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179

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How to
Multiply Matrices Faster



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Some Notation and Abbreviations

Notation	Meaning, Comments	Defined or First Used In Section(s) (see also Index)
A	algorithm	19,23
ar(A);ar(P)	number of arithmetical operations involved in A; required in order to solve a problem P	19,23
as(A)	number of additions/subtractions involved in A	32,33
AAPR	accumulation of the accelerating power via recursion	6
$b_Y(X,Y)$	bilinear form in X,Y	2
BA(n)	bilinear algorithm for nXn MM	2,22,23
BA(n, λ)	bilinear λ -algorithm for nXn MM	23
BBM	Boolean MM	18
bs(A)	bit-space used by A	23
bt(A)	bit-time used by A	23
bt(s),bt(*,s),bt(\pm ,s)		18
bs(P),bt(P)	bit-time and bit-space of a computational problem P	23
C	the field of complex numbers	2
$c\rho$	commutative rank	32
$c\rho_\lambda$	commutative λ -rank	33
$C(g,h)$	$g!/(h!(g-h)!)$	8,9
cond	condition	25
D	domain of definition of problem or algorithm	Part 2. (Summary); 23
d	degree of λ -algorithm	6
d	shortest distance	18 only

VIII

$\det(W)$	determinant of a matrix W	19
$\text{Det}(n)$	the problem of the evaluation of the determinant of an $n \times n$ matrix	Part 2 (Summary); 19
DFT(n)	discrete Fourier transform,	38,39
E	extension of a ring (field)	5
$E, E(n), e(n), E(A, D, h),$ $E(Z(V), D, h)$	error bounds	Part 2 (Summary); 23-30
F	ring, field	2
$F[\lambda]$	ring of polynomials over F	6
$f(i, j, q), f'(j, k, q),$	constant coefficients (from F)	2
$f''(k, i, q),$ $f(\alpha, q), f'(\beta, q),$ $f''(\gamma, q)$	of bilinear algorithms	
$f(i, j, q, \lambda),$ $f'(j, k, q, \lambda),$ $f''(k, i, q, \lambda),$ $f(\alpha, q, \lambda),$ $f'(\beta, q, \lambda),$ $f''(\gamma, q, \lambