CANONICAL REALIZATIONS OF TRANSFER OPERATORS *

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Abstract. We wish to present in this paper the realization of a class of transfer operators of infinite dimensional state space discrete-time systems. The realization will be carried out on a functional state space constructed from a given transfer operator. Our method here is based on the canonical model theory of contraction Hilbert Space operators of Nagy and Foias.

It will be shown that the state space in this case has beside the output component, a second component which characterizes the energy dissipated in the system. Furthermore, the realization will be automatically canonical when one uses cyclic subspaces of the restricted shift operator on a Nagy-Foias Space. Relationships between the realization here and the scattering synthesis of passive networks will also be discussed.

I. Introduction.

We study in this paper the realization of linear discrete-time systems whose transfer functions are contractive analytic functions from the unit disc to the operators from one Hilbert Space to another Hilbert Space. Our method here is based on the operator model theory of Nagy and Foias.

A model of an operator is, plainly speaking, another operator (or operators) which is simpler in some suitable sense, and at the same time, has richer structure. In their theory, Nagy and Foias have shown that every Hilbert Space contraction is unitarily equivalent to a shift operator compressed to a functional space, called a Nagy-Foias Space.

In this paper we shall show that given a contractive analytic transfer function, the Nagy-Foias space constructed from this function is a state-energy type space, and the realization on this space, using cyclic subspaces of the compressed shift operator will naturally be canonical.

In Section 2 we present the basic mathematical preliminaries and background motivation. Structures of a Nagy-Foias space will be discussed in detail in Section 3. Section 4 is devoted to the realization problem. Relationships between the Nagy-Foias theory and scattering realization of networks will also be discussed here.

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II. Mathematical Preliminaries. In this paper we shall be, unless otherwise stated, dealing with linear bounded operators on separable Hilbert spaces. Norm and inner product are denoted by $|| \ ||$ and by $[\ , \ ]$ respectively.

We briefly present in this section some basic notations and definitions which will be needed throughout the paper. The Nagy-Foias Space will then be constructed while its structures and applications to system realizations will be dealt with in Section 3.

Given a Hilbert Space H_1 the space of power series: $f(z) = \sum_{n=0}^{\infty} f_n z^n$, where the f_n's are in H, $\sum_{n=0}^{\infty} ||f_n||^2 < \infty$, and $|z| < 1$, is denoted by $H^2(H)$. The norm and inner product in $H^2(H)$ are defined by $||f||^2 = \sum_{n=0}^{\infty} ||f_n||_H^2$, and $[f,g] = \sum_{n=0}^{\infty} [f_n,g_n]_H$, where $g(z) = \sum_{n=0}^{\infty} g_n z^n$. Clearly, $H^2(H)$ can be identified with the space of square summable H-sequences $\{f_0,f_1,f_2,\ldots; f_i \in H\}$.

We can associate with each $f(z)$ in $H^2(H)$ its "boundary function" $f(e^{it}) = \sum_{n=0}^{\infty} f_n e^{int}$, and consequently, $H^2(H)$ can be identified with the space $L_+^2(H)$ of Fourier series with non-negative powers of e^{it}. This space, and therefore $H^2(H)$ also, are in turn, a subspace of the space $L^2(H)$ of Fourier Series with all powers of e^{it}. We have the orthogonal decomposition $L^2(H) = L_-^2(H) \oplus L_+^2(H)$, where $L_-^2(H)$ is the set of Fourier Series with negative powers of e^{it}.

Given two Hilbert Spaces H_1 and H_2, a function $\theta(z)$ from the unit disc to the operators from H_1 to H_2 is denoted by $\{\theta(z), H_1,H_2\}$. Such a function is bounded analytic when

$$\theta(z) = \sum_{n=0}^{\infty} \theta_n z^n, \ \theta_n\colon \ H_1 \to H_2, \ |z| < 1$$

and

$$||\theta(z)h_1|| \leq M||h_1||, \ h_1 \text{ in } H_1.$$

θ is said to be contractive when $M = 1$, and purely contractive, if in addition, $||\theta(0)h_1|| < ||h_1||$ for any h_1 in H_1.

Given a bounded analytic function $\{\theta(z), H_1, H_2\}$ we can associate with it the following spaces and operators:

a) The space $H^2(H_1) \subset L^2(H_1)$ and $H^2(H_2) \subset L^2(H_2)$.

b) As in the above, we can associate with $\theta(z)$ its "boundary function" $\theta(e^{it})$ defined almost everywhere. Hence, we have the following operators

$$\theta_z\colon \ H^2(H_1) \to H^2(H_2)$$
$$(\theta_z f)(z) = \theta(z)f(z)$$

$$\theta_t:\ L^2(H_1) \to L^2(H_2)$$

$$(\theta_t f)(t) = \theta(e^{it})f(t)$$

$$\theta_t^*:\ L^2(H_2) \to L^2(H_1)$$

$$(\theta_t^* f)(t) = \theta(e^{it})^* f(t),\ \theta(e^{it})^* = \sum_{n=0}^{\infty} \theta_n^* e^{-int}$$

If $\theta(z)$ is contractive, then we can, in addition, define the operator

$$\Delta_t:\ L^2(H_1) \to L^2(H_1)$$

$$(\Delta_t f)(t) = \Delta(t)f(t)$$

where $\Delta(t)$ is the unique positive square root $[I - \theta(e^{it})^*\theta(e^{it})]^{\frac{1}{2}}$, and it is bounded between 0 and 1.

