OPTIMAL POLLUTION CONTROL OF A LAKE (+)

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ABSTRACT

This paper considers a lake in which a pollutant is dumped at a rate whose maximum value is constant. We assume that the quantity of pollutant eliminated by natural processes is proportional to the total amount of pollutant contained into the lake. With this process we associate a cost which is the sum of two terms : the first one represents the cost of cleaning up a fraction of the pollutant and the second term is a measure of the damage done to the environment.

We then determine the optimal dumping policy, i.e., the policy which minimizes that cost integrated over a fixed period of time by solving an optimal control problem

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1. Introduction

We are concerned with the time history of the pollution of a lake by a pollutant dumped at a time rate where maximum value is constant.

Two cases are examined : either the pollutant is non-degradable (purely cumulative system), or the pollutant is disappearing exponentially with time (phenomenon of sedimentation, renewal of lake's water or radio-active decay).

The goal is to compute the cleaning policy such that the discounted sum of the costs (cleaning plus damage to the environment) extended over a fixed period of time is minimal. That goal is achieved by solving an optimal control problem. This is done in the four distinct situations obtained by combining constant or linear marginal cost of cleaning up with constant or linear marginal cost of damage to the environment.

We shall find that while in conventional environmental economics (static case), the optimal strategy consists in setting a cleaning up standard, the optimal strategy in this case amounts to a fixed standard in a few instances and to selecting time varying clean up standards in most cases.

2. Statement of the problem

The sources of pollution are emitting the pollutant at a constant time rate q_o. At time τ the fraction $q(\tau)$ is dumped into the lake and the fraction $q_o - q(\tau)$ is cleaned up. This gives rise to the cleaning cost :

$$c q_o \left[\frac{q_o - q(\tau)}{q_o}\right]^p, \qquad c > 0, \quad p \geqslant 1 \tag{1}$$

and $q(\tau)$ is constrained according to :

$$0 \leqslant q(\tau) \leqslant q_o \tag{2}$$

On the other hand, if $Q(\tau)$ is the total amount of pollutant contained into the lake at time τ, the cost due to the damage done to the environment will be :

$$b Q_M \left[\frac{Q(\tau)}{Q_M}\right]^n, \qquad b > 0, \quad n \geqslant 1 \tag{3}$$

where the quantity Q_M will be defined later.

The total cost extended over a fixe period of time $[o, T]$ will then be :

$$\int_o^T \left\{ cq_o\left[\frac{q_o - q(\tau)}{q_o}\right]^p + bQ_M\left[\frac{Q(\tau)}{Q_M}\right]^n \right\} e^{-at}dt \qquad (4)$$

where $a > 0$ is the discount factor.

If we assume that the quantity of pollutant disappearing by a natural process (e.g. sedimentation, renewal of lake's water or radio-active decay) is proportional to the total amount of pollutant $Q(\tau)$ contained into the lake, $Q(\tau)$ and $q(\tau)$ are related by the following ordinary differential equation :

$$\frac{dQ(\tau)}{d\tau} = -fQ(\tau) + q(\tau) \qquad (5)$$

where $f \geqslant 0$ and $f = 0$ for the purely cumulative system.

We call social horizon, the inverse $1/a$ of the discount factor. Through the relation :

$$\frac{1}{a} = \int_o^\infty e^{-at}d\tau \qquad (6)$$

the social horizon can be interpreted as a discounted sum of intervals of time : the largest the discount factor, the smallest is the social horizon.

Next, we define the quantity Q_M of (3) : it is the amount of pollutant contained into the lake at the social horizon for the purely cumulative system when no clean up is performed and when the initial amount $Q(o)$ is zero :

$$Q_M = \int_o^{1/a} q_o d\tau = \frac{q_o}{a} \qquad (7)$$

We are looking for a policy $q(\tau)$, $\tau \varepsilon [o,T]$, which minimizes (4) under the constraints (2) and (5).

Defining the non dimensionnal variables :

$$t = a\tau, \quad u = \frac{q_o - q}{q_o}, \quad x = \frac{Q}{Q_M} \qquad (8)$$

and the parameters :

$$t_f = aT, \qquad k = \frac{b}{ac}, \qquad \ell = \frac{f}{a} \tag{9}$$

we obtain the following optimal control problem :

Find the optimal control $u^*(t)$, $t\varepsilon[0,t_f]$ *and the corresponding optimal trajectory* $x^*(t)$, $t\varepsilon[0,t_f]$ *which minimizes*

$$\int_0^{t_f} [kx^n(t) + u^p(t)]e^{-t}dt \tag{10}$$

with $k > 0$, $n \geqslant 1$, $p \geqslant 1$, *under the constraints :*

$$\dot{x}(t) = -\ell x(t) + 1 - u(t), \qquad 0 \leqslant u(t) \leqslant 1, \; t\varepsilon[0,t_f] \tag{11}$$

with $\ell \geqslant 0$, *starting with initial condition* $x(0) = x^o \geqslant 0$. *We ask further that the optimal control* $u^*(t)$, $t\varepsilon[0,t_f]$ *belongs to the class of piecewise continuous functions.*

Recall that $u(t) = 1$ corresponds to no pollution and $u(t) = 0$ to maximum pollution; also, $\ell = 0$ for a purely cumulative system.

