varepsilon_{11})^2 + 6(\varepsilon_{12}^2+\varepsilon_{23}^2+\varepsilon_{31}^2) \leq 2(1+\nu)^2\varepsilon_e^2$$

(ν is Poisson's ratio). For anisotropic materials the quadratic form has more complicated coefficients but remains essentially positive definite.
This explains that in any given component (finite element) of the structure the elastic

limit is nowhere exceeded if the set $e_{(e)}$ of generalized strains in this component is subjected to suitable constraints (finite in number) of the form

$$e_{(e)}^T E_e e_{(e)} \leq \alpha_e \qquad E_e \text{ positive definite matrix} \qquad \alpha_e > 0. \tag{9}$$

As $e_{(e)}$ is a subset of e, we may write

$$e_{(e)} = B_e e \qquad B_e \text{ a Boolean matrix}$$

and, in view of equation (1), each constraint of this type is translated in a constraint on the displacement vector

$$q^T S B_e^T E_e B_e S^T q \leq \alpha_e \tag{9'}$$

Such constraints are independent of the values of the design variables but depend on the material properties.
A constraint of global rigidity type is equivalent to the requirement of a minimum value for the strain energy under the prescribed load system; hence it can be presented in the form

$$\frac{1}{2} q^T g = \frac{1}{2} g^T F g \geqslant \beta > 0 \tag{10}$$

where g is known and F, the global flexibility matrix

$$F = K^{-1} \rightarrow q = Fg \tag{11}$$

depends non linearly on our definition of design parameters.
If, under a given loading system g, a linear constraint

$$m^T q = \gamma$$

is imposed on the displacements, we obtain the constraint

$$m^T Fg = \gamma$$

that involves again the flexibility matrix.
The elastic stability constraints will be analyzed later.

ISOSTATICITY

The property of a structure to be isostatic is well known from simple examples of pin-jointed trusses. The concept can be extended to a continuum[5], the degree of hyperstaticity being identified with the degree of linear connectivity. The definition of isostaticity from the view point of a discretized model is that the homogeneous equation associated to (2) has only the trivial solution

$$S\ s = 0 \quad \rightarrow \quad s = 0$$

Then, provided the structure is isostatically supported, S is square and non singular and the generalized stresses can be directly determined for any loading conditions from the equilibrium equations as

$$s = S^{-1}\ g$$

We may note that this situation is seldom met in practice for more general thin-walled structures, because discretization induces artificial hyperstaticity, even if the continuum is simply connected.
An isostatic structure can be designed to be fully stressed under a single loading case. The case of constant strain elements (corresponding to first degree polynomial approximations to the displacement field) is particularly obvious in that respect. The plate thickness or bar cross-sectional area is taken to be constant within the element so that a single design parameter c_e and a single constraint (9) are to be considered. The generalized stress $s_{(e)}$ is known from statics and its relation to the generalized strain is

$$s_{(e)} = c_e \frac{\partial J_e}{\partial c_e}\ e_{(e)}$$

where $\partial J_e\ /\ \partial c_e$ is a positive definite matrix independant from c_e.
The weight of the element is $p_e\ c_e$ where p_e is some positive constant. Clearly, since c_e should be minimized, its minimum value is obtained by satisfying the constraint (9)

$$\frac{1}{c_e^2}\ s_{(e)}^T\ \left(\frac{\partial J_e}{\partial c_e}\right)^{-1} E_e\ \left(\frac{\partial J_e}{\partial c_e}\right)^{-1}\ s_{(e)} = \frac{1}{c_e^2}\ h_e\ \leq\ \alpha_e$$

as an equality.
If several loading cases are to be considered it is also clear that in each element the design parameter has to be choosen by the same equality constraint for the largest of the h_e values generated by the different loading cases. Hence, in general, for each case, at least one of the elements will be stressed to its limit capacity.
This concept of fully stressed design [6] has been extended to hyperstatic structures as an approximation to real minimum weight design under stress constraints alone.

HYPERSTATICITY

Isostatic structures are not efficient when, as is mostly the case, several types of loadings are to be taken into account. Cooperation of all the resisting members due to redundant coupling helps to reduce local peak stresses and is finally conducive to lighter and stiffer structures.

Hyperstatic structure possess self-stressing states, each of which is an s vector, solution of the homogeneous equation associated to (2) (g=0).

If X is a matrix, whose columns form a basis for the subspace of self-stressings, we may write

$$SX = 0 \quad \rightarrow \quad X^T S^T = 0 \tag{12}$$

and, as general solution to equation (2),

$$s = S^{\#} g + Xx \tag{13}$$

Where $S^{\#} g$ is any particular stress vector in equilibrium with the loads and x an arbitrary vector of intensities of self-stressings, usually termed <u>redundancies</u>.

