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WHEN ARE TWO EFFECTIVELY GIVEN DOMAINS IDENTICAL?

(EXTENDED ABSTRACT)

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(Extended Abstract)

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ABSTRACT

In this paper, we will observe that the notion of computability in an effectively given domain is dependent on the indexing of its basis. This indicates that we cannot identify two effectively given domains just because they are order isomorphic. We propose a suitable notion of effective isomorphism to compensate for this deficiency. Also we show that, for every recursive domain equation, there is an effectively given domain which is an initial solution to within effective isomorphism.

1. Effectively Given Domains

The fundamental idea of effectively given domains is to assume effectiveness of finite join operations on a basis of each countably based cpo and to define computable elements as the least upper bounds (lub) of r.e. chains of basis elements. For details of results based on this idea see Scott [7], Tang [6], Egli-Constable [1], Markowsky-Rosen [3] and Smyth [8].

In this theory it is tempting to avoid questions of indexing. In fact, initially it is not clear whether an effectively given domain is to be a domain which can be effectively given in some unspecified manner or is a domain where this is specified. One could ask if it makes any difference. One of the main purposes of this paper is to show it does. This calls for rather "tedious" definitions of effectively given domains (see definition 1.1).

A poset is directed complete iff every directed subset has a lub. A directed complete poset with a least element (called bottom) is called a complete partial ordering (cpo). An element x of a poset D is compact iff for every directed subset $S \subseteq D$, s.t. $\sqcup S \in D$, $x \sqsubseteq \sqcup S \Rightarrow x \sqsubseteq s$ for some $s \in S$. A directed complete poset D is countably algebraic iff the set E_D of all compact elements of D is countable and for every $x \in D$, the set $J_x = \{e \mid e \in E_D, e \sqsubseteq x\}$ is directed and $x = \sqcup J_x$. In this case E_D is called the basis of D. The following extension property of bases is well-known : Let D be countably algebraic, then for any cpo Q, every monotone $m : E_D \to Q$ has a unique continuous extension $\bar{m} : D \to Q$ given by $\bar{m}(x) = \sqcup\{m(e) \mid e \in E_D, e \sqsubseteq x\}$. A poset is said to have bounded joins iff every bounded finite subset has a lub. If every bounded subset has a lub, we say that the poset is bounded complete. It can readily be seen that a countably algebraic cpo D has bounded joins iff E_D has bounded joins iff D is bounded complete.

Definition 1.1 (1) Let D be a countably algebraic bounded complete cpo (countably algebraic domain) with the basis E_D. A (total) indexing $\varepsilon : N \to E_D$ is effective (or is an effective basis of D) iff the following relations are recursive in indices :

1. $\{\varepsilon(i_1), \ldots, \varepsilon(i_n)\}$ is bounded in E_D $\qquad n \geq 0$
2. $\varepsilon(k) = \sqcup\{\varepsilon(i_1), \varepsilon(i_n)\}$

Notice that $\sqsubseteq$ and $\perp$ are effective according to this definition.

(2) An indexed domain is an ordered pair $<D,\varepsilon>$ when D is a countably algebraic domain and $\varepsilon : N \to E_D$ is a total indexing of E_D. An indexed domain $<D,\varepsilon>$ is effectively given iff ε is an effective basis of D. We will write D^ε for $<D,\varepsilon>$.

(3) Given an effectively given domain D^ε, $x \in D$ is computable w.r.t.ε (or is computable in D^ε) iff there exists a recursive function $\rho : N \to N$ s.t. $\varepsilon \cdot \rho : N \to E_D$ is an ω-chain and $x = \sqcup \varepsilon \cdot \rho(n)$. The set of all computable elements of D^ε will be denoted by Comp (D^ε).

(4) Given effectively given domains D^{ε} and $D'^{\varepsilon'}$, a function $f : D \to D'$ is computable w.r.t. $\langle\varepsilon,\varepsilon'\rangle$ iff the graph of f, which is $\{\langle n,m\rangle \mid \varepsilon'(m) \sqsubseteq f.\varepsilon(n)\}$, is an r.e. set.

Notice that an indexed domain D^{ε} is effectively given iff there exists a pair of recursive predicates $\langle b,\ell\rangle$, which will be called the characteristic pair of D^{ε} s.t. :

$$b(x) \Leftrightarrow \{\varepsilon(i_1),\ldots,\varepsilon(i_n)\} \text{ is bounded in } E_D \text{ and}$$

$$\ell(k,x) \Leftrightarrow \varepsilon(k) = \sqcup\{\varepsilon(i_1),\ldots,\varepsilon(i_n)\}$$

where f_s is the standard enumeration of finite subsets of N and $f_s(x) = \{i_1,\ldots,i_n\}$. Notice that if D^{ε} and $D'^{\varepsilon'}$ have the same characteristic pair, then D^{ε} is merely a "renaming" of $D'^{\varepsilon'}$. More formally, there exists an order isomorphism $f : D \to D'$ s.t. $f.\varepsilon = \varepsilon'$. We will denote this relation by $D^{\varepsilon} \overset{r}{\cong} D'^{\varepsilon'}$.

To within $\overset{r}{\cong}$ we can introduce the following partial indexing of the set of all effectively given domains. Let $\langle\phi_i\rangle$ and $\langle W_i\rangle$ to fixed (throughout this paper) acceptable indexings [5] of partial recursion functions and r.e. sets respectively s.t. range $(\phi_i) = W_i$. We say that an effectively given domain D^{ε} has an acceptable index $\langle i,j\rangle$ iff $\langle\phi_i,\phi_j\rangle$ is a characteristic pair of D^{ε}. We will denote this partial indexing of effectively given domains by $\bar{\xi}$. We will write $\xi(i)$ to denote the effective basis of $\bar{\xi}$ (i). Notice that for a partial indexing τ, we write $\tau(i)$ iff i is a τ-index, i.e. $\tau(i)$ is defined.