We shall study the 4 cases corresponding to p and n equal to 1 or 2, for finite and infinite terminal time.

3. Technique of analysis

To solve the problem, we use the maximum principle of Pontryagin [1]. Hence, consider the hamiltonian

$$H(t,\lambda,x,u) = \lambda_o(kx^n + u^p)e^{-t} + \lambda(-\ell x + 1 - u) \tag{12}$$

where $\lambda_o = -1$ if $t_f \neq \infty$ and $\lambda_o \leqslant 0$ if $t_f = \infty$ [2].

The optimal control $u^*(t)$ must satisfy the condition

$$H(t,\lambda(t),x(t),u^*(t)) \geqslant H(t,\lambda(t),x(t),u) \qquad \forall u\varepsilon[0,1] \tag{13}$$

whenever $\lambda(t)$ satisfy the ordinary differential equation :

$$\dot{\lambda}(t) = -\frac{\delta H}{\delta x} = -\lambda_o nkx^{n-1}(t)e^{-t} + \ell\lambda(t) \tag{14}$$

Furthermore, since $x(t_f)$ is free, $\lambda(t_f)$ must satisfy the transversality condition :

$$\lambda(t_f) = 0 \tag{15}$$

The problem is thus reduced to the solution of the two points boundary value problem given by relation (11), (13), (14) and (15), when t_f is finite. Indeed, condition (15) does not hold when t_f is infinite [2]. In this last case, we must integrate the system for arbitrary initial condition $\lambda(0)$ in order to compute the cost and find which $\lambda(0)$ minimizes it : the problem is reduced to parameter optimization.

4. Results

The solution of the two points boundary value problem (t_f finite), as well as the solution of the parameter optimization (t_f infinite) are straightforward, so that the details will not be given here. More details can be found in [3]-[5].

In the sequel, MCD will stand for marginal cost of damage and MCC for marginal cost of cleaning up.

4.1. Constant MCD (n = 1) and constant MCC (p = 1)

For a finite t_f we get :

$$u^*(t) = 0, \quad x^*(t) = e^{-\ell t}(x^o - \frac{1}{\ell}) + \frac{1}{\ell}$$

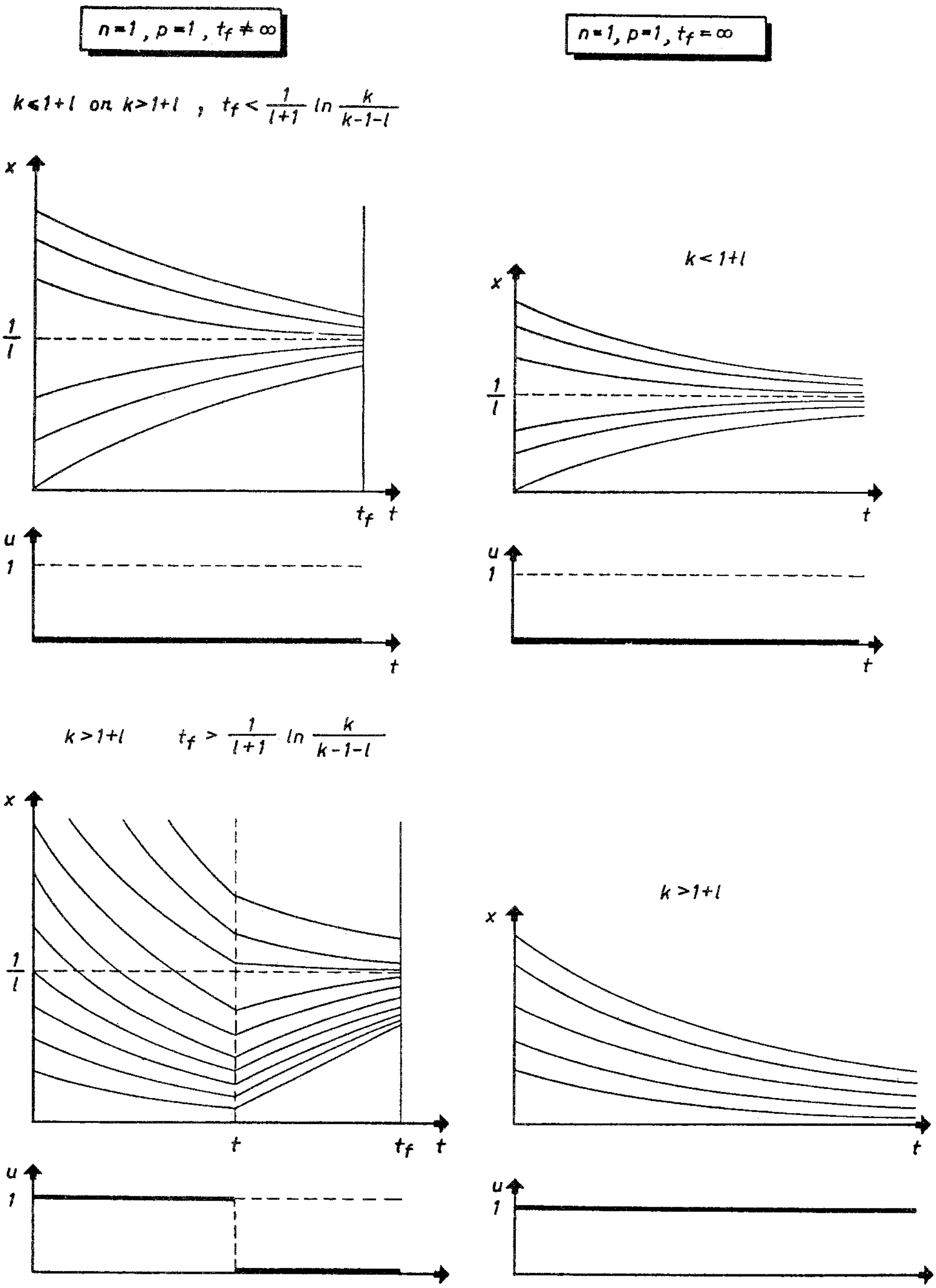
if $k \leqslant \ell + 1$ or $k > \ell + 1$ and $t_f \leqslant \frac{1}{\ell+1} \ell n \frac{k}{k-1-\ell}$,