To each contractive analytic function $\{\theta(z), H_1, H_2\}$ there corresponds a Nagy-Foias Space which can be constructed as follows [1]:

First let H be the Hilbert space of pairs of functions $\big(v(z), \Delta(t)u(t)\big)$, for v in $H^2(H_2)$ and u in $L^2(H_1)$

$$H = H^2(H_2) \oplus \overline{\Delta(t)L^2(H_1)} \tag{2-1}$$

where $\overline{}$ indicates the closure. The inner product and norm in H are defined in the usual way

$$[(v_1,\Delta u_1), (v_2,\Delta u_2)]_H = [v_1,v_2]_{H^2(H_2)} + [\Delta u_1,\Delta u_2]_{L^2(H_1)} \tag{2-2}$$

and

$$||(v,\Delta u)||_H^2 = ||v||^2_{H^2(H_1)} + ||\Delta u||^2_{L^2(H_1)} \tag{2-3}$$

The space H is clearly a subspace of the space

$$K = L^2(H_2) \oplus \overline{\Delta(t)L^2(H_1)} \tag{2-4}$$

$$= \{(f_-,0),\ f_- \in L_-^2(H_2)\} \oplus H \tag{2-5}$$

Next, in H, consider the set

$$M = \{\big(\theta(z)w(z),\ \Delta(t)w(e^{it})\big),\ w \in H^2(H_1)\} \tag{2-6}$$

It is clear that

$$||(\theta w,\Delta w)||^2 = ||\theta w||^2 + ||w||^2 - ||\theta w||^2 = ||w||^2$$

Hence the map $w \to (\theta w, \Delta w)$ is an isometry, and therefore M is closed in H.

Definition

The orthogonal complement $M^\perp$ of M in H:

$$M^\perp = H^2(\mathcal{H}_2) \oplus \overline{\Delta(t)L^2(\mathcal{H}_1)} \ominus \{(\theta w, \Delta w),\ w \in H^2(\mathcal{H}_1)\} \tag{2-7}$$

is called the Nagy-Foias space of $\{\theta(z), \mathcal{H}_1, \mathcal{H}_2\}$.

It is not at all clear from this definition what are the meanings and structures of $M^\perp$. In the next section we shall investigate these from a system-network viewpoint.

In what follows we will be concerned with linear, fixed, discrete-time system which has the state space description:

$$\begin{cases} x_n = Ax_n + Bu_n & (2\text{-}8) \\ v_n = Cx_n + Du_n & (2\text{-}9) \end{cases}$$

where $n = 0, 1, 2, \ldots$, A, B, C, and D are appropriate operators; $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are the states, inputs and outputs respectively. The $\{x_n\}$ and $\{v_n\}$ are chosen to be square-summable $\mathcal{H}_2$-sequences, and $\{u_n\}$ is taken to be square-summable $\mathcal{H}_1$-sequences.

Given a square-summable sequence $\{f_0, f_1, \ldots\}$, its discrete Fourier transform is the map

$$\{f_n\} \to \sum_{n=0}^{\infty} f_n z^n = f(z) \text{ say} \tag{2-10}$$

Hence taking the transforms of equations (2-8) and (2-9) with $x_0 = 0$, we find

$$v(z) = \left[D + zC[I - zA]^{-1}B\right]u(z) \tag{2-11a}$$

$$= \theta(z)\, u(z) \text{ say} \tag{2-11b}$$

The function $\theta(z)$ is called the transfer operator of the system. Again, we assume that $\theta(z)$ is contractive. Given a discrete-time signal $\{\ldots f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots\}$, if 0 is taken to be the present instant of time, then the sequence $\{f_0, f_1, \ldots\}$ is the present-future segment of the signal, while the sequence $\{\ldots f_{-3}, f_{-2}, f_{-1}\}$ is the past segment of the signal. It then follows that, the space $H^2(\mathcal{H}_1)$ (or $L^2_+(\mathcal{H}_1)$) is just the space of (transform of) present-future inputs, and $L^2_-(\mathcal{H}_1)$ is the space of (transform of) past inputs, while $L^2(\mathcal{H}_1)$ is the space of allowable inputs over all (discrete) time. Similarly for the output spaces $H^2(\mathcal{H}_2)$, $L^2_-(\mathcal{H}_2)$ and $L^2(\mathcal{H}_2)$.

