ON THE OPTIMAL CONTROL OF VARIATIONAL INEQUALITIES

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I. An existence theorem.

We are given a real and reflexive Banach space V, with dual V', an operator $A:V \longrightarrow V'$ and a function $j:V \longrightarrow (-\infty,+\infty]$.

We consider variational inequalities of the following type: find $y \in V$ such that:

$$\text{(I.I)} \qquad \langle Ay, z-y\rangle + j(z) - j(y) \geq \langle g, z-y\rangle \qquad \forall\, z \in V,$$

where $g \in V'$ and $\langle \cdot, \cdot \rangle$ is the pairing between V' and V.

We assume A a pseudo-monotone operator, i.e:

$$\text{(I.2)} \begin{cases} \text{a)} & A \text{ is bounded (on bounded subsets)} \\ \text{b)} & y_n \rightharpoonup y \text{ and } \overline{\lim\limits_{n\to\infty}} \langle Ay_n, y_n - y\rangle \leq 0 \text{ imply} \\ & \underline{\lim\limits_{n\to\infty}} \langle Ay_n, y_n - z\rangle \geq \langle Ay, y-z\rangle \quad \forall\, z \in V. \end{cases}$$

j is a convex, proper (i.e. $j \not\equiv +\infty$) and lower semicontinuous function.

It is well known that inequalities of the type (I.I), for any given $g \in V'$, have a solution (see J.L.Lions [4]), if the following coercivity hypothesis is satisfied:

$$\text{(I.3)} \begin{cases} \exists\, z_o \in V \text{ such that } j(z_o) < +\infty \text{ and} \\ \dfrac{\langle Az, z-z_o\rangle + j(z)}{\|z\|} \longrightarrow +\infty \qquad \text{when} \quad \|z\| \longrightarrow \infty . \end{cases}$$

We shall deal with the following control problem. The space of controls is U, a real and reflexive Banach space; the set of admissible controls is U_{ad}, a closed convex (non empty) subset of U; $B:U \longrightarrow V'$ is a map such that $u_n \xrightarrow[U]{} u$ implies $Bu_n \xrightarrow[V']{} Bu$.

For any given $u \in U_{ad}$, the state $y=y(u)$ is given by the solution (not necessarily unique) of the following inequality($f\in V'$ is fixed):

(I.4) $\langle Ay,z-y\rangle+j(z)-j(y)\geq\langle f+Bu,z-y\rangle \quad \forall z\in V.$

The cost is assumed to be quadratic:

$J(u,y(u))=\|y(u)-z_d\|_V^2+\nu\|u\|_U^2$, $\nu\geq 0$ (a linear and continuous observation operator might as well be considered).

The control problem is: minimize J on U_{ad}.

We have the following theorem:

Theorem I.I Under the hypotheses above, if we have:

(I.5) either $\nu>0$ or U_{ad} is bounded,

there exists an optimal pair $(u,y(u))$.

We give here a brief sketch of the proof. We take a minimizing sequence $\{u_n,y_n\}_{n\in N}$, where y_n is selected among the solutions of (I.4) corresponding to u_n. It follows from hypothesis (I.5) and the coercivity hypothesis (I.3) that we can extract a subsequence $\{u_{n_i},y_{n_i}\}_{i\in N}$ such that $u_{n_i}\longrightarrow u$ and $y_{n_i}\longrightarrow y$ in V. The compactness of B, the lower semicontinuity of j and the pseudo-monotonicity of A allow us to pass to the limit in the inequality and to prove that y is a solution corresponding to u of (I.4). To conclude, the weak lower semicontinuity of J on $U\times V$ assure us that $\{u,y\}$ is an optimal control-state pair. ■

Remark I A particular case of inequality (I.I) is:

(I.6) $\langle Ay,z-y\rangle\geq\langle g,z-y\rangle \quad \forall z\in K,$

where K is a closed convex subset of V. It suffices to define $j(z)=\delta_K(z)$ where δ_K is the indicator function of K, i.e.:

$$\delta_K(z)=\begin{cases}0 & \text{if } z\in K\\ +\infty & \text{if } z\notin K\end{cases}$$

Remark 2 Generally we have not uniqueness of the optimal control. A very simple counterexample is the following ($V=U=R$). The inequality is given by ($Bu=u+\sqrt{2}$):

(I.7) $\quad y\cdot(z-y)\geq(u+\sqrt{2})\cdot(z-y) \qquad \forall z\in[0,I]$

The (unique) solution of (I.7) is given by $y(u)=Pr_K(u+\sqrt{2})$, the projection of $u+\sqrt{2}$ on $K=[0,I]$.

If we set $U_{ad}=U$ and $J(u)=u^2+(y(u))^2$, both $u=0$ and $u=-\frac{\sqrt{2}}{2}$ are optimal controls.