and

$$u^*(t) = \begin{cases} 1 & t \varepsilon [0, t_1[\\ 0 & t \varepsilon [t_1, t_f] \end{cases}$$

$$u^*(t) = \begin{cases} e^{-\ell t} x^o & t \varepsilon [o, t_1) \\ e^{-\ell(t-t_1)}(e^{-\ell t_1} x^o - \frac{1}{\ell}) + \frac{1}{\ell} & t \varepsilon [t_1, t_f] \end{cases}$$

where $t_1 = \frac{1}{\ell+1}(t_f - \ell n \frac{k}{k-1-\ell})$



$n=1,\ p=1,\ t_f \neq \infty$
$n=1,\ p=1,\ t_f = \infty$
$k \leq 1+l$ or $k>1+l$, $t_f < \frac{1}{l+1} \ln \frac{k}{k-1-l}$
$k < 1+l$
$k > 1+l$ $t_f > \frac{1}{l+1} \ln \frac{k}{k-1-l}$
$k > 1+l$
x
$\frac{1}{l}$
u
1
t_f
t

Fig. 1

if $k > \ell + 1$ and $t_f > \frac{1}{\ell+1} \ln \frac{k}{k-1-\ell}$

For $t_f = \infty$, we get :

$u^*(t) = 0$ if $k < \ell + 1$, and $u^*(t) = 1$ if $k > \ell + 1$.
When $k = \ell + 1$, any control is optimal; indeed in that case it is possible to integrate the cost by parts and then realize that it depends only upon x^o.

We see that the optimal policy does not depend upon the initial level of pollution x^o but depends upon the parameters k, ℓ and t_f. For $k > 1 + \ell$ and t_f sufficiently large but finite, there is a switch from $u = 1$ to $u = 0$; that switch disappears when t_f becomes infinite. For all other cases the optimal policy is constant.

The situation is described on fig. 1.

4.2. Constant MCD ($n = 1$) and linear MCC ($p = 2$)

For a finite t_f we get :

$$u^*(t) = \frac{k}{2(\ell+1)} \left[1 - e^{-(\ell+1)(t_f-t)}\right]$$

if $k \leqslant 2(\ell + 1) / \left[1 - e^{-(\ell+1)t_f}\right]$

$$u^*(t) = \begin{cases} 1 & t \epsilon [0, t_1) \\ \frac{k}{2(\ell+1)} \left[1 - e^{-(\ell+1)(t_f-t)}\right] & t \epsilon [t_1, t_f] \end{cases}$$

