A MIXT RELAXATION ALGORITHM APPLIED TO QUASI-VARIATIONNAL INEQUATIONS

J.C. MIELLOU
Faculté des Sciences et des Techniques
La Bouloie - Route de Gray

25030 - BESANCON CEDEX

Introduction -

In the first paragraph we consider a class of finite dimensionnal quasi-variationnal inequalities problems, for which we recall a "Theoretical" algorithm allowing the approximation of a maximal inequalities. Then we recall an heuristic formulation of a method which mixt point relaxation method and the "Theoretical" algorithm mentionned above.

In the second paragraph we introduce in a more general frame (Block-relaxation), a precise formulation and set convergence results for this mixt method. For this purpose we use a notion of "delays" previously introduced by CHAZAN and MIRANKER, in the formulation of multiprocessor relaxation algorithms.

§ I - BACKGROUND : definitions, a finite dimensional I.Q.V. problem ; a "Theoretical" algorithm ; First formulation of a mixt GAUSS-SEIDEL algorithm. Let $\beta \in N$, $E=R^\beta$, $K=R^\beta_+$ -

We note $\leqslant$ the partial order relation induced on E by the cone K.

We consider the application $\mathcal{A}$:

$v = \{v_1,\dots,v_k,\dots,v_\beta\} \in R^\beta \xrightarrow{\mathcal{A}} \mathcal{A}(v) = \{\mathcal{A}_1(v),\dots,\mathcal{A}_\ell(v),\dots,\mathcal{A}_\beta(v)\} \in R^\beta$.

For $k \in \{1,\dots,\beta\}$ let $\ell_k = \{0,\dots,0,1,0,\dots,0\}$ $\forall k,l \in \{1,\dots\beta\}$ we consider the function $\forall t \in R \xrightarrow{\varphi_{k,l}} \varphi_{k,l}(t) = \mathcal{A}_l(v+te_k)$.

Definition 1 -

The application $\mathcal{A}$ is an M-application (or M-function) if :
For $k \neq l$ $t \to \varphi_{k,l}(t)$ is antitone and moreover :

$$\mathcal{A}(u) \leqslant \mathcal{A}(v) \implies u \leqslant v.$$

Definition 2 -

The application $\mathcal{A}$ is an L-application if :

$\exists \{\lambda_1,\dots,\lambda_\beta\} \in K$ such that $\mathcal{A}+\Lambda$ be an M-application from R^β onto R^β where is defined by :

$$v = \{v_1,\dots,v_l,\dots,v_\beta\} \xrightarrow{\Lambda} \Lambda v = \{\lambda_1 v_1,\dots,\lambda_l v_l,\dots,\lambda_\beta v_\beta\}.$$

Let $\{u^p\}$ be a sequence of vectors ($\forall p \in N$ $u^p \in R^\beta$), we note the fact that $\{u^p\}$ is isotone (resp. antitone) and converges towards u, by $u^p \uparrow u$ (resp. $u^p \downarrow u$).

Définition 3 -

Let a mapping $g : R^\beta \to R^\beta$ be isotone, g is 0-half-continuous on the right (resp. the left) if :

(1.1) $u^p \downarrow u$ (resp $u^p \uparrow u$) $\quad g(u^p) \downarrow g(u)$ $\quad$ (resp $g(u^p) \uparrow g(u)$).

Remark - Definition 1 is due to W.C. RHEINBOLDT, property (1.1) that we call here 0-half-continuity is due to BENSOUSSAN-GOURSAT-LIONS.

we consider now the hypothesis :

(1.2) Let $\mathcal{A} : R^\beta \to R^\beta$ be a continuous mapping which is an L-application ; and let $f \in R^\beta$.

(1.3) Let $\Phi \; K \to K$ be isotone for the order induced on R^β by K, and 0-half-continuous on the right on K.

(1.4) $\forall w' \in K$ Let $\mathcal{U}_{(w')} = \{v \in K \mid v \leq \Phi(w')\}$.