Given an effectively given domain D^{ε}, an r.e. set W is ε-directed iff $\varepsilon(W)$ is directed in E_D. In this case we say that $\varepsilon(W)$ is effectively directed via ε. It can readily be seen that $x \in \mathrm{Comp}(D^{\varepsilon})$ iff $x = \sqcup\varepsilon(W)$ for some ε-directed r.e. set W. Furthermore we can effectively "ε-direct" every r.e. set. More formally:

Lemma 1.2 For every effective given domain D^{ε}, there is a recursive function $d_{\varepsilon} : N \to N$ s.t. for every $j \in N$, $W_{d_{\varepsilon}(j)}$ is ε-directed, and s.t. in case W_j is ε-directed, $\sqcup\varepsilon(W_j) = \sqcup\varepsilon(W_{d_{\varepsilon}(j)})$. This lemma gives us the following total indexing δ_{ε} of $\mathrm{Comp}(D^{\varepsilon})$. If $x = \sqcup\varepsilon(W_{d_{\varepsilon}(j)})$ then we say that x has a directed index j and denote it by $x = \delta_{\varepsilon}(j)$. ◪

Since we took the view that an effectively given domain is a domain with a specified effective basis, domain constructors must relate not only po structure but also effective structure. Thus we have to be explicit about constructed effective bases.

Definition 1.3 Given indexed domains D^{ε} and $D'^{\varepsilon'}$, define $D^{\varepsilon} \times D'^{\varepsilon'}$, $D^{\varepsilon} + D'^{\varepsilon'}$, and $[D^{\varepsilon} \to D'^{\varepsilon'}]$ to be the following indexed domains :

(1) $D^{\varepsilon} \times D'^{\varepsilon'} \stackrel{\text{def}}{=} \langle D \times D', (\varepsilon \times \varepsilon') \rangle$ where $(\varepsilon \times \varepsilon')(n) = \langle \varepsilon.\pi_1(n), \varepsilon.\pi_2(n) \rangle$.

(2) $D^{\varepsilon} \times D'^{\varepsilon'} \stackrel{\text{def}}{=} \langle D + D', (\varepsilon + \varepsilon') \rangle$ where $(\varepsilon + \varepsilon')(n) =$ <u>if</u> $n = 0$ <u>then</u> $\perp$ <u>else</u> <u>if</u> $n = 2m+1$ <u>then</u> $\langle 0, \varepsilon(m) \rangle$ else if $n = 2m$ <u>then</u> $\langle 1, \varepsilon'(m) \rangle$

(3) $[D^{\varepsilon} \to D'^{\varepsilon'}] \stackrel{\text{def}}{=} \langle [D \to D'], [\varepsilon \to \varepsilon'] \rangle$ where $[\varepsilon \to \varepsilon'](n) =$ <u>if</u> $\sigma(n)$ has a lub <u>then</u> $\sqcup\sigma(n)$ <u>else</u> $\perp$, and $\sigma(n) = \{[\varepsilon(i), \varepsilon(j)] \mid \langle i,j \rangle \in P(n)\}$ where P is the standard enumeration of finite subsets of $N \times N$, and $[e,e'](x) =$ <u>if</u> $x \sqsupseteq e$ <u>then</u> e' <u>else</u> $\perp$. □

It is well known that if D^{ε} and $D'^{\varepsilon'}$ are effectively given domains then so are $D^{\varepsilon} \times D'^{\varepsilon'}$, $D^{\varepsilon} + D'^{\varepsilon'}$, and $[D^{\varepsilon} \to D'^{\varepsilon'}]$. The following theorem says that $\times, +, \to$ are "effective" constructors :

Theorem 1.4 There are recursive functions Prod, Sum, Func: $N \times N \to N$ s.t. if i, and j are acceptable indices of D^{ε} and $D'^{\varepsilon'}$, then Prod (i,j), Sum (i,j), and Func (i,j) are acceptable indices of $D^{\varepsilon} \times D'^{\varepsilon'}$, $D^{\varepsilon} + D'^{\varepsilon'}$, and $[D^{\varepsilon} \to D'^{\varepsilon'}]$ respectively. ◪

Smyth [7] showed that a function $f : D \to D'$ is computable w.r.t. $\langle \varepsilon, \varepsilon' \rangle$ iff $f \in \mathrm{Comp}\ ([D^{\varepsilon} \to D'^{\varepsilon'}])$. We can show that this equivalence is "effective". Let $f : D \to D'$ to computable w.r.t. $\langle \varepsilon, \varepsilon' \rangle$. If W_j is the graph of f, then we say that j is a ($\langle \varepsilon, \varepsilon' \rangle$-) graph index of f.

Lemma 1.5 There are recursive functions d_g, $g_d : N \times N \to N$ s.t.,

(1) If k is a graph index of f which is computable w.r.t. $\langle \xi(i), \xi(j) \rangle$ then $f = \delta_{[\xi(i) \to \xi(j)]}(d_g(k, \langle i,j \rangle))$

(2) If $f = \delta_{[\xi(i) \to \xi(j)]}(k)$ then f has a graph index $g_d(k, \langle i,j \rangle)$ ☐

In addition to 1.5, we have further evidence to convince us that our notion of computability is really satisfactory.

Lemma 1.6 (1) A function from an effectively given domain to another is computable w.r.t. their effective bases iff it maps computable elements to computable elements recursively in directed indices.