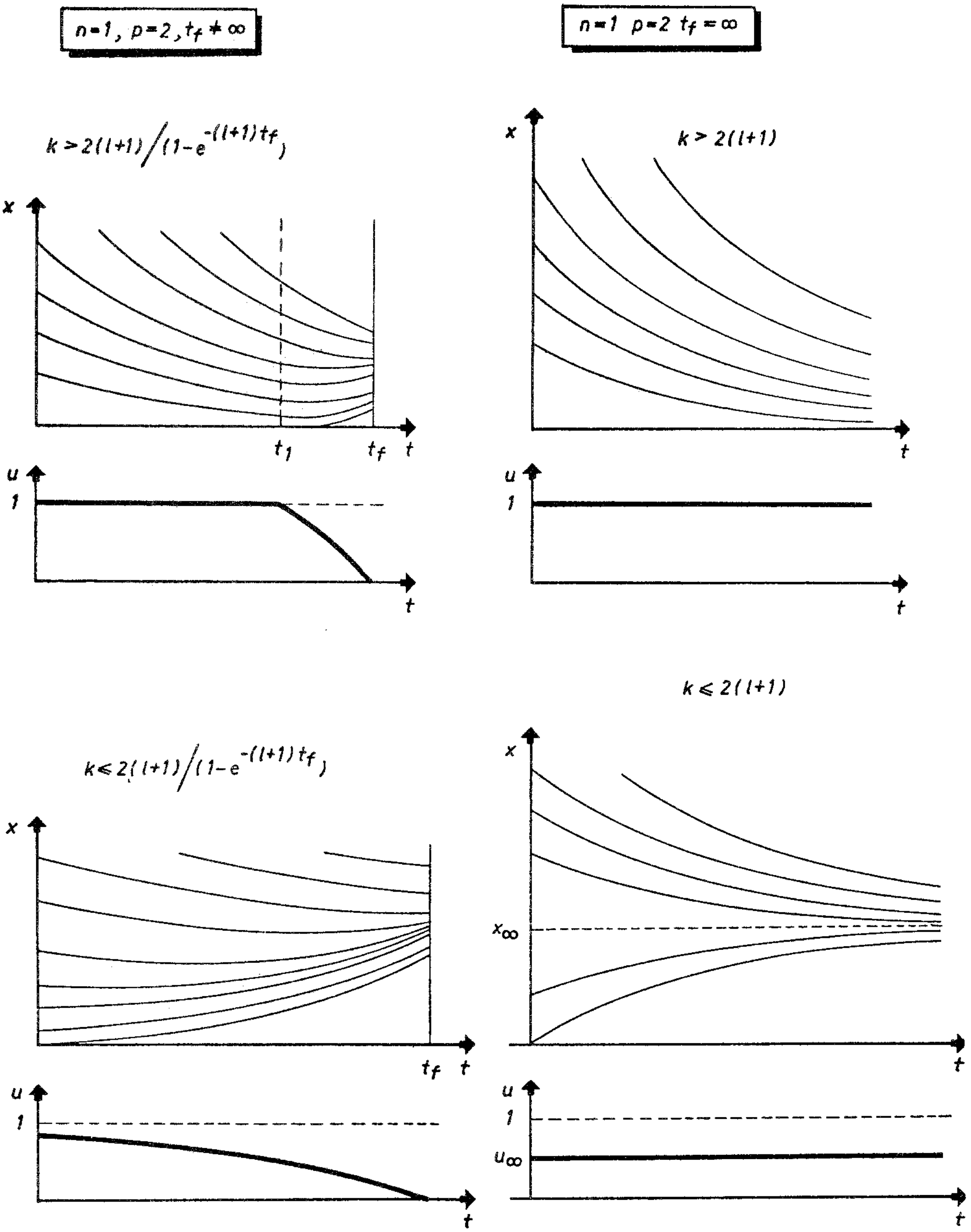
where $t_1 = t_f + \frac{1}{\ell+1} \; n \frac{k-2(\ell+1)}{k}$

if $k > 2(\ell + 1) / \left[1 - e^{-(\ell+1)t_f}\right]$

For $t_f = \infty$, we get :

$$u^*(t) = \frac{k}{2(\ell+1)} = u_\infty \qquad \text{if } k \leqslant 2(\ell + 1)$$

$$u^*(t) = 1 \qquad \text{if } k > 2(\ell + 1)$$



$n=1,\ p=2,\ t_f \neq \infty$
$n=1\ \ p=2\ \ t_f=\infty$
$k>2(l+1)/(1-e^{-(l+1)t_f})$
x
t_1
t_f
t
u
1
t
x
$k>2(l+1)$
t
u
1
t
$k\leq 2(l+1)$
$k\leq 2(l+1)/(1-e^{-(l+1)t_f})$
x
t_f
t
u
1
t
x
x_∞
t
u
1
u_∞
t

Fig. 2

Again, the optimal policy does not depend upon the initial level of pollution x^o, but depends upon the parameters k, ℓ and t_f. The optimal policy is constant for an infinite t_f but is partly or totally of exponential type when t_f is finite.

The situation is described on fig. 2.

4.3. Linear MCD (n = 2) and constant MCC (p = 1)

For a finite t_f, there is in the plane (t,x) a locus AB (see fig. 3 and 4) along which the optimal control will switch from $u = 1$ to $u = 0$. The equation of that locus is :

$$x = \frac{\ell[(\ell+1)(2\ell+1)-2k]-2k[(\ell+1)e^{-(2\ell+1)(t_f-t)}-(2\ell+1)e^{-(\ell+1)(t_f-t)}]}{2k\ell(\ell+1)[1-e^{-(t_f-t)}]}$$

That locus has a vertical asymptote given by $t = t_f$ and an horizontal one given by $x = \tilde{x}$ with

$$\tilde{x} = \frac{(\ell+1)(2\ell+1)-2k}{2k(\ell+1)}$$

For $t_f \to \infty$, the limit of the locus is its horizontal asymptote.

There is further, under certain conditions, a singular arc given by :

$$u_s = 1 - \frac{\ell(\ell+1)}{2k}, \qquad x_s = \frac{\ell+1}{2k}$$

(i) $x_s \geqslant 1/\ell \Rightarrow$ no singular arc.

For a finite t_f, if we define $\hat{x}$ as the intersection of the locus AB and the x axis, we get :

$$u^*(t) = 0 \qquad \text{if } x^o \leqslant \hat{x}, \text{ and}$$

$$u^*(t) = \begin{cases} 1 & t\varepsilon[0,t_1) \quad \text{if } x^o > \hat{x} \\ 0 & t\varepsilon[t_1,t_f] \end{cases}$$

where $(t_1,x(t_1))$ is a point of the locus AB.