We formulate the I.Q.V. problem - (I.Q.V. for quasi-variationnal inequalities)

$$(1.5)_a \quad \begin{cases} \text{Find } u \in \mathcal{U}_{(u)} \text{ such that :} \\ (\mathcal{A}(u), v-u) \geq (f, v-u) \quad \forall v \in \mathcal{U}_{(u)} \end{cases}$$

(where (,) notes the usual scalar product on R^β).

Other formulation of problem $(1.5)_a$: $\forall w' \in K$, we mark $\Phi(w') = \{\ldots, \Phi^l_{w'}, \ldots\}$ for $l \in \{1,\ldots,\beta\}$.

Let $\Psi_{[0,\Phi^l_{w'}]}$ is the indicatrix function of the segment $[0, {}^l_{w'}]$, and let $\partial\Psi_{[0,\Phi^l_{w'}]}$ be the sub-differential of $\Psi_{[0,\Phi^l_{w'}]}$.

$$(1.6) \quad \begin{cases} \text{Then } \forall w' \in K, \ \forall l \in \{1,\ldots,\beta\} \text{ we consider the multiapplication } M_l^{w'} \\ \text{which :} \\ \forall u_l \in R \xrightarrow{M_l^{w'}} M_l^{w'}(u_l) = \partial\Psi_{[0,\Phi^l_{w'}]}(u_l) \subset R \\ \text{and we note :} \\ \quad u = \{u_1,\ldots,u_l,\ldots,u_\beta\} \in R^\beta \xrightarrow{M^{w'}} M^{w'}_{(u)} = \{\ldots, M_l^{w'}(u_l), \ldots\},\ l \in \{1,..,\beta\} \end{cases}$$

($M^{w'}$ is an operator of diagonal type).

The problem $(1.5)_a$ can now be formulated in the following manner :

$$(1.5)_b \quad \begin{cases} \text{Find } u \in D(M^u) \text{ such that} \\ 0 \in M^u(u) + \mathcal{A}(u) - f. \end{cases}$$

A "Theoretic" algorithm for the approximation of a solution of problem $(1.5)_{a/\text{or } b/}$

Starting from a convenient initial vector u^o, we consider a sequence of I.V. problems (for Variationnal Inequalities) :

$(1.7)_a \quad (\mathcal{A}(u^{p+1}), v-u^{p+1}) \geq (f, v-u^{p+1}) \quad \forall v \in \mathcal{U}(u^p), \quad p=0,1,\ldots$

These kind of algorithm has been proposed and studied by BENSOUSSAN, GOURSAT, LIONS in the continuous case (A being then an elliptic second order, partial differential operator).

We can also formulate the problems $(1.7)_a$ in the following manner.

$(1.7)_b$ $\quad 0 \in M^{u^p}(u^{p+1}) + \mathcal{A}(u^{p+1}) - f \qquad p = 0,1,\ldots$

First formulation of a mixt GAUSS-SEIDEL algorithm -

BENSOUSSAN - LIONS (see also the work of COMMINCIOLI and alt for analogous free boundary problems) had proposed for the resolution of each problem $(1.5)_{a/\text{ or } b/}$ the point GAUSS-SEIDEL method with projection, which can be formulated by : let $u^{o,p} = u^p$ we approximate u^{p+1} by a sequence $\{u^{q,p}\}$ defined by :

$$(1.8)_a \quad \begin{cases} \forall q \in N, \quad \forall l \in \{1,\ldots,\beta\} \quad \mathcal{A}_l(u_1^{q+1,p},\ldots,u_{l-1}^{q+1,p},\tilde{u}_l^{q+1,p}\ u_{l+1}^{q,p},\ldots,u_l^{q,p}) = f_l \\ u_l^{q+1,p} = \mathrm{Proj}_{[0,\Phi^l_{u^p}]}\ \tilde{u}_l^{q+1,p} \end{cases}$$

or equivently :

$$(1.8)_b \quad \forall q \in N, \quad \forall l \in \{1,\ldots,\beta\}\ 0 \in M_l^{u^p}(u_l^{q+1,p}) + \mathcal{A}_l(u_1^{q+1,p},\ldots,u_l^{q+1,p},u_{l+1}^{q,p},\ldots) - f_l .$$

For q "great" we replace in $(1.8)_{a/\text{ or } b/}$ u_p by $u^{q,p}$ and start again, in (1.8) a/ or b/, with $u^{o,p+1} = u^{q,p}$.