The boundary function $\theta(e^{it})$ is called the system frequency operator. For our purpose, $\theta(e^{it})$ is represented by the matrix operator

$$\theta(e^{it}) = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \qquad (2\text{-}12)$$

with respect to the decompositions $L^2(H_i) = L_-^2(H_i) \oplus L_+^2(H_i)$, $i = 1, 2$. Clearly $\theta_{11} = P_2^-\theta(e^{it})P_1^-$, $\theta_{12} = P_2^-\theta(e^{it})P_1^+$, $\theta_{21} = P_2^+\theta(e^{it})P_1^-$ and $\theta_{22} = P_2^+\theta(e^{it})P_1^+$, $P_i^{\pm}$, $i = 1, 2$ are projection operators from $L^2(H_i)$ onto $L_{\pm}^2(H_i)$. We note that θ_{22} can actually be identified with $\theta(z)$.

The system is said to be causal when $\theta(e^{it})$ is lower triangular [2], that is when $\theta_{12} = 0$, and this certainly is the case when θ is analytic.

The space H (equation 2-1) whose elements are pairs of functions $(v(z), \Delta(t)u(t))$ can be considered as the space of 'present-future output - "input" pairs' of a system whose transfer operator is $\theta(z)$. The function Δu is not really an input although it is in the input space $L^2(H_1)$, while u itself is an input. However, since

$$||\Delta u||^2_{L^2(H_1)} = ||u||^2_{L^2(H_1)} - ||\theta u||^2_{L^2(H_2)} \qquad (2\text{-}13)$$

which can be regarded as the amount of energy which the system absorbed from the outside world. Thus, in some sense, Δu characterizes the passage of energy through the system.

We conclude the section by noting that, in a pair $(v, \Delta u)$, v and u are quite arbitrary, in the sense that the output v may or may not result from the input u. When $(v = \theta w, \Delta u = \Delta w)$ which is an element in M(equation 2-6), then the present-future output θw in this case is the response of the system to the present-future input w.

III. Structures of A Nagy-Foias Space. The Nagy-Foias Space $M^{\perp}$ of a contractive analytic function $\{\theta(z), H_1, H_2\}$ was introduced by Nagy and Foias [1] in the study of models of Hilbert space contractions (i.e. operators with norms less than or equal to 1). They showed that such an operator is unitarily equivalent to the compressed shift operator on $M^{\perp}$.

Here, in this section we shall investigate the structures of $M^{\perp}$, and in particular, its roles in the realization of $\theta(z)$.

Let $(v,\Delta u)$ be an element in H, its component $P_{M^{\perp}}(v,\Delta u)$, where $P_{M^{\perp}}$ is the orthogonal projection onto $M^{\perp}$, is clearly

$$P_{M^{\perp}}(v,\Delta u) = (v,\Delta u) - (\theta w,\Delta w) \tag{3-1}$$

where w is in $H^2(H_1)$ and is such that $P_{M^{\perp}}(v,\Delta u) \perp M$, that is

$$[(v-\theta w,\ \Delta u-\Delta w),\ (\theta\tilde{w},\Delta\tilde{w})]_H = 0,\ \tilde{w} \text{ in } H^2(H_1)$$

or

$$[\theta^* v + \Delta^2 u - w,\ \tilde{w}]_{L^2(H_1)} = 0$$

Hence, as a function in $L^2(H_1)$

$$(\theta^* v + \Delta^2 u - w) \perp H^2(H_1)$$

For this to be ture, we must have

$$P_1^+[\theta^* v + \Delta^2 u - w] = 0$$

Therefore

$$w = P_1^+[\theta^* v + \Delta^2 u] \tag{3-2}$$

From which it is evident that $(v,\Delta u)$ in H is in $M^{\perp}$ if and only if

$$P_1^+[\theta^* v + \Delta^2 u] = 0 \tag{3-3}$$

This condition can also be expressed differently as follows. To each pair $(v,\Delta u)$ in H there corresponds the pair $(\theta_{21}u_-,\Delta u_-)$ in $M^{\perp}$ and $(\theta_{22}u_+,\Delta u_+)$ in M, where u_- is in $L_-^2(H_1)$, u_+ in $L_+^2(H_1)$ and $u_- + u_+ = u$. From these pairs, we form the pair

$$(\hat{v},\Delta u) = (\theta_{21}u_- + \theta_{22}u_+,\Delta u) \tag{3-4}$$

which is an output-input pair in which the present-future output $\hat{v}$ resulted from the input (over all time) u. We can therefore write

$$(v,\Delta u) = (\theta_{21}u_-,\Delta u_-) + (\theta_{22}u_+,\Delta u_+) + (v-\hat{v},\ 0)$$

hence

<u>Lemma 1</u>

$(v,\Delta u)$ in H is in $M^{\perp}$ if and only if

$$P_1^+(\theta^* v + \Delta^2 u) = 0 \tag{3-3}$$

or alternately

$$(v,\Delta u) = (\theta_{21}u_-,\Delta u_-) + P_{M^{\perp}}(v-\hat{v},\ 0) \tag{3-5}$$

We note that $(v-\hat{v})$ can be regarded as the error between the two outputs v and $\hat{v}$.