(2) The composition of computable functions is recursive in directed indices. More formally there exists a recursive function d-Compose : $N \times N \times N \times N \times N \to N$ s.t. :

$$\delta_{[\xi(k) \to \xi(\ell)]}(i) \cdot \delta_{[\xi(\ell) \to \xi(m)]}(j) = \delta_{[\xi(k) \to \xi(m)]}(\text{d-Compose}(i,j,k,\ell,m)).$$

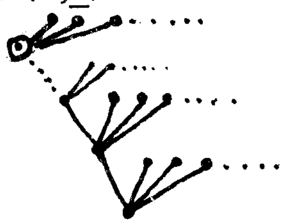
2. Effective Embeddings

In this section, we will observe why the indexing of a basis of an effectively given domain must be specified.

Theorem 2.1 (1) There is a countably algebraic domain D with two different effective bases ε and ε' s.t. $\mathrm{Comp}\ (D^{\varepsilon}) = \mathrm{Comp}\ (D^{\varepsilon'})$ but s.t. $\mathrm{Comp}\ ([D^{\varepsilon} \to \mathbb{O}^{\pi}])$ is not isomorphic to $\mathrm{Comp}\ ([D^{\varepsilon'} \to \mathbb{O}^{\pi}])$, when $\mathbb{O}$ is the two point lattice and π is an arbitrary effective basis of $\mathbb{O}$.

(2) There is a countably algebraic domain D with two different effective bases ε and ε' s.t. $\mathrm{Comp}\ (D^{\varepsilon}) \nsubseteq \mathrm{Comp}\ (D^{\varepsilon'})$.

proof (outline) (1) Let $(D,\sqsubseteq)$ be the following countably algebraic domain:



Note that D has only one limit point $\odot$. Thus the basis E_D of D is the poset obtained from D by removing $\odot$. Think of the following poset : $(N \cup (N\times N),\sqsubseteq)$ where $i\sqsubseteq j$ iff $i\leq j$, $i\sqsubseteq\langle m,n\rangle$ iff $\phi_m(n)$ takes at least i steps, and $\langle m,n\rangle\sqsubseteq\langle m',n'\rangle$ iff $m=m'$ and $n=n'$. Evidently the partial ordering $\sqsubseteq$ is decidable in terms of the Gödel numbering of $N \cup (N\times N)$. Thus this Gödel numbering provides an effective indexing ε' of E_D. Now think of the following poset: $(N \cup (N\times N) \cup (\{\omega\}\times N),\sqsubseteq)$ s.t. $i\sqsubseteq j$ iff $i\leq j$, $i\sqsubseteq\langle m,n\rangle$ iff $i\leq m$, and $\langle m,n\rangle\sqsubseteq\langle m'n'\rangle$ iff $m=m'$ and $n=n'$. It is also easy to observe that the Gödel numbering of $N \cup (N\times N) \cup (\{\omega\}\times N)$ provides an effective indexing ε of E_D. Obviously Comp (D^{ε}) = Comp $(D^{\varepsilon'})$ = D. Now let $f : D\to\mathbb{C}$ be a continuous function s.t. $f(x)$ = <u>if</u> $x\sqsubseteq\odot$ <u>then</u> $\bot$ <u>else</u> $\top$. Then f is computable w.r.t. $\langle\varepsilon,\pi\rangle$ but not so w.r.t. $\langle\varepsilon',\pi\rangle$. For let $M = \{h\in[D\to\mathbb{C}] \mid \{g\in[D\to\mathbb{C}] \mid g\sqsupseteq h\}$ is finite$\}$. Then $M = \{h_X \mid X$ is a finite set of leaves above compact elements of $D\}$ where $h_X(x)$ = <u>if</u> $x\sqsubseteq y\in X$, <u>then</u> $\bot$ <u>else</u> $\top$. It can be readily seen that $M\subseteq$Comp $([D^{\varepsilon}\to C^{\pi}])$ and $M\subseteq$Comp $([D^{\varepsilon'}\to\mathbb{C}^{\pi}])$. Let Φ: Comp $([D^{\varepsilon}\to\mathbb{C}^{\pi}]) \to$ Comp $([D^{\varepsilon'}\to\mathbb{C}^{\pi}])$ be a monotone isomorphism. Then $\Phi(M) = M$. Notice that $f=\sqcap M$. Therefore $\sqcap M\in$Comp $([D^{\varepsilon}\to\mathbb{C}^{\pi}])$. Since Φ is an isomorphism, $\Phi(\sqcap M) = \sqcap' M\in$Comp $([D^{\varepsilon'}\to\mathbb{C}^{\pi}])$. But it is easy to see that M has no greatest lower bound in Comp $([D^{\varepsilon'}\to\mathbb{C}^{\pi}])$. For suppose g is a lower bound of M in Comp $([D^{\varepsilon'}\to\mathbb{C}^{\pi}])$. Then $g(x) = \bot$ for some $x\not\sqsupseteq\odot$, since $g\sqsubseteq f$. But then $h(y)$ = <u>if</u> $y=x$ <u>then</u> $\top$ <u>else</u> $g(y)$ is also a lower bound of M in Comp $([D^{\varepsilon'}\to\mathbb{C}^{\pi}])$ and above g. ☒

Notice that 2.1 is more than a counter-example to a careless definition of effectively given domains. In fact (1)-2.1 indicates that Comp (D^{ε}) = Comp $(D^{\varepsilon'})$ is not sufficient to identify ε and ε'. Remember that in domain theory, domain constructors must preserve equality of domains, more technically, they must be functors. But if we assume that D^{ε} and $D^{\varepsilon'}$ are equivalent iff Comp (D^{ε}) = Comp $(D^{\varepsilon'})$, then "$\hookrightarrow$" does not preserve this equality as shown in (1)-2.1. We claim that the following equivalence of effectively given domains is appropriate :

Definition 2.2 Let D^{ε} and $D^{\varepsilon'}$ be indexed domains. We say that ε and ε' are effectively equivalent (in symbols, $\varepsilon \overset{e}{=} \varepsilon'$) iff there are recursive functions, r,s, : $N \to N$ s.t. $\varepsilon' = \varepsilon.s$ and $\varepsilon = \varepsilon'.r$.