Here it must be observed that $u^{o,p+1} \neq u^{p+1}$ so we cannot by the use of GAUSS-SEIDEL algorithm for the approximation of I.V. subproblems obtain exactly the "theoretical" algorithm (1.7).

§ II - FORMULATION OF A MIXT RELAXATION ALGORITHM USING A NOTION OF DELAYS -

Our main interest is now : To give a precise formulation of a mixt algorithm allowing the association of the "Theoretical" algorithm (1.7), and relaxation methods (namely under-relaxation and GAUSS-SEIDEL) ; moreover we place ourselves in a little more general frame, than a bove : that is to say, block-relaxation-methods.

Subproblems associated to problem (1.5) -

$$(2.1) \quad \begin{cases} \text{Let } \alpha \in N \text{ such that } \alpha \leq \beta, \text{ and a family of integers } \{\beta_1,\ldots,\beta_i,\ldots,\beta_\alpha\} \\ \text{such that } \sum_{i=1}^{\alpha} \beta_i = \beta . \end{cases}$$

$$(2.2) \quad \begin{cases} \forall i \in \{1,\ldots,\alpha\} \qquad E_i = R^{\beta_i} \text{ and } \leq_i \text{ is the relation of partial order indu-} \\ \text{ced by the cone } K_i = R_+^{\beta_i} \text{ then } E = \prod_{i=1}^{\alpha} E_i \ ;\ K = \prod_{i=1}^{\alpha} K_i . \end{cases}$$

$\forall w' \in E, \quad \forall i \in \{1,\ldots,\alpha\} \qquad$ Let :

$\mathcal{U}^i_{(w')} = \{v_i \in K_i \subset E_i \mid v_i \leq_i \Phi^i_{w'} = \{\ldots,\Phi^l_{w'},\ldots\}$ for $l \in \{\beta_1+\ldots+\beta_{i-1}-1,\ldots,\beta_1+ +\ldots+\beta_i\}$.

we note moreover :

$\forall i \in \{1,\ldots,\alpha\} \quad u_i \in E_i \xrightarrow{M_i^{w'}} M_i^{w'}(u_i) = \{\ldots,M_l^{w'}(u_l),\ldots\}$, for

$$1 \in \{\beta_1+\dots+\beta_{i-1}+1,\dots,\beta_1+\dots+\beta_i\}.$$

$\forall u = \{u_1,\dots,u_i,\dots,u_\alpha\} \in \prod_{i=1}^{\alpha} E_i$

Let $\mathcal{A}(u) = \{\mathcal{A}_1(u),\dots,\mathcal{A}_j(u),\dots,\mathcal{A}_\alpha(u)\} \in \prod_{j=1}^{\alpha} E_j$

$\forall w = \{w_1,\dots,w_j,\dots,w_\alpha\} \in \prod_{j=1}^{\alpha} E_j$, $\forall u_j \in E_i$

Let $a_{i,w}(u_i) = \mathcal{A}_i(w_1,\dots,w_{i-1},u_i,w_{i+1},\dots,w_\alpha)$

Let $\Theta = \{\theta_1,\dots,\theta_j,\dots,\theta_\alpha\} \in \prod_{j=1}^{\alpha} [0,+\infty[$ such that :

$$(2.3) \qquad \forall i \in \{1,\dots,\alpha\} \quad \theta_j \geq \max_{\beta_1+\dots+\beta_{j-1}<1\leq\beta_1+\dots+\beta_j} \lambda_1$$

where $\{\lambda_1,\dots,\lambda_\beta\}$ has been introduced in <u>definition 2</u>.