In what follows we shall concentrate on two special subspaces of $M^{\perp}$:

$$M_1^{\perp} = \text{closure } \{(\theta_{21}u_-, \Delta u_-),\ u_- \text{ in } L_-^2(H_1)\} \tag{3-6}$$

and

$$M_2^{\perp} = \text{closure } \{P_{M^{\perp}}(y,0),\ y \text{ in } H^2(H_2)\} \tag{3-7}$$

Plainly speaking, in $M_1^{\perp}$, the set of present-future outputs comes entirely from past inputs, while in $M_2^{\perp}$, the inputs are not specified.

To proceed further we define

Definition 1

The shift operator S on H is defined by

$$S(v,\Delta u) = (zv,\ e^{it}\Delta u),\ (v,\Delta u) \text{ in } H \tag{3-8}$$

and its adjoint S^* is given by

$$S^*(v,\Delta u) = \left(\frac{v(z)-v(0)}{z},\ e^{-it}\Delta u\right),\ (v,\Delta u) \text{ in } H \tag{3-9}$$

We note that S is an isometry on H, and M is invariant [1] under S, as a consequence, $M^{\perp}$ is invariant under S^*.

Since $M^{\perp}$ is not invariant under S, the restriction of S onto $M^{\perp}$ is called the compressed shift and is defined by

Definition 2

The operator T: $M^{\perp} \to M^{\perp}$ given by

$$T(v,\Delta u) = P_{M^{\perp}}(zv,\ e^{it}\Delta u),\ (v,\Delta u) \text{ in } M^{\perp} \tag{3-10}$$

is called the compressed shift operator on the Nagy-Foias space $M^{\perp}$.

The adjoint operator T^* is just S^* restricted to $M^{\perp}$:

$$T^*(v,\Delta u) = \left(\frac{v(z)-v(0)}{z},\ e^{-it}\Delta u\right),\ (v,\Delta u) \text{ in } M^{\perp}, \tag{3-11}$$

Now we assume that θ is causal, that is $\theta_{12} = 0$, in this case $M_1^{\perp}$ and $M_2^{\perp}$ have very nice structures, and as we shall see, they play a crucial role in the realization of θ as well as in the Nagy-Foias model theory.

Structures of $M_1^{\perp}$

As we have defined above

$$M_1^{\perp} = \text{closure } \{(\theta_{21}u_-, \Delta u_-),\ u_- \text{ in } L_-^2(H_1)\}$$

setting

$$u_- = e^{-it}\alpha_1 + e^{-2it}\alpha_2 + \dots + e^{-int}\alpha_n + \dots, \ \alpha_i \text{ in } H_1, \tag{3-12a}$$

and

$$\theta(e^{it}) = \theta_0 + \theta_1 e^{it} + \theta_2 e^{2it} + \dots \theta_n e^{int} + \dots \ , \quad \theta_i: \ H_1 \to H_2 \tag{3-12b}$$

Then $\theta_{21}u_-$ can be calculated as follows

$$\begin{aligned} \theta_{21}u_- &= P_2^+\{\theta(e^{it})u_-\} \\ &= (\theta_1\alpha_1 + \theta_2\alpha_2 + \theta_3\alpha_3 + \dots)\, 1 \\ &+ (\theta_2\alpha_1 + \theta_3\alpha_2 + \theta_4\alpha_2 + \dots)\, e^{it} \\ &+ \dots \end{aligned} \tag{3-13}$$

Hence the matrix of θ_{21} with respect to the orthonormal basis $\{1, e^{it}, e^{2it}, \dots\}$ is

$$[\theta_{21}] = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \cdots \\ \theta_2 & \theta_3 & \theta_4 & \cdots \\ \theta_3 & \theta_4 & \theta_5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{3-14}$$

This infinite matrix is called the Hankel matrix of the Hankel operator generated by θ, denoted by $\mathscr{H}_\theta$. Hence $\theta_{21} \equiv \mathscr{H}_\theta$.