It can readily be seen that if either ε or ε' is effective then $\varepsilon \overset{e}{=} \varepsilon'$ implies both ε and ε' are effective and Comp (D^{ε}) = Comp $(D^{\varepsilon'})$.

Notice that D^{ε} and $D^{\varepsilon'}$ of (1)-2.1 are not effectively equivalent. In fact, if ε and ε' were effectively equivalent then there could exist a recursive function r : $N \to N$ s.t.$\phi_m(n)$ terminates iff $r(<m,n>) = <m',n'>$ with $m' \neq \infty$, and we could solve the Halting problem.

We can easily extend the notion of effective equivalence to isomorphism.

Definition 2.3 Let D^{ε} and $D'^{\varepsilon'}$ be indexed domains. A function $f : E_D \to E_{D'}$ is an effective imbedding from ε to ε' (in symbol $f : \varepsilon \to \varepsilon'$) iff

1. f is injective
2. there exists a recursion function r_f : $N \to N$ s.t. $f.\varepsilon = \varepsilon'.r_f$
3. $\{\varepsilon(i_1),\ldots,\varepsilon(i_n)\}$ is bounded iff $\{f.\varepsilon(i_1),\ldots,f.\varepsilon(i_n)\}$ is bounded
4. $f(\sqcup\{\varepsilon(i_1),..,\varepsilon(i_n)\}) = \sqcup\{f.\varepsilon(i_n),\ldots,f.\varepsilon(i_n)\}$, $n \geq 0$. □

In case both D^{ε} and $D'^{\varepsilon'}$ are effectively given domains, then we have : $\overline{f}(\text{Comp } (D'^{\varepsilon'})) \subseteq \text{Comp } (D'^{\varepsilon'})$, where $\overline{f}$ is the continuous extension of f.

Remember that a continuous function $f : D \to D'$ is an embedding iff there exists a continuous function $g : D' \to D$ s.t. $f.g. \subseteq id_{D'}$, and $g.f = id_D$. Every embedding f uniquely determines such g, which will be called the adjoint of f. Also every embedding is strict. In case D and D' are algebraic cpo's, we have $f(E_D) \subseteq E_D$, and $g(E_{D'}) \subseteq E_D$.

Theorem 2.4 (1) Let D^{ε}, $D'^{\varepsilon'}$ be indexed domains and f be an effective imbedding from ε to ε', then $f : D \to D'$ is an embedding with the adjoint $g : D' \to D$ given by $g(y) = \sqcup\{e \in E_D | f(e) \subseteq y\}$. Furthermore $g \restriction f(E_D) = f^{-1}$.

(2) In case D^{ε} and $D'^{\varepsilon'}$ are effectively given domains, $\bar{f}$ is computable w.r.t. <ε,ε'> and g is computable w.r.t. <ε',ε>. ☒

We will call $\bar{f}$ an effective embedding when f is an effective imbedding. A pair-wise computable embedding (p-computable embedding) is an embedding which is computable as well as its adjoint. Thus by (2)-2.4, an effective embedding from an effectively given domain to an effectively given domain is a p-computable embedding. The converse of this is also true.

Theorem 2.5 Let D^{ε} and $D'^{\varepsilon'}$ be effectively given domains s.t. $\bar{f} : D \to D'$ is a p-computable embedding, then $\bar{f}$ is an effective embedding.

proof Let $\bar{g} : D' \to D$ be the adjoint of $\bar{f}$. Then both $\varepsilon'(n) \sqsubseteq f.\varepsilon(m)$ and $\varepsilon(n) \sqsubseteq g.\varepsilon'(m)$ are r.e. in indices. We will show the existence of a recursive function $r : N \to N$ s.t. $f.\varepsilon = \varepsilon'.r$. We claim that the following terminating program computes such r(m) for each $m \in N$:

- enumerate n s.t. $\varepsilon'(n) \sqsubseteq f.\varepsilon(m)$.
- for each enumerated n, enumerate k s.t. $\varepsilon(k) \sqsubseteq g.\varepsilon'(n)$.
- continue this process until we obtain a k s.t. $\varepsilon(k) = \varepsilon(m)$.

The n for which this k is produced is r(m).

By a "dove-tailing" technique [5], we can compute the above process. We can check that such r is actually the one desired. Assume k,n are the values when the above process terminates. Then $\varepsilon(k) \sqsubseteq g.\varepsilon'(n) \sqsubseteq g.f(\varepsilon(m)) = \varepsilon(m)$. Since $\varepsilon(k) = \varepsilon(m)$, we have $g.\varepsilon'(n) = \varepsilon(m)$. But $\varepsilon'(n) \sqsupseteq f.g.\varepsilon'(n) = f.\varepsilon(m)$. Therefore $\varepsilon'(n) = f.\varepsilon(m)$. ☒

In fact, we can observe that the equivalence of effective embeddings and p-computable embeddings is "effective". Given an effective imbedding $f : \varepsilon \to \varepsilon'$, if $r_t = \phi_j$ we say that f has a recursive index j. In this case we say that the effective embedding $\bar{f}$ has a recursive index j.