The results are the same for $t_f = \infty$ provided we replace $\hat{x}$ by $\tilde{x}$ and

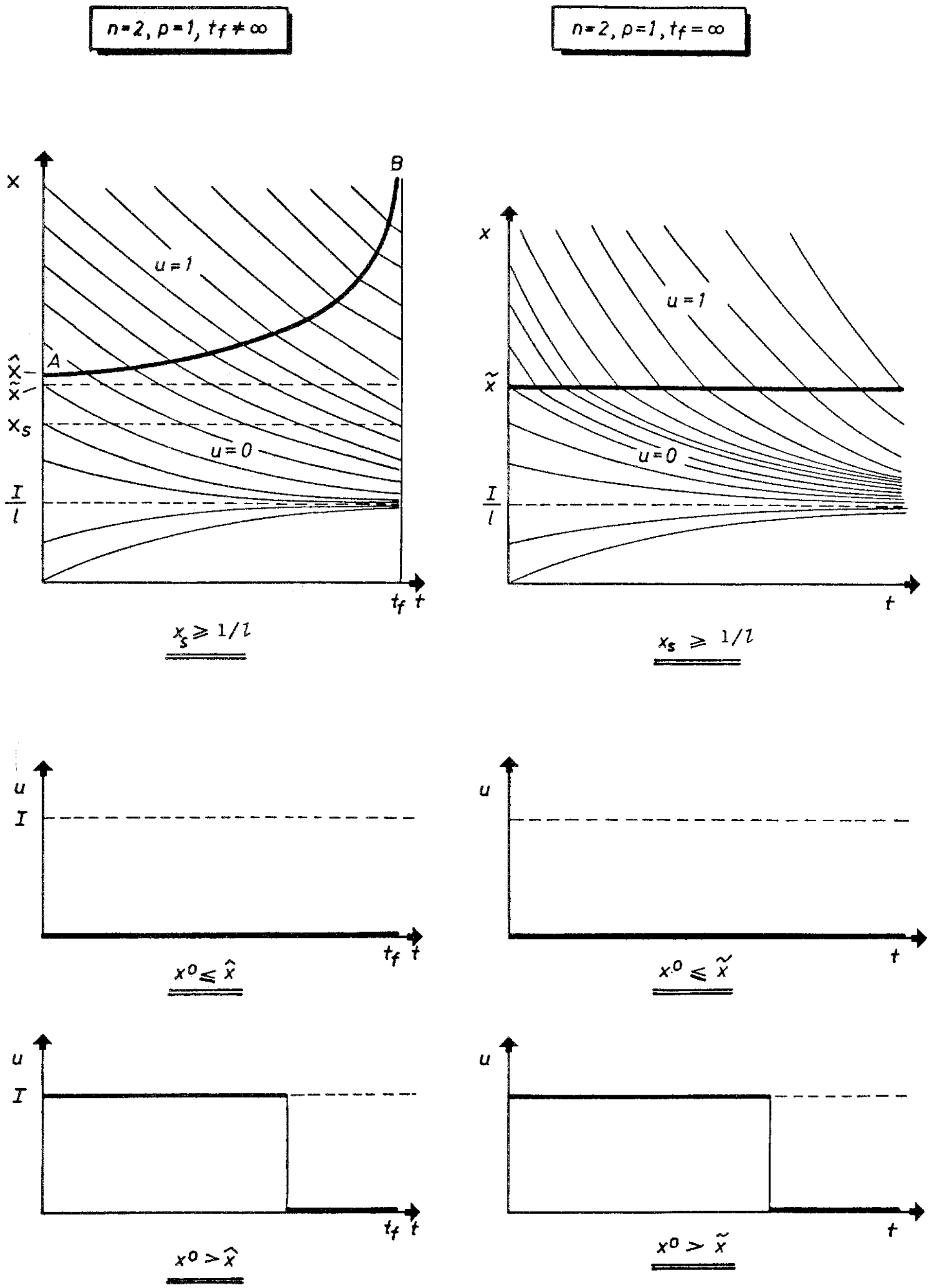


Fig. 3

the curve AB by its horizontal asymptote $x = \tilde{x}$.

(ii) $x_s < 1/\ell \Rightarrow$ there is a singular arc.

For a finite t_f, we get :

$$u^*(t) = \begin{cases} 0 & t \in [0,t_1) \\ u_s & t \in [t_1,t_2) \\ 0 & t \in [t_2,t_f] \end{cases} \qquad \text{if } x^o \leqslant x_s$$

where $t_1 = -\frac{1}{\ell} \ln \frac{\ell x_s - 1}{\ell x^o - 1}$ and t_2 is at the intersection of the locus AB and the horizontal x_s.

$$u^*(t) = \begin{cases} 1 & t \in [0,t_1) \\ u_s & t \in [t_1,t_2) \\ 0 & t \in [t_2,t_f] \end{cases} \qquad \text{if } x_s < x^o < x_s e^{\ell t_2}$$

$$u^*(t) = \begin{cases} 1 & t \in [0,t_1) \\ 0 & t \in [t_1,t_f] \end{cases} \qquad \text{if } x_s e^{\ell t_2} \leqslant x^o$$

where $(t_1,x(t_1))$ is a point on the locus AB.