Let $\big(\theta(z)w(z),\ \Delta(t)w(e^{it})\big)$ be an element of M, and since M is not invariant under S^*, $S^*(\theta w, \Delta w)$ will just be in H. What are the elements of M which are mapped entirely into $M^\perp$ under S^*? We have

$$S^*(\theta w, \Delta w) = \Big(z^{-1}\big(\theta(z)w(z) - \theta(0)w(0)\big),\ e^{-it}\Delta w(e^{it})\Big), \quad w \text{ in } H^2(H_1) \tag{3-15}$$

Using (3-3) of Lemma 1, we find

$$P_{M^\perp} S^*(\theta w, \Delta w) = S^*(\theta w, \Delta w) - \Big(\theta e^{-it}\big(w(z) - w(0)\big),\ \Delta e^{-it}\big(w(z) - w(0)\big)\Big) \tag{3-16}$$

Thus for $S^*(\theta w, \Delta w)$ to be in $M^\perp$, we must have

$$w(z) = w(0) = \alpha \ \text{ say}$$

that is $w(z)$ must be a constant function in $L^2(H_1)$, in other words w is in H_1. Conversely, for α in H_1,

$$S^*(\theta\alpha, \Delta\alpha) = \Big(\frac{\theta(z) - \theta(0)}{z}\ \alpha,\ e^{-it}\Delta\alpha\Big) \tag{3-17}$$

Using (3-12a) and (3-13) we find

$$\frac{\theta(z)-\theta(0)}{z}\alpha = (\theta_1 + \theta_2 z + \theta_3 z^2 + \dots)\alpha$$
$$= \theta_{21}\, e^{-it}\, \alpha \tag{3-18}$$

Therefore

$$S^*(\theta\alpha, \Delta\alpha) = (\theta_{21}\, e^{-it}\alpha,\ \Delta e^{-it}\alpha) \tag{3-19}$$

which shows that $S^*(\theta\alpha,\Delta\alpha)$ is in $M_1^\perp$ and therefore it is in $M^\perp$.

<u>Lemma 2</u>

S^* sends $(\theta w,\Delta w)$ into $M_1^\perp$ if and only if $w(z) = \alpha$ in H_1, in which case

$$S^*(\theta\alpha,\Delta\alpha) = \left(\frac{\theta(z)-\theta(0)}{z},\ \Delta e^{-it}\right)\alpha$$
$$= (\theta_{21}\, e^{-it}\alpha,\ \Delta e^{-it}\alpha)$$

This lemma suggests that in $M_1^\perp$ we can have a sequence of subspaces which are constructed as follows.

First set

$$\phi_0(z) = \theta(z) \tag{3-20}$$

Then for $n \geq 0$, define

$$\phi_{n+1}(z) = \frac{\phi_n(z)-\phi_n(0)}{z} \tag{3-21}$$

It is clear that $\phi_n(0) = \theta_n$, the nth coefficient in the power series of $\theta(z)$, and for α in H_1, $\phi_n(z)\alpha = \theta_{21}e^{-int}\alpha$. Next, we define

$$K_n = [\phi_n(z),\ \Delta e^{-int}],\quad n = 0, 1, 2, \dots \tag{3-22a}$$

Then for α in H_1, $K_0\alpha = [\theta\alpha,\Delta\alpha]$ is an element of M, while for $n \geq 1$

$$K_n\alpha = [\phi_n,\ \Delta e^{-int}]\alpha \tag{3-22b}$$
$$= (\theta_{21}e^{-int}\alpha,\ \Delta e^{-int}\alpha) \tag{3-22c}$$

which are in $M_1^\perp$.

It follows at once from the above that

$$(\theta_{21}u_-,\Delta u_-) = K_1\alpha_1 + K_2\alpha_2 + \dots + K_n\alpha_n + \dots \tag{3-23}$$

and therefore

$$M_1^\perp = \overline{\text{span}}\{K_1\alpha, K_2\alpha, \dots, K_n\alpha, \dots\} \tag{3-24}$$

Furthermore, for $n \geq 1$

$$T^*K_n\alpha = \left(\frac{\phi_n(z)-\phi_n(0)}{z}, \Delta e^{-i(n+1)t}\right)\alpha$$

$$= K_{n+1}\,\alpha \tag{3-25}$$

Hence, for $n \geq 1$

$$K_{n+1}\alpha = T^{*n}K_1\alpha \tag{3-26}$$

Thus we have shown that

Theorem 1

$\overline{K_1\alpha}$ is a cyclic subspace [1] of $T^*\big|_{M_1^\perp}$, the restriction of T^* to $M_1^\perp$.

Structures of $M_2^\perp$

The subspace $M_2^\perp$ was defined above:

$$M_2^\perp = \text{closure } \{P_{M^\perp}(y,0),\ y \text{ in } H^2(H_2)\}$$

straightforward calculations give

$$P_{M^\perp}(y,0) = (I - \theta(z)\theta_{22}^*, -\Delta\theta_{22}^*)y$$

$$= k\,y \quad \text{say, for } y \text{ in } H^2(H_2) \tag{3-27}$$

We note that for any β in H_2,

$$\theta_{22}^*\beta = P_1^+\{\theta(e^{it})^*\beta\} = \theta(0)^*\beta \tag{3-28}$$

Therefore

$$P_{M^\perp}(\beta,0) = k\beta = \left(I-\theta(z)\theta(0)^*, -\Delta\theta(0)^*\right)\beta = k_0\beta \quad \text{say, for } \beta \text{ in } H_2 \tag{3-29}$$