Theorem 2.6 (1) There is a recursive function $\xi_d : N \times N \times N \to N$ s.t. if i and j are directed indices of a p-computable embedding $\bar{f} \in \mathrm{Comp}\ ([\bar{\xi}(k) \to \bar{\xi}(\ell)])$ and its adjoint $\bar{g} \in \mathrm{Comp}\ ([\bar{\xi}(k) \to \bar{\xi}(\ell)])$ respectively, then $\xi_d(i,j,\langle k,\ell\rangle)$ is a recursive index of $\bar{f}$.

(2) There are recursive functions d_p, $d_a : N \times N \to N$ s.t. if i is a recursive index of an effective embedding $\bar{f} \in \mathrm{Comp}\ ([\bar{\xi}(j) \to \bar{\xi}(k)])$ then $d_p\ (i,\langle j.k\rangle)$ is a directed index of $\bar{f}$ and $d_a\ (i,\langle j,k\rangle)$ is a directed index of the adjoint $\bar{g} \in \mathrm{Comp}\ ([\bar{\xi}(k) \to \bar{\xi}(j)])$ of $\bar{f}$.

Now we can define what an effective isomorphism is about.

Definition 2.7 Let D^ε and $D'^{\varepsilon'}$ be indexed domains. We say ε and ε' are effectively isomorphic iff there exists an effective imbedding $f : \varepsilon \to \varepsilon'$ s.t. f^{-1} is also an effective imbedding from ε' to ε. We will denote this by $\varepsilon \overset{e}{\cong}\ '$. In this case we also say that D^ε and $D'^{\varepsilon'}$ are effectively isomorphic and denote it by $D^\varepsilon \overset{e}{\cong} D'^{\varepsilon'}$. Evidently $(\bar{f},\bar{f}^{-1})$ is a continuous isomorphic pair. We will call $\bar{f}$ (or $\bar{f}^{-1}$) an effective isomorphism.

If $D^\varepsilon \overset{e}{\cong} D'^{\varepsilon'}$ and either of them is an effectively given domain, then both of them are effectively given and $\mathrm{Comp}(D^\varepsilon) \cong \mathrm{Comp}\ (D'^{\varepsilon'})$. Also an isomorphism between two effectively given domains is an effective isomorphism iff both itself and its adjoint are computable.

Remember that we have claimed that the notion of effective isomorphism gives an appropriate criterion for identifying two effectively given domains. We can provide quite convincing evidence to this claim. First, evidently $\overset{e}{\cong}$ is an equivalence relation. Furthermore we can show that $\overset{e}{\cong}$ is invariant under the domain constructions $\times$, $+$, and $\to$.

More formally:

Theorem 2.8 Let A^{α}, B^{β}, C^{γ}, and D^{δ} be indexed domains s.t. $A^{\alpha} \overset{e}{\cong} C^{\gamma}$ and $B^{\beta} \overset{e}{\cong} D^{\delta}$. Then we have

(1) $A^{\alpha} \times B^{\beta} \overset{e}{\cong} C^{\gamma} \times D^{\delta}$ (2) $A^{\alpha} + B^{\beta} \overset{e}{\cong} C^{\gamma} + D^{\delta}$

(3) $[A^{\alpha} \to B^{\beta}] \overset{e}{\cong} [C^{\gamma} \to D^{\delta}]$.

Note that if A^{α}, B^{β}, C^{γ}, D^{δ} are effectively given domains, then the invariance of $\overset{e}{\cong}$ immediately follows from 1.6.

3. Algebraic Completion

Smyth showed (in [8]) that for continuous cpo's, we cannot introduce effectiveness as we did for algebraic cases in 1.1. He characterized an effectively given continuous domain as a continuous domain which is "isomorphic" to the completion of an effective R-structure but this characterization ignores the precise indexing of the effectively given domain. We will provide an algebraic version of Smyth's characterization, taking care of effective isomorphisms. In fact we will observe that this characterization is an alternative characterization to 1.1.

By the (algebraic) completion of a poset $(E,\sqsubseteq)$, we mean a poset $(\bar{E},\subseteq)$ where $\bar{E}$ is the set of all directed subsets of A which are downward closed i.e. $x \in X$ & $y \sqsubseteq x$ implies $y \in X$. In case $(E,\sqsubseteq)$ is a countable poset with a bottom and bounded joins, then there exists an embedding $\tau : E \to \bar{E}$ s.t. $(\bar{E},\subseteq)$ is a countably algebraic domain with the basis $(\tau(E),\subseteq)$. In fact $\tau(x) = \{e \in E \mid e \sqsubseteq x\}$. Conversely if D is a countably algebraic domain then the basis E_D is a countable poset with a bottom and bounded joins, and $D \cong \bar{E}_D$.

Definition 3.1 Let $(E,\sqsubseteq)$ be a countable poset with a bottom and bounded joins and $\varepsilon : N \to E$ be a total indexing. We call $\langle E,\varepsilon \rangle$ indexed poset. In case ε is effective, which means ε satisfies (1)-1.1, we call $\langle E,\varepsilon \rangle$ an effective poset. The (algebraic) completion of an indexed poset $<E,\varepsilon>$ is an indexed domain $<\bar{E},\bar{\varepsilon}>$ where $\bar{\varepsilon}:N\to\tau(E)$ is given by $\bar{\varepsilon}(n)=\tau.\varepsilon(n)$. □

Theorem 3.2 (1) Let $<E,\varepsilon>$ be an effective poset. Then the completion of it is an effectively given domain.

(2) Given an effectively given domain D^{ε}, E_D^{ε} is an effective poset and $<\bar{E}_D,\bar{\varepsilon}>=<D,\varepsilon>$.