For $t_f = \infty$, we get :

$$u^*(t) = \begin{cases} 0 & t \in [0,t_1) \\ u_s & t \geqslant t_1 \end{cases} \qquad \text{if } x^o \leqslant x_s$$

where $t_1 = -\frac{1}{\ell} \ln \frac{\ell x_s - 1}{\ell x^o - 1}$

$$u^*(t) = \begin{cases} 1 & t \in [0,t_1) \\ u_s & t \geqslant t_1 \end{cases} \qquad \text{if } x_s < x^o$$

where $t_1 = -\frac{1}{\ell} \ln \frac{x_s}{x_o}$.

We see that in this case, the optimal control depends upon both the initial level of pollution and the parameters of the problem.

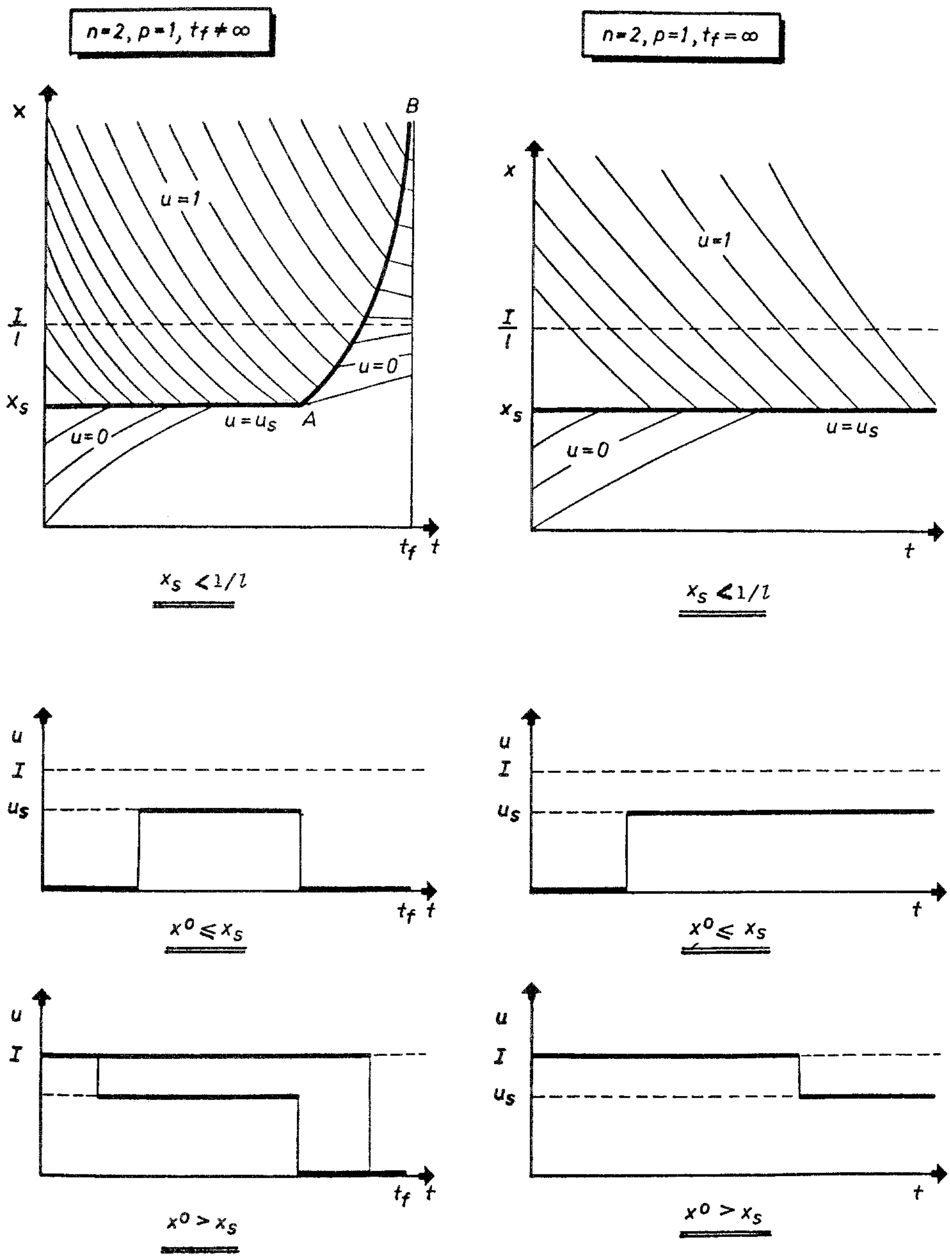

n=2, p=1, $t_f \neq \infty$
n=2, p=1, $t_f = \infty$
x
B
u=1
I/l
x_s
u=0
$u=u_s$
A
t_f
t
$x_s < 1/l$
$x_s \leq 1/l$
u
I
u_s
$x^0 \leq x_s$
$x^0 > x_s$