Next, let us calculate $P_{M^\perp}(\beta e^{it},0)$, we have

$$P_{M^\perp}(\beta e^{it},0) = k\beta e^{it} = (I-\theta\theta_{22}^*, -\Delta\theta_{22}^*)\beta e^{it} \tag{3-30}$$

where

$$\theta_{22}^*\beta e^{it} = P_1^+\{\theta(e^{it})^*\beta e^{it}\}$$

$$= P_1^+\{\theta(0)^*\beta e^{it} + \theta_1^*\beta + \theta_2^*\beta\, e^{-it} + \ldots\}$$

$$= \theta(0)^*\beta e^{it} + \theta_1^*\beta$$

Therefore

$$P_{M^\perp}(\beta e^{it},0) = k\beta e^{it}$$

$$= \left(I-\theta\theta(0)^*, -\Delta\theta(0)^*\right)\beta e^{it} - (\theta\theta_1^*\beta,\Delta\theta_1^*\beta)$$

$$= k_0\beta e^{it} - (\theta\theta_1^*\beta,\Delta\theta_1^*\beta)$$

we have

$$P_{M^\perp}(\beta e^{it},0) = k\beta e^{it} = P_{M^\perp}(k_0\beta e^{it}) = k_1\beta \quad \text{say} \tag{3-31}$$

But, $k_0\beta e^{it} = e^{it}\, k_0\,\beta$, hence

$$k_1\beta = P_{M^\perp}(e^{it}k_0\beta) = T\, k_0\,\beta \tag{3-32}$$

Similarly, setting

$$k_2\beta = k\beta e^{2it} = P_{M^\perp}(\beta e^{2it},0) \tag{3-33}$$

we find

$$k_2\beta = P_{M^\perp}(e^{it}k\beta e^{it}) = T(k_1\beta) = T^2(k_0\beta)$$

Thus, in general for $n \geq 0$, and β in H_2

$$k_n\beta = k\beta e^{int} = P_{M^\perp}(\beta e^{int},0) = T^n(k_0\beta) \tag{3-34}$$

Consequently

$$M_2^\perp = \overline{\text{span}}\{k_0\beta,\, k_1\beta,\, k_2\beta,\, \ldots,\, k_n\beta,\, \ldots\} \tag{3-35}$$

We have therefore shown that

Theorem 2

$\overline{k_0\beta}$ is a cyclic subspace [1] of $T\big|_{M_2^\perp}$, the restriction of T to $M_2^\perp$.

Our derivations of $K_n\alpha$ and $k_n\beta$ above were motivated by the work of D. N. Clark [3] on one dimensional perturbations of the restricted shift on a Nagy-Foias space associated with a scalar inner function θ. In his work Clark was using only K_1 (which he denoted by K_0) and k_0. It is a pleasure to thank Doug. Clark for introducing this work to me. Clark's work was subsequently generalized by Fuhrmann [4], Ball and Lubin [5].

To proceed further, we now consider the necessary and sufficient condition for an element $(v,\Delta u)$ in $M^\perp$ to be orthogonal to $K_n\alpha$.

The case $n = 0$ is trivial since $K_0\alpha$ is in M. For $n \geq 1$, we have:

$$[(v,\Delta u),\ K_n\alpha] = [(v,\Delta u),\ (P_2^+\theta(e^{it})e^{int}\alpha,\ \Delta e^{-int}\alpha)]$$

$$= [(Z^n v, \Delta e^{int}u),\ (\theta\alpha,\Delta\alpha)]$$

$$= [e^{int}(\theta^* v + \Delta^2 u,\ \alpha]_{L^2(H_1)},\quad \alpha \text{ in } H_1$$

Now since $(v,\Delta u)$ is in $M^\perp$, $\theta^* v + \Delta^2 u$ has an expansion in negative powers of e^{it}

$$\theta^* v + \Delta^2 u = \gamma_1 e^{-it} + \gamma_2 e^{-2it} + \ldots + \gamma_n e^{-int} + \ldots,\quad \gamma_i \text{ in } H_1$$

Consequently,

$$[(v,\Delta u),\ K_n\alpha] = [\gamma_n,\alpha]_{H_1},\ \gamma_n,\alpha \quad \text{in } H_1,\ n \geq 1 \tag{3-36}$$

Hence $(v,\Delta u)$ in $M^\perp$ is orthogonal to $K_n\alpha$ if and only if $\gamma_n = 0$.