(3) An indexed domain is an effectively given domain iff it is effectively isomorphic to the completion of some effective poset. ▨

4. Inverse Limits

Given an ω-sequence $<D_m,f_m>$ of embeddings of countably algebraic domains, the inverse limit of the sequence, in symbols $\underleftarrow{\lim}<D_m,f_m>$, is the poset $\{<x_m> \mid x_m=g_m(x_{m+1})\}$ with the coordinate-wise ordering, where g_m is the adjoint of f_m. It is well-known that $\underleftarrow{\lim}<D_m,f_m>$ is again a countably algebraic domain (see Plotkin [4]). We will write D_∞ for $\underleftarrow{\lim}<D_m,f_m>$. Define $f_{n\infty}:D_n\to D_\infty$ and $g_{\infty n}:D_\infty\to D_n$ by:

$$f_{n\infty}(x)=<g_o.g_1....g_{n-1}(x),...,g_{n-1}(x),x,f_n(x),f_{n+1}.f_n(x),... > ,$$

$$g_{\infty n}(<x_o,x_1,...>)=x_n.$$

We call $<f_{n\infty}>$ the universal cone of $<D_m,f_m>$. Evidently $f_{n\infty}$ is an embedding with the adjoint $g_{\infty n}$.

As an obvious extension of this notion, we have the inverse limit of ω-sequences of embeddings of indexed domains. Let $<D_m^{\varepsilon}m,f_m>$ be an ω-sequence of embeddings of indexed domains. By the inverse limit of this sequence, in symbols $\underleftarrow{\lim}<D_m^{\varepsilon}m,f_m>$, we mean an indexed domain $<D_\infty,\varepsilon_\infty>$ where $\varepsilon_\infty:N\to E_{D\infty}$ is given by :

$$\varepsilon_\infty(o) = f_{o\infty}(\varepsilon_o(o)) \qquad \varepsilon_\infty(1) = f_{o\infty}(\varepsilon_o(1))$$

$$\varepsilon_\infty(2) = f_{1\infty}(\varepsilon_1(o)) \qquad \varepsilon_\infty(3) = f_{o\infty}(\varepsilon_o(2))$$

$$\varepsilon_\infty(4) = f_{1\infty}(\varepsilon_1(o)) \qquad \varepsilon_\infty(5) = f_{2\infty}(\varepsilon_2(o))$$

In case $D_m^{\varepsilon_m}$ are effectively given domains, $\varprojlim\langle D_m^{\varepsilon_m}, f_m\rangle$ need not be an effectively given domain. Smyth [8] showed that if $\langle D_m^{\varepsilon_m}, f_m\rangle$ is "effective" then $\varprojlim\langle D_m^{\varepsilon_m}, f_m\rangle$ is effectively given. We observe that Smyth's effectiveness of ω-sequences is essentially equivalent to the constraint that the sequence of approximate domains can be obtained in a uniform way.

Definition 4.1 Let $\langle D_m^{\varepsilon_m}, f_m\rangle$ be an ω-sequence of effective embeddings of effectively given domains. In case there exists a recursive function $q: N \to N$ s.t. $\pi_1 . q(m)$ is a recursive index of $f_m \in \mathrm{Comp}\ ([D_m^{\varepsilon_m} \to D_{m+1}^{\varepsilon_{m+1}}])$ and $\pi_2 . q(m)$ is an acceptable index of $D_m^{\varepsilon_m}$, we say that this sequence is effective. □

From 2.6 and 4.1, we immediately have the following alternative characterization of effective sequences of effective embeddings.

Lemma 4.2 An ω-sequence $\langle D_m^{\varepsilon_m}, f_m\rangle$ of effective embeddings is effective iff there exists a recursive function $q: N \to N$ s.t. $\pi_1 . \pi_1 . q(m)$ is a directed index of $f_m \in \mathrm{Comp}\ ([D_m^{\varepsilon_m} \to D_{m+1}^{m+1}])$, $\pi_2 . \pi_1 . q(m)$ is a directed index of the adjoint g_m, and $\pi_2 . q(m)$ is an acceptable index of $D_m^{\varepsilon_m}$. ☑

Theorem 4.3 (The Inverse Limit Theorem)

Let $\langle D_m^{\varepsilon_m}, f_m\rangle$ be an effective sequence of effective embeddings of effectively given domains. Then $\langle D_\infty, \varepsilon_\infty\rangle$ is an effectively given domain. Also $f_{m\infty}: D_m \to D$ is an effective embedding from ε_m to ε_∞. Therefore $f_{m\infty} \in \mathrm{Comp}\ ([D_m^{\varepsilon_m} \to D_\infty^{\varepsilon_\infty}])$ and $g_{\infty m} \in \mathrm{Comp}\ ([D_\infty^{\varepsilon_\infty} \to D_m^{\varepsilon_m}])$. Furthermore there exists recursive functions λ_d, $\delta_d: N \to N$ s.t. $\lambda_d(m)$ and $\delta_d(m)$ are directed indices of $f_{m\infty}$ and $g_{\infty m}$ respectively. ☑

To obtain further affirmative evidence for the notion of effective isomorphism, let us examine if it is invariant under the inverse limit construction. Notice that unlike previously studied domain constructors the inverse limit constructor works not only on domains but also on embeddings among them. Thus we need the following notion to be preserved under the inverse limit construction.