Neither the particular pseudo-inverse $S^{\#}$, nor the matrix X depend on the design parameters, they depend only on the topology of interconnexions.

The determination of the redundancies rests on compatibility conditions for the strains

$$e = J^{-1} s$$

They are the existence conditions for inversion of (1), that is, in view of (12)

$$X^T e = 0 \quad \rightarrow \quad X^T J^{-1} S^{\#} g + X^T J^{-1} X x = 0 \tag{14}$$

Because X is a base matrix (independent columns), this set of equations for x has a positive definite, hence invertible, matrix.

The presence of J^{-1} causes the redundancies to depend non linearly on the design parameters. The satisfaction of the stressing constraints becomes therefore difficult and iterative search techniques are needed, [7,8,9].

STRESS CONSTRAINTS AND CONVEXITY OF THE SET OF ADMISSIBLE LOADS

When several loading cases are considered, the following question arises : to which extent may the loads be linearly combined without overstressing a given design ? Consider the general linear combination

$$g = \sum_{1}^{n} \lambda_m \, g_{(m)} \tag{15}$$

where the "design" loads $g_{(m)}$ are specified.

The λ_m, positive or negative, are loading factors. It is easily shown that, when all the constraints (9) are satisfied, they belong to a convex set of λ space. Observe that in g-space each form (9') of the constraints requires the q-vectors to belong to a convex, but generally unbounded set (even independent of the design variables). The intersection of all these convex sets is itself convex and bounded (again provided the kinematic degrees of freedom have been removed). The linear transformation (5) maps this convex set into a convex bounded set of g space. Hence if all the stressing constraints are satisfied for each design load, they remain satisfied for the linear combination (15) if (sufficient condition) the combination is convex

$$\lambda_m \geqslant 1 \qquad m = 1,2 \ldots n \qquad \sum_1^n \lambda_m = 1. \tag{16}$$

Indeed each $g_{(m)}$ lies in the convex set of admissible loads and the convex combination being the smallest convex set containing the $g_{(m)}$, is also contained in the admissible set. The convex admissible set of loads depends of course through the mapping (5) on the values of the design parameters.

STABILITY OF EQUILIBRIUM AND CONVEXITY OF THE SET OF ADMISSIBLE LOADS

Under a given loading vector λg a stability matrix $\hat{S}$ (not to be confused with the kinematical matrix) can be obtained that enables the criterium of elastic stability to be placed in the form

$$u^T \hat{S} u + u^T K u \geqslant 0 \quad \text{for every } u \tag{17}$$

where u is a vector of perturbation of displacements. Assuming the gradients of the displacements at equilibrium in the continuum to be negligible before unity (small strains and rotations), the $\hat{S}$ matrix may be taken to be proportional to be loading factor λ , we write

$$\hat{S} = - \lambda S$$

Changing the stability criterium to

$$\lambda \mu \leqslant 1 \qquad\qquad \mu = \frac{u^T Su}{u^T Ku} \tag{18}$$

Let $\bar{\mu}$ and $\underline{\mu}$ be respectively the maximum and minimum of the Rayleigh quotient μ.

Case 1 $\qquad \underline{\mu} < \bar{\mu} < 0$

For every u $\qquad \mu < 0 \quad$ and, as $\dfrac{1}{\bar{\mu}} < \dfrac{1}{\underline{\mu}} < 0$

the structure is unconditionally stable for positive loading factors, the negative values being limited by $\quad \lambda \geqslant 1/\underline{\mu}$.

Case 2 $\quad 0 < \underline{\mu} < \overline{\mu}$

For every u $\quad \mu > 0$ and, as $\frac{1}{\underline{\mu}} > \frac{1}{\overline{\mu}} > 0$
the structure is unconditionally stable for negative loading factors, the positive values being limited by $\lambda \leq 1/\overline{\mu}$

Case 3 $\quad \underline{\mu} < 0 < \overline{\mu}$
is the general one as compression stresses prevail usually somewhere for positive as well as negative loading factors.
The loading factors are bounded in both directions

$$\frac{1}{\underline{\mu}} \leq \lambda \leq 1/\overline{\mu}$$

Consider now again the case of a linear combination (15) of several loading cases
We have $\quad \hat{S} = -\sum_{1}^{n} \lambda_m S_m$

and the stability condition

$$-\sum_{1}^{n} \lambda_m u^T S_m u + u^T K u \geq 0 \quad \text{for any perturbation } u$$

In the positive hyperoctant of λ-space we solve the eigenvalue problem for given $\lambda_m \geq 0$

$$\alpha K u = \sum_{1}^{n} \lambda_m S_m u$$

the stability condition being then $u^T Ku\ (1-\alpha) \geq 0$ or,
Since $u^T K u > 0, \quad \alpha \leq 1$.