Similarly, for $(v,\Delta u)$ in $M^\perp$ and for $n \geq 0$

$$[(v,\Delta u),\ k_n\beta] = [(v,\Delta u),\ P_{M^\perp}(z^n\beta,\ 0)]$$

$$= [v,\ z^n\beta]$$

$$= [v_n,\beta] \tag{3-37}$$

where v_n is the coefficient of z^n in the power series expansion of v_n. Hence $(v,\Delta u)$ in $M^\perp$ is orthogonal to $k_n\beta$ if and only if $v_n = 0$. It follows from the above that

<u>Lemma 3</u>

For $(v,\Delta u)$ in $M^\perp$,

$$(v,\Delta u) \perp M_1^\perp \quad \text{if and only if} \quad v = 0$$

$$(v,\Delta u) \perp M_2^\perp \quad \text{if and only if} \quad \theta^* v + \Delta^2 u = 0$$

The orthogonal complement (in $M^\perp$) of $M_1^\perp$ will be denoted by M_1, while that of $M_2^\perp$ will be denoted by M_2.

For any $(v,\Delta u)$ in $M^\perp$, straightforward calculations give

$$T^{*n}(v,\ \Delta u) = (\frac{v}{Z^n},\ e^{-int}\Delta u) - (\frac{V_0}{Z^n} + \frac{V_1}{Z^{n-1}}) + \ldots + V_{n-1} \quad n \geq 1, \tag{3-38}$$

and

$$T^n T^{*n}(v,\Delta u) = (v,\Delta u) - \sum_{\ell=0}^{n-1} k_\ell v_\ell, \quad n \geq 1 \tag{3-39}$$

Hence

$$(I-T^n T^{*n})\ (v,\Delta u) = \sum_{\ell=0}^{n-1} k_\ell v_\ell,\ n \geq 1 \tag{3-40}$$

Similarly

$$(I-T^{*n+1}T^{n+1})(v,\Delta u) = \sum_{\ell=0}^{n} K_{\ell+1}\ \gamma_{\ell+1},\ n \geq 0, \tag{3-41}$$

It then follows that

<u>Lemma 4</u>

For $(v,\Delta u)$ in $M^\perp$:

(i) $T^{*n}(v,\Delta u) = (\frac{v}{z^n}, e^{-int}\Delta u), \quad n \geq 1$

or equivalently

$(I-T^n T^{*n})\ (v,\Delta u) = 0\ , \quad n \geq 1$

if and only if $(v,\Delta u) \perp k_\ell\ \beta$ for all $\ell = 0, 1, 2, \ldots, n-1$.

(ii) $T^{n+1}(v,\Delta u) = \left(z^{n+1}v,\ e^{i(n+1)t}\Delta u\right), \quad n \geq 0$

or equivalently

$(I-T^{*n+1}T^{n+1})\ (v,\Delta u) = 0, \quad n \geq 0$

if and only if $(v,\Delta u) \perp K_\ell\ \alpha$ for all $\ell = 0, 1, \ldots, n+1$.

This lemma is a generalization of Clark's results for the scalar case [3].

The following results can be easily verified:

<u>Lemma 5</u>

$T^{n+1}K_1\alpha \quad = -\ k_n\theta(0)\alpha\ , \qquad n \geq 0$

and

$T^{*n+1}k_0\beta = -\ K_{n+1}\theta(0)^*\ \beta, \quad n \geq 0$

In the next Section, we shall use the above results for system operators realizations.

IV. Realizations of Transfer Operators. In this Section we shall discuss the realization of a given contractive analytic transfer operator $\{\theta(z), H_1, H_2\}$. Thus, our problem is to find operators A, B, C and D such that

$$\theta(z) = D + z\, C[I-zA]^{-1}B, \quad |z| < 1, \tag{4-1}$$

We shall use results of previous Sections. First, let us see what are the meanings of a Nagy-Foias space associated with a given transfer operator $\{\theta(z), H_1, H_2\}$.

As we have seen above, the Hilbert Space H (equation 2-1) is the space of present-future outputs (together with elements of the form Δu, which characterize the net energy absorbed by the system) while its subspace M (equation 2-6) is the set of all present - future outputs, resulted entirely from present - future inputs. Thus, the orthogonal complement $M^{\perp}$ of M (in H), can be regarded [6] as a state-energy type space.

The subspace $M_1^{\perp}$ (equation 3-6) consists of all present - future outputs $\theta_{21}u_-$ resulted entirely from past inputs u_-, further more we have

$$||(\theta_{21}u_-,\Delta u_-)||^2 = ||\theta_{21}u_-||^2 + ||\Delta u_-||^2 = ||u_-||^2 - ||\theta_{11}u_-||^2$$

which can be regarded as energy stored in the system - due to inputs in the past.