Definition 4.4 Given two effective sequences $\langle D^{\varepsilon}_{m}m, f_{m}\rangle$ and $\langle D'^{\varepsilon'}_{m} f'_{m}\rangle$ of effective embeddings, we say that they are effectively isomorphic (in symbols $\langle D^{\varepsilon}_{m}m, f_{m}\rangle \overset{e}{\cong} \langle D'^{\varepsilon'}_{m}m, f'_{m}\rangle$) iff there exist recursive functions $u,v: N \to N$ s.t. $u(m)$ is a recursive index of an effective isomorphism $i_m \in$ Comp $([D^{\varepsilon}_{m}m \to D'^{\varepsilon'}_{m}m])$ and $v(m)$ is a recursive index of the adjoint $j_m \in$ Comp $([D'^{\varepsilon}_{m}m \to D^{\varepsilon}_{m}m])$; and $f'_m \cdot i_m = i_{m+1} \cdot f_m$, $g_m \cdot j_{m+1} = j_m \cdot g'_m$ where g_m and g'_m are the adjoints of f_m and f'_m respectively.

Theorem 4.5 Let $\langle D^{\varepsilon}_{m}m, f_{m}\rangle \overset{e}{\cong} \langle D'^{\varepsilon'}_{m}m, f'_{m}\rangle$ then $\underleftarrow{\lim}\langle D^{\varepsilon}_{m}m, f_{m}\rangle \overset{e}{\cong} \underleftarrow{\lim}\langle D'^{\varepsilon'}_{m}m, f'_{m}\rangle$.□

5. Effective Categories and Effective Functors

Smyth-Plotkin [9] proposed a theory of ω-categories and ω-functors which admits an initial solution to each recursive object equation $X = F(X)$ where F is an ω-functor, but without consideration of effectiveness. By showing that the category of cpo's and continuous embeddings is an ω-category where $\times, +, \to$ are ω-categories, they guaranteed an initial solution to each recursive domain equation which involves these domain constructors. We will play an effective version of this game.

Definition 5.1 An E-category is a category $\underline{K}$ together with (possibly partial) object indexing κ, and a morphism indexing $\partial(K,K'): N \to \mathrm{Hom}(K,K')$ for each pair (K,K') of objects, s.t. the composition of morphisms is effective, i.e. there is a recursive function ∂-compose s.t.:
$\partial(\kappa(i),\kappa(k))\ (\partial\text{-compose}(i,j,k,\ell,m)) = \partial(\kappa(j),\kappa(k))\ (m) . \partial(\kappa(i),\kappa(j))\ (\ell)$.

Definition 5.2 (1) $\underline{\omega}$ is the category of non-negative integers and ≤, pictorially: $0 \le 1 \le 2 \le \ldots\ldots$.

(2) An effective diagram in an E-category $(\underline{K},\kappa,\partial)$ is a functor $G: \underline{\omega} \to \underline{K}$

s.t. $G(n)=\kappa(\pi_1 . q(n))$ and $G(n \leq n+1)=\partial(G(n),G(n+1))\ (\pi_2 . q(n))$ for some recursive function $q: N \to N$.

(3) Given an effective diagram G in an E-category $(\underline{K},\kappa,\partial)$, an <u>effective cone</u> of G is a cone $\langle\lambda_n : G(n) \to K\rangle$ of G s.t. $\lambda_n=\partial(G(n),K)\ (c)n))$ for some recursive function c.

(4) An effective diagram G in an E-category $(\underline{K},\kappa,\partial)$ has an <u>effective colimit</u>, in symbols ef-colim G, iff there exists an effective cone $\langle\delta_n : G(n) \to \text{ef-colim } G\rangle$ of G s.t. for every effective cone $\langle\lambda_n : G(n) \to K\rangle$ of G, there exists a unique morphism $\sigma : \text{ef-colim } G \to K$ s.t. the following diagram commutes:

$$\begin{tikzcd} G = G(o) \arrow[r] \arrow[d, "\lambda_0"'] \arrow[dr] & G(1) \arrow[r] \arrow[dl] \arrow[d] & G(2) \arrow[r] \arrow[dll] \arrow[dl, "\delta_2"] & \cdots \\ K & \text{ef-colim } G \arrow[l, bend left, "\sigma"] & & \end{tikzcd}$$

where $G_n = G(n \leq n+1)$.

$\langle\delta_n\rangle$ will be called an <u>effective colimiting cone</u>.

(5) An E-category is an <u>effective category</u> iff every effective diagram has an effective colimit.

<u>Definition 5.3</u> Given effective categories $(\underline{K},\kappa,\partial)$ and $(\underline{K}',\kappa',\partial')$ a functor $F:\underline{K} \to K'$ is an effective functor iff it maps effectively on both objects and morphisms, and it preserves effective colimits and effective colimiting cones. More formally, iff there are recursive functions f_o and f_m s.t. $(F(\kappa(n))=\kappa'\ (f_o(n))$ and $F(\partial(\kappa(i),\kappa(j))\ (n)=\partial(\kappa(\pi_1 . \pi_1 . f_m(n)),$ $\kappa(\pi_2 , \pi_1 . f_m(n)))\ (\pi_2 . f_m(n))$ and F preserves effective colimits and effective colimiting cones.