But if $\overline{\mu}_m = \max \dfrac{u^T S_m u}{u^T Ku}$

$$u^T S_m u \leq \overline{\mu}_m u^T Ku \qquad \text{for any } u$$

and $\quad \alpha u^T Ku = \sum_{1}^{n} \lambda_m u^T S_m u \leq u^T Ku \left(\sum_{1}^{n} \lambda_m \overline{\mu}_m \right)$

Whence the stability criterion is certainly satisfied if

$$\Sigma \lambda_m \overline{\mu}_m \leq 1 \tag{19}$$

When all the upper bounds $\overline{\mu}_m$ are positive this condition bounds the positive hyperoctant in λ-space by a hyperplane passing through the coordinates $1/\overline{\mu}_m$ on each axis. If one or several of the upper bounds are negative, the positive part of the

hyperplane is a boundary but the hyperoctant itself is unbounded.
In the hyperoctant $\lambda_1 \leq 0$, other $\lambda_m \geq 0$, it is sufficient to replace $u^T S_1 u$ by its minimum $\underline{\mu}_1 u^T K u$ and the stability conditions is seen to be satisfied by

$$\lambda_1 \underline{\mu}_1 + \sum_2^m \lambda_m \overline{\mu}_m \leq 1 \tag{20}$$

This produces the bounding hyperplane for this hyperoctant. The generalization to the other hyperoctants is obvious.
In the usual case where $\underline{\mu}_m < 0 < \overline{\mu}_m$ for all m, it is seen that the convex polyedron defined by its vertices $1/\overline{\mu}_m$ and $1/\underline{\mu}_m$ on the m axis is a domain of stability in λ-space.

The domain of stability is in fact a larger one. The characteristic surface bounding the domain in the positive hyperoctant will be shown to be convex. Suppose we know the critical perturbation shape u that, for a given set $\lambda_m \geq 0$ belonging to the characteristic surface, yields the critical eigenvalue $\alpha = 1$. Thus

$$K u = \Sigma \lambda_m S_m u \tag{21}$$

A first order perturbation gives

$$(1 + d\alpha)K(u + du) = \Sigma (\lambda_m + d\lambda_m) S_m (u + du)$$

Or, after simplifying by (21) and keeping first order terms,

$$(K - \Sigma\lambda_m S_m) du = \Sigma d \lambda_m S_m u - d\alpha Ku \tag{22}$$

The homogeneous equation, identical to the homogeneous adjoint since the matrix is symmetrical, has the non trivial solution

$$du = d\rho\, u \qquad d\rho \quad \text{arbitrary}$$

Hence the existence condition for a solution to the non homogeneous problem is

$$\Sigma d \lambda_m u^T S_m u - d\alpha u^T K u = 0 \tag{23}$$

For $d\alpha = 0$ we stay on the tangent plane

$$\Sigma d \lambda_m u^T Su = 0 \tag{24}$$

to the characteristic surface. Keeping first order perturbations on the loading factors, let us now examine how the critical perturbation on displacements and the eigenvalue are affected to second order

$$(1 + d\alpha + d\beta)K(u + du + dv) = \Sigma\ (\lambda_m + d\lambda_m)\ S_m\ (u + du + dv)$$

In view of (21) and (22) this already reduces to the second order terms balance

$$(K - \Sigma\ \lambda_m\ S_m)\ dv = \Sigma\, d\lambda_m\ S_m\ du - d\alpha\ K\ du - d\beta\ K\ u$$

The existence condition for dv, or a simple cancellation of terms obtained from (21) as

$$dv^T K u = \Sigma\lambda_m\ d\ v^T\ S_m\ u$$

yields

$$d\beta\ u^T K u = \Sigma\ d\lambda_m\ u^T S_m\ du - d\alpha\ u^T\ K\ du$$

The right-hand side can be transformed by premultiplication of (22) by u^T, hence

$$d\beta\ u^T\ K\ u = du^T\ K\ du - \Sigma\lambda_m\ du^T\ S_m\ du$$

Now as $u^T\ K\ u > 0$ and, by hypothesis

$$\max_{v} \frac{\Sigma\lambda_m\ v^T\ S_m v}{v^T\ Kv} = 1$$

We obtain $d\beta \geq 0$

This shows in particular that, when we move in the tangent plane to the characteristic surface, the eigenvalue $\alpha = 1$ receives a positive second order increase and we penetrate into the unstable region. The characteristic surface is therefore convex.

A similar conclusion is reached for the characteristic surfaces of the other hyperoctants.

This constitutes another proof that the domain of stability is convex in λ-space.

The two preceding convexity properties provide a justification for considering a finite number of loading cases, the vertices of a convex polyedron.

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