Fig. 4

4.4. Linear MCD ($n = 2$) and linear MCC ($p = 2$)

For a finite t_f, there is in the plane (t,x) a locus AB (see fig. 5) along which the optimal control will change from the boundary value $u(t) = 1$ to values in the interior of the interval $[0,1]$. The equation of that locus is :

$$x = \frac{k(s_1-s_2)-\ell(\ell+1)\left[s_2 e^{s_1(t_f-t)} - s_1 e^{s_2(t_f-t)}\right]}{kh\left[e^{s_1(t_f-t)} - e^{s_2(t_f-t)}\right]} + \frac{\ell+1}{k}$$

where $h = \ell(\ell+1) + k$, $s_1 = \frac{1+\sqrt{4h+1}}{2}$, $s_2 = \frac{1-\sqrt{4h+1}}{2}$.

That locus has a vertical asymptote given by $t = t_f$ and an horizontal one given by $x = \overline{x}$ with :

$$\overline{x} = \frac{\ell+1}{kh}(h - \ell s_2)$$

For $t_f \to \infty$, the limit of the locus is its horizontal asymptote.

For a finite t_f, if we define $\bar{\bar{x}}$ as the intersection of the locus AB and the x axis, we get :

$$u^*(t) = \frac{k}{h} + \frac{ka_1}{\ell+s_2} e^{s_1 t} + \frac{ka_2}{\ell+s_1} e^{s_2 t} \qquad \text{if } x^o \leqslant \bar{\bar{x}}$$

$$\text{where } a_1 = \frac{h-\left[hx^o-(\ell+1)\right](\ell+s_2)e^{s_2 t_f}}{h\left[(\ell+s_1)e^{s_1 t_f}-(\ell+s_2)e^{s_2 t_f}\right]}$$

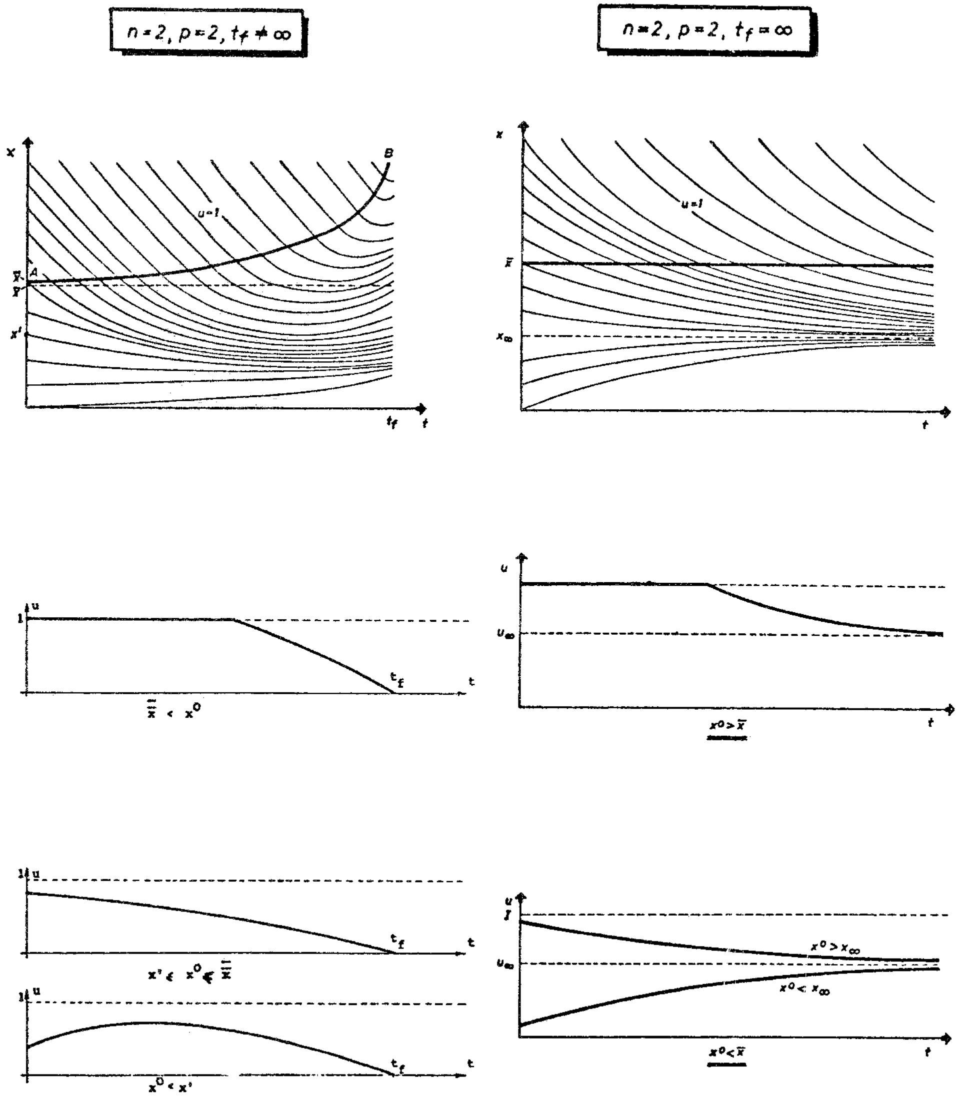
$$a_2 = \frac{h-\left[hx^o-(\ell+1)\right](\ell+s_1)e^{s_1 t_f}}{h\left[(\ell+s_1)e^{s_1 t_f}-(\ell+s_2)e^{s_2 t_f}\right]}$$