Let $(\underline{K},\kappa,\partial)$ be an effective category and $F:K \to K$ be an effective functor. For every $K \in \underline{K}$ and $\theta : K \to FK$, define an ω-diagram $\Delta_{(F,K,\theta)} : \underline{\omega} \to \underline{K}$ by $\Delta_{(F,K,\theta)}\ (n)=F^n(K)$ and $\Delta_{(F,K,\theta)}\ (n \leq n+1)=F^n(\theta)$. Evidently $\Delta=\Delta_{(F,K,\theta)}$ is an effective diagram. Let $\langle\delta_n : \Delta(n) \to \text{ef-colim}\Delta\rangle$ be an effective colimiting cone. Then by the effectiveness of F, $\langle F(\delta_n) : F(\Delta(n)) \to F(\text{ef-colim}\Delta)\rangle$ is an effective colimiting cone of $F.\Delta$. Since $<\delta_n>_{n\geq 1}$ is an effective cone of $F.\Delta$, there exists a unique morphism $\rho: F(\text{ef-colim}\Delta) \to \text{ef-colim}\Delta$. Now define $<\lambda_n>$ by $\lambda_n = F(\delta_{n+1})$ for $n\geq 1$ and $\lambda_o = F(\delta_o).\theta$. Then $<\lambda_n>$ is an effective cone of Δ. Thus there is a unique morphism $\eta: \text{ef-colim}\Delta \to F(\text{ef-colim}\Delta)$. Therefore (ρ, η) is an isomorphism pair. In summary we have observed $F(\text{ef-colim}\Delta) \cong \text{ef-colim}\Delta$.

Given an effective category $(\underline{K}, \kappa, \partial)$ and an effective functor $F: K \to K$ an <u>F_θ-algebra</u> is a triple (α, x, γ) s.t.:

$$\begin{tikzcd} K \arrow[r, "\theta"] \arrow[d, "\alpha"'] & FK \arrow[d, "F\alpha"] \\ x & Fx \arrow[l, "\gamma"] \end{tikzcd}$$ commutes

An <u>F_θ-homomorphism</u> from an F_θ-algebra (α, x, γ) to an F_θ-algebra (α', x', γ') is an K-morphism $\pi: x \to x'$ s.t. the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\theta"] \arrow[d, "\alpha"'] \arrow[dd, bend right, "\alpha'"'] & FK \arrow[d, "F\alpha"] \arrow[dd, bend left, "F\alpha'"] \\ x \arrow[d, "\pi"'] & Fx \arrow[l, "\gamma"'] \arrow[d, "F\pi"] \\ x' & Fx' \arrow[l, "\gamma'"] \end{tikzcd}$$

It can readily be seen that the class of all F_θ-algebras and the class of all F_θ-homomorphisms form a category, which we will denote by AF_θ.

<u>Theorem 5.4</u> Let $(\underline{K}, \kappa, \partial)$ be an effective category and $F: \underline{K} \to \underline{K}$ be an effective functor. Let $\theta \in \text{Hom}(K, FK)$ and $<\delta_n>$ be the effective colimiting cone of $\Delta_{(F,K,\theta)}$. Then $(\delta_o, \text{ef-colim}\Delta, \rho)$ is an initial object in the category AF_θ, where ρ is as above. ◪

<u>Lemma 5.5</u> Given two effective categories $(\underline{K}, \kappa, \partial)$ and $(\underline{K}', \kappa', \partial')$, the product category $\underline{K} \times \underline{K}'$ together with the evidently induced objects indexing and morphism indexing is an effective category. ◪

<u>Lemma 5.6</u> Let $(\underline{K}, \kappa, \partial)$, (K', κ', ∂'), $(K'', \kappa'', \partial'')$ be effective categories. A bi-functor $F: \underline{K} \times \underline{K}' \to \underline{K}''$ is effective iff it is effective in both $\underline{K}$ and $\underline{K}'$.

Lemma 5.7 The composition of two effective functor is an effective functor. ∎

Theorem 5.8 The category of effectively given domains and effective embeddings together with $\bar{\xi}$ as an object indexing and the recursive (or directed) indexing as a morphism indexing is an effective category. The effective diagrams are effective sequences and effective colimits are the inverse limits of effective sequences. We will denote this category by ED^E without explicitly mentioning the indexings. ∎

Definition 5.9 The arrow functor $\to : ED^E \times ED^E \to ED^E$ is defined on objects by $\to(D^{\varepsilon}, D'^{\varepsilon'}) = [D^{\varepsilon} \to D'^{\varepsilon'}]$, and on morphisms by $\to(p:D_1' \to D_2', q:D_1 \to D_2) = \lambda f.q.f.p'$ where p' is the adjoint of p. We can similarly define product functor and sum functor from the domain constructors $\times$ and $+$. □

Theorem 5.10 The arrow functor, product functor, and sum functor, are effective functors. ∎

Notice that 1.6 coincides with 5.10. In fact 1.6 is a part of a proof of this theorem.

In summary we have guaranteed initial solutions, which are effectively given domains, to recursive domain equations. In fact these solutions are up to effective isomorphisms. This is very satisfactory for we have observed that we should identify two effectively given domains iff they are effectively isomorphic.

Notice that the theory of effective categories developed here is not unconditionally satisfactory. In fact the abstract notion of effective categories does not include effectiveness (or acceptability) constraints to the object indexing. There seems to be no easy way to axiomatize this effectiveness. A fundamentally different approach for defining more appropriate notion of effective categories, which does not have this problem is currently being developed by Smyth.

There are more examples of effective categories (in our sense).

Theorem 5.11 (1) Let D^{ε} be an effectively given domain. Comp(D^{ε}) together with the directed indexing as an object indexing and the evident morphism indexing is an effective category.

(2) Let D^{ε} and $D'^{\varepsilon'}$ be effectively given domains. Every computable function $f \in$ Comp ($D^{\varepsilon} \to D'^{\varepsilon'}$) restricted to Comp(D^{ε}) is an effective functor.

This indicates that Comp(D^{ε}) is more substantial than D^{ε}, and suggests a theory of effective domains (see Kanda [2]) . Furthermore we can show that the category of effectively given SFP objects (and effective embeddings), the category of effective domains, and the category of effective SFP are effective categories where $\times$, $+$, are effective functors. Thus we can solve recursive domain equations within these categories up to effective isomorphisms. Details of these results will appear elsewhere.

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