We consider also $\Omega = \{\omega_1,\dots,\omega_j,\dots,\omega_\alpha\} \in \prod_{j=1}^{\alpha} [0,1]$ such that :

$\forall j \in \{1,\dots,\alpha\}$ either $\omega_j = 1$, or $d_j \in R$, $d_j > 0$ such that

$$(2.4) \qquad \forall w \in R^\beta, \quad \forall v_j^1, v_j^2 \in R^{\beta_j} \quad \text{with} \quad v_j^1 \lesssim_j v_j^2.$$

$$a_{j,w}(v_j^2) - a_{j,w}(v_j^1) \leq_j d_j\,(v_j^2 - v_j^1) \text{ and } (1-\omega_j)d_j \leq \theta_j.$$

we consider now the subproblems :

$$(2.5)_a \begin{cases} \forall w, w' \in K, \quad \forall j \in \{1,\dots,\alpha\} \quad \text{find } u_j \in \mathcal{U}^j_{(w')} \text{ such that :} \\ (a_{j,w}(\omega_j u_j+(1-\omega_j)w_j) + \theta_j(u_j-w_j), v_j-u_j)_j \geq (f_j, v_j-u_j) \quad \forall v_j \in \mathcal{U}^j_{(\bar{w}')} \\ \text{where } (,)_j \text{ is the usual scalar product on } E_j = R^{\beta_j}). \end{cases}$$

which can also be formulated by :

$\forall w, w' \in K \quad \forall j \in \{1,\dots,\alpha\}$ find $u_j \in D(M_j^{w'})$ such that :

$$(2.5)_b \qquad f_j \in M_j^{w'}(u_j) + a_{j,w}(\omega_j u_j+(1-\omega_j)w_j) + \theta_j(u_j-w_j)$$

and we wright :

$$(2.5)_c \begin{cases} \forall j \in \{1,\dots,\alpha\} \quad G_{j,\omega_j,\theta_j}(w,w') = u_j \\ G_{\Omega,\Theta}(w,w') = \{u_1,\dots,u_j,\dots,u_\alpha\}. \end{cases}$$

<u>Proposition 1</u> -

The hypothesis (1.2), (1.3), (1.4), (2.1), (2.2), (2.3) being satisfied, $w, w' \in K \xrightarrow{G_{\Omega,\Theta}} G_{\Omega,\Theta}(w,w')$ *is well defined (i.e.* $\exists$ *a unique solution of each subproblem* $(2.5)_{a/ \text{ or } b/}$*), is an isotone mapping of each of its two arguments* w, w'*, and is continuous relatively to the first argument* w*, and half-continuous on the right relatively to the second argument* w'*. Moreover* $u \in K$ *is a solution of the I.Q.V. problem* $(1.5)_{a/ \text{ or } b/}$ *iff* u *is a fixed point of the application :*

$$w \xrightarrow{F} F(w) = G_{\Omega,\Theta}(w,w)$$

Mixt algorithm associating "theoretical" algorithm (1.7) and relaxation methods -

For the sake of simplicity we suppose that we want to perform r relaxation iterations for the approximation of the solution of each I.V. occuring in the "Theoretical" algorithm $(1.7)_a$. u^o being conveniently choiced (the precise choice shall begiven in proposition 2/) we introduce now three formulation of the same following mixt algorithm :

let $h(p) = p \bmod(\alpha) + 1$; $k(p) = p \bmod(r\alpha)$.

$$(2.6)_a \begin{cases} \text{If} \quad j \neq h(p) & u_j^{p+1} = u_j^p \\ \text{If} \quad j = h(p) & \text{find } u_j^{p+1} \in \mathcal{U}^j(u^{p-k(p)}) \\ (\mathcal{A}_{j,u^p}(\omega_j u_j^{p+1} + (1-\omega_j)u_j^p) + \theta_j(u_j^{p+1} - u_j^p), v_j - u_j^{p+1})_j \geqslant & \\ \qquad\qquad (f_j, v_j - u_j^{p+1})_j \quad \forall v_j \in \mathcal{U}^j_{(u^{p-k(p)})} & \end{cases}$$