Former results on the optimal control of variational inequalities may be found in J.P.Yvon [6], or in J.L.Lions [3] (here a special case is considered); see also R.Kluge [2], and the bibliography listed there for further references.

2. Further results on the control of variational inequalities.

The existence theorem previously given can be extended to more general cost functionals, provided that the operator A is actually a monotone hemicontinuous and bounded operator. To do this, we need the following theorem of F.E.Browder [I] on the sequential lower semicontinuity of certain types of functionals.

Theorem 2.I We are given three real Banach spaces X, X_1, X_2 ; a map $h:X_1\ X_2\longrightarrow R$ such that:

a) $h(x_1,x_2)$ is convex and strongly continuous in x_1 when x_2 is fixed

b) $h(x_1,x_2)$ is strongly continuous in x_2 , when x_1 is fixed, and uniformly continuous in x_2 when x_1 varies in bounded subsets.

We are given also $L:X\longrightarrow X_1$, a linear and continuous map, and $M:X\longrightarrow X_2$ a map sequentially continuous from the weak to the strong topology. Then, setting $J(u)=h(Lu,Mu):X\longrightarrow R$, J is weakly sequentially lower semicontinuous.

We note that in Browder's theorem it is required that M is a linear map, but this hypothesis can be suppressed, as can be easily verified.

If we require that A is a strongly monotone operator, we can apply Browder's theorem to our control problem: to do this we note that under this assumption the map $G:U \longrightarrow V$, which associates to the control u the corresponding (unique) state $y(u)$, is sequentially continuous from weak to strong. If we define $X=U$, $X_2=V$ and $M=G$, while h, X_1 and L are as in Browder's theorem, we obtain that the map $J(u)=h(Lu,Mu)$ is sequentially weakly lower semicontinuous on U. This fact gives us an existence theorem if we assume that the set of admissible controls U_{ad} is bounded.

To conclude, we give here briefly a result on the sensitivity of the control problem. We consider variational inequalities of type (I.6), where A is a strictly monotone hemicontinuous and bounded operator. We perturb A, K, z_d and f. More precisely, we consider a sequence of problems like this:

(P_n) : minimize $J_n(v)=\|y_n(v)-z_{d_n}\|_V^2+\nu\|v\|_U^2$

for $v\in U_{ad}$ and $y_n(v)$ the solution of:

$$\langle A_n y_n(v), z-y_n(v)\rangle \geq \langle f_n+Bv, z-y_n(v)\rangle \qquad \forall\, z\in K_n$$

The initial problem is:

(P_o) : minimize $J_o(v)=\|y_o(v)-z_{d_o}\|_V^2+\nu\|v\|_U^2$

for $v\in U_{ad}$ and $y_o(v)$ the solution of:

$$\langle A_o y_o(v), z-y_o(v)\rangle \geq \langle f_o+Bv, z-y_o(v)\rangle \qquad \forall\, z\in K_n$$

If we assume that $f_n \longrightarrow f_o$, $z_{d_n} \longrightarrow z_{d_o}$, and that A_n, K_n and A_o, K_o verify the hypotheses introduced by U.Mosco in [5] to prove the strong convergence of solutions of perturbed variational inequalities, then we obtain the following result:

<u>Theorem 2.2</u> Given a sequence of optimal controls u_n for P_n, we can extract a subsequence u_{n_i} weakly converging to an optimal control $\bar{u}$ for P_o; the corresponding states y_{n_i} converge strongly to $\bar{y}=y_o(\bar{u})$.

Let us give here a brief sketch of the proof. Our aim is to prove the boundedness of u_n, and this is obvious if we prove that $J_n(u_n)$ is such. To prove this fact, we note that $J_n(u_n) \leq J_n(u_o)$, where u_o is an optimal control for P_o. But $J_n(u_o)$ is bounded because the states $y_n'=y_o(u_n)$ converge strongly to y_o; more precisely we obtain that:

(2.I) $\quad \forall \varepsilon > 0, \quad \exists\, n(\varepsilon) \quad$ s.t. $\quad \left|J_n(u_o)-J_o(u_o)\right| < \varepsilon$.

This fact implies the boundedness of $J_n(u_n)$, so of u_n. If we consider a subsequence $u_{n_i} \longrightarrow \bar{u}$, we have $Bu_{n_i} \longrightarrow B\bar{u}$: from this follows that $y_{n_i}(u_{n_i}) \longrightarrow y_o(\bar{u})$. To conclude, the weak lower semicontinuity of the costs J imply that $\underline{\lim}_{n\to\infty} J_n(u_n) \geq J_o(\bar{u})$. From this, and from (2.I), we get that $J_o(\bar{u}) \leq J_o(u_o)+\varepsilon$ for every $\varepsilon>0$, so $J_o(\bar{u}) \leq J_o(u_o)$. But $J_o(u_o) \leq J_o(\bar{u})$ for definition of u_o: this means that $\bar{u}$ is actually an optimal control for P_o. ∎

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