To proceed with the realization problem, we first observe, from Lemma 4, with $n = 1$, that

$$(I-T^*T)\ (v,\Delta u) = 0 \iff (v,\Delta u) \perp K_1\alpha, \tag{4-2}$$

and

$$(I-TT^*)\ (v,\Delta u) = 0 \iff (v,\Delta u) \perp k_0\beta, \tag{4-3}$$

Also using Lemma 5 with $n = 0$ we get,

$$(I-T^*T)\ K_1\alpha = K_1[I-\theta(0)^*\ \theta(0)]\alpha \quad , \tag{4-4}$$

and

$$(I-TT^*)\ k_0\beta = k_0[I-\theta(0)\ \theta(0)^*] \quad , \tag{4-4}$$

Now, since $\{\theta(z), H_1, H_2\}$ is purely contractive, that is $||\theta(0)|| < 1$, it can be shown that [1] the ranges of $[I-\theta(0)^*\ \theta(0)]$ and of $[I-\theta(0)\ \theta(0)^*]$ are dense in H_1 and in H_2 respectively. Hence

$$[I-T^*T]\ \overline{K_1H_1} = \overline{K_1H_1} \tag{4-6}$$

$$[I-TT^*]\ \overline{k_0 H_2} = \overline{k_0 H_2} \tag{4-7}$$

From which it follows that

$$[I-T^*T]^p \overline{K_1 H_1} = \overline{K_1 H_1} \tag{4-8}$$

$$[I-TT^*]^p \overline{k_0 H_2} = \overline{k_0 H_2} \tag{4-9}$$

for $p = 1, 2, \ldots$

The operators $[I-T^*T]$ and $[I-TT^*]$ are both positive and bounded between 0 and 1, consequently we can define its positive square roots $[I-T^*T]^{\frac{1}{2}}$ and $[I-TT^*]^{\frac{1}{2}}$. Furthermore, it follows from (4-8) and (4-9) that the range of $[I-T^*T]^{\frac{1}{2}}$ is dense in $\overline{K_1 H_1}$ while that of $[I-TT^*]^{\frac{1}{2}}$ is dense in $\overline{k_0 H_2}$.

The following model theorem of Nagy-Foias gives a solution to the realization problem.

<u>Nagy-Foias Model Theorem [1]</u>

Let $\{\theta(z),\ H_1,\ H_2\}$ be a purely contractive analytic function, and let T be be the restricted shift operator on the Nagy-Foias space $M^{\perp}$ generated by $\theta(z)$, then

$$\theta(z) = U\ \theta_T(z)\ V \tag{4-10}$$

where $\theta_T(z)$ is called the characteristic operator function of T and is defined by

$$\theta_T(z) = -T + z(I-TT^*)^{\frac{1}{2}}\ [I-zT^*]^{-1}\ (I-T^*T)^{\frac{1}{2}} \tag{4-11}$$

and maps $\overline{K_1 H_1}$ into $\overline{k_0 H_2}$.

U and V are unitary maps

$$U:\ \overline{k_0 H_2} \to H_2 \quad \text{and} \quad V:\ H_1 \to \overline{K_1 H_1}$$

It follows at once from this theorem that the operators A, B, C, and D which realize $\theta(z)$ are

$$A = T^*$$
$$B = (I-T^*T)^{\frac{1}{2}}\ V$$
$$C = U(I-TT^*)^{\frac{1}{2}}\Big|_{\overline{k_0 H_2}}$$

$$D = U\,\theta(0)V = -\,UT\Big|_{\overline{K_1 H_1}}\,V$$

Moreover, since $\overline{K_1 H_1}$ and $\overline{k_0 H_2}$ are cyclic for $T^*\Big|_{M_1^{\perp}}$ and $T\Big|_{M_2^{\perp}}$ (Theorems 1 and 3 of Section 3), we conclude that the Nagy-Foias realization is both controllable and observable.

We note that the Nagy-Foias model theory was developed via the unitary dilations of contraction operators [1], this is why they used the two operators $(I-T^*T)$ and $(I-TT^*)^{\frac{1}{2}}$, since the operator

$$\Sigma = \begin{bmatrix} -T & (I-TT^*)^{\frac{1}{2}} \\ (I-T^*T)^{\frac{1}{2}} & T^* \end{bmatrix} \tag{4-12}$$

is unitary and is a unitary dilation of $-T$.

One can of course obtain other realization schemes, using T, T^*, $(I-T^*T)^p$ and $(I-TT^*)^p$. For instance, if we form

$$\Phi(z) = -\,T + z(I-TT^*)\,[I-zT^*]^{-1}\,(I-T^*T)$$

Then we have

$$\tau_2\,\theta(z) = \Phi(z)\tau_1$$

where τ_1 and τ_2 are bounded invertible operators: τ_1: $H_1 \to \overline{K_1 H_1}$ and τ_2: $H_2 \to \overline{k_\theta H_2}$. In this case θ and Φ are said to be quasi - similar [1].

Finally, we note that if we set $z = \frac{p-1}{p+1}$, then $\theta(z = \frac{p-1}{p+1})$ can be taken to be the scattering operator of a linear passive multiport network [7], and the Nagy-Foias characteristic operator function $\theta_T(z)$ (equation 4-11) can be gotten by cascade loading the lossless network whose scattering operator is Σ (equation 4-12) in unit inductors.

Thus, for multiport passive networks, the Nagy-Foias model theorem results in the cascade load synthesis procedure. For a complete discussion of this, we refer to [8].

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