$$u^*(t) = \begin{cases} 1 & t\in[0,t_1) \\ \frac{\ell+1}{h} + a_1 e^{s_1(t-t_1)} + a_2 e^{s_2(t-t_1)} & t\in[t_1,t_f] \end{cases} \qquad \text{if } x^o > \bar{\bar{x}}$$

$$\text{where } a_1 = \frac{\ell+1+s_1(\ell+s_2)x(t_1)}{s_1(s_1-s_2)}, \qquad a_2 = \frac{\ell+1+s_2(\ell+s_1)x(t_1)}{s_2(s_1-s_2)}$$

and $(t_1,x(t_1))$ is a point of the locus AB.

For $t_f = \infty$, we get :



n = 2, p = 2, $t_f \neq \infty$
n = 2, p = 2, $t_f = \infty$
x
A
B
u=1
x'
t_f
t
u
1
$\bar{\bar{x}} < x^0$
$x' \leq x^0 \leq \bar{\bar{x}}$
$x^0 < x'$
$\bar{x}$
x_∞
u_∞
$x^0 > \bar{x}$
$x^0 > x_\infty$
$x^0 < x_\infty$
$x^0 < \bar{x}$

Fig. 5

$$u^*(t) = \frac{k}{h}\left[1 + \frac{hx^o-(\ell+1)}{\ell+s_1} \quad e^{s_2 t}\right] \qquad \text{if } x^o \leqslant \overline{x}$$

$$u^*(t) = \begin{cases} 1 & t \varepsilon [0,t_1) \\ \frac{k}{h} + \frac{k[\ell+1+s_2(\ell+s_1)\overline{x}}{s_2(s_1-s_2)(\ell+s_1)} e^{s_2(t-t_1)} & t \geqslant t_1 \end{cases}$$

if $x^o > \overline{x}$, and t_1 is defined by : $t_1 = -\frac{1}{\ell}\ell n \frac{\overline{x}}{x^o}$

All the above results have been written for $\ell > 0$, but the results for the limit case $\ell = 0$ (purely cumulative system) can be everywhere obtained by taking the limit of the above results when $\ell \to 0$. For more details, the interested reader is refered to [3].

5. Conclusions

When the MCD is constant ($n = 1$), the optimal policy does not depend upon the initial level of pollution but depends only upon the parameters of the problem in the following way : for a given initial level of pollution, a large value of the ratio b/c (cost of damage/cost of cleaning up) leads to a more severe policy (more cleaning up) while large values of either the discount factor (a) or the disappearing coefficient (f) lead to a less severe policy.

Moreover, in that case ($n = 1$), the optimal policy for an infinite t_f is always constant. This last fact can be checked a priori. Indeed, by performing an integration by parts on the cost of damage, it is easy to see that the total cost takes on the form :

$$J = \frac{k}{\ell+1}\,[x(0)+1] + \int_o^\infty [u^p(t)-\frac{k}{\ell+1}u(t)]\,e^{-t}dt$$

Hence we shall find the optimal control by solving :

$$\min_{0\leqslant u\leqslant 1} [u^p - \frac{k}{\ell+1}\,u]$$

So the case MCD constant leads, like for the static case, to setting a cleaning up standard.

When the MCD is linear (n = 2), the optimal policy is influenced by both the initial level of pollution and the parameters of the problem. In this case the optimal strategy varies with time but monotonically.

This study gives quantitative results : if the parameters of the problem can be known, then we just have to apply some formula to find what the optimal strategy should be. An important factor is the discount coefficient a. It is probably the most uncertain parameter in the problem because it is not a physical parameter. Rather it has to be chosen according to what value we give to the future. The formulae we gave allow us to measure the impact of that parameter on the optimal strategy.

An other parameter we have to decide upon a priori, is the terminal time t_f. We can see from the results that given the same initial level of pollution, the optimal policy for finite t_f is in all cases at least as polluting and in most cases more polluting than the optimal policy for infinite t_f.

Finally we can say that the results do not contain any revolutionary idea : rather they are in accordance with common sense.

References

[1] PONTRYAGIN L.S., et al., "The mathematical Theory of Optimal Processes", *Interscience*, New York, 1967.

[2] HALKIN H., "Necessary conditions for optimal control problems with infinite horizons", *Econometrica*, vol. 42, No. 2, 1974.

[3] LITT F.X., "Politique optimale de dépollution dans un système accumulatif", *RART 74/01*, Université de Liège, juillet 1974.

[4] LITT F.X., "Politique optimale de dépollution dans un système non strictement cumulatif", *RART 74/03*, Université de Liège, septembre 1974.

[5] LITT F.X., "Politique optimale de dépollution dans un système non strictement cumulatif : suite. Conclusions de l'étude", *RART 74/05*, Université de Liège, novembre 1974.