$$(2.6)_b \begin{cases} \text{If} \quad j \neq h(p) & u_j^{p+1} = u_j^p \\ \text{If} \quad j = h(p) & \text{find } u_j^{p+1} \in D(M_j^{u^{p-k(p)}}) \text{ such that} \\ f_j \in M_j^{u^{p-k(p)}}(u_j^{p+1}) + \mathcal{A}_{j,u^p}(\omega_j u_j^{p+1} + (1-\omega_j)u_j^p) + \theta_j(u_j^{p+1} - u_j^p) & \end{cases}$$

and :

$$(2.6)_c \begin{cases} \text{if} \quad j \neq h(p) & u_j^{p+1} = u_j^p \\ \text{if} \quad j = h(p) & u_j^{p+1} = G_{j,\omega_j,\theta_j}(u^p, u^{p-k(p)}). \end{cases}$$

Proposition 2 -

The hypothesis are the some that those of proposition 1, and let $u^o \in K$, be such that $f \leqslant \mathcal{A}(u^o)$, then the sequence $\{u^p\}$ produced by algorithm $(2.6)_{a/\ b/\ or\ c/}$ is such that $u^p \downarrow u$ maximal solution of problem $(1.5)_{a/\ or\ b/}$, and fixed point of

$$w \xrightarrow{F} F(w) = G_{\Omega, \textcircled{H}}(w,w).$$

Remarks - Il we cancel the constraints of our problem and :

- take $\alpha = 1$ and so $\beta_1 = \beta$ we can give an interpretation of algorithm (2.6) by the scheme :

$$\frac{u^{p+1} - u^p}{k} + \mathcal{A}(\omega u^{p+1} + (1-\omega)u^p) - f = 0$$

(Taking $\theta = \frac{1}{k}$).

as the discretisation of :

$$\frac{du}{dt} + \mathcal{A}(u) - f = 0 \qquad u(0) = u^o$$

- take $\alpha = \beta$ and suppose that $\mathcal{A}$ is a matrix with diagonal elements all equals to 1 then we find the classical relaxation method with relaxation parameters $\frac{1}{\omega_j + \theta_j}$ and hypothesis (2.4) implies that we are here in the case of under relaxation and GAUSS-SEIDEL methods.

- The sequence $\{k(p)\}$ constitutes the "delays" : notion introduced by CHAZAN-

MIRANKER for multiprocessors relaxation algorithms which can also be mixted with "Theoretical algorithm" (1.7) in the following manner :

$\forall p \in N$ let $h(p) \subset \{1,\ldots,\alpha\}$; $k^1(p) = \{k_1^1(p),\ldots,k_i^1(p),\ldots$ $k_\alpha(p)\} \in N^\alpha$, $k^2(p) \in N$ and we suppose that :

- $\forall i \in \{1,\ldots,\alpha\}$ $\{p \in N \mid i \in h(p)\}$ is indefinite
- $\forall s \in N$ such that $\forall i \in \{1,\ldots,\alpha\}$ $0 \leqslant k_i^1(p) \leqslant s(p)$ $0 \leqslant k^2(p) \leqslant s(p)$
- $\forall i \in \{1,\ldots,\alpha\}$ $p \to p-k_i^1(p)$ is isotone, and $p \to p-k^2(p)$ is also isotone.

Then we consider the algorithm :

$$u_j^{p+1} = G_j(\ldots,u_i^{p-k_i^1(p)},\ldots, u^{p-k^2(p)}) \text{ if } j=h(p)$$

$$u_j^{p+1} = u_j^p \quad \text{if } j \neq h(p)$$

Proposition 2 extends without difficulties to this more general situation. It seems that a particular interest of block relaxation methods in quasi-variational inequation problems lies in the fact that, these problems being often of great size, it can be necessary to decompose the arrays, which contain iterates vectors, in sub arrays corresponding to block sub problems, which can be contained in central memory of the processor used. Block relaxation can be interpreted as an iterative coordination method between these subproblems, which needs relatively simple (sequential), exchanges between central memory, and peripheral memories (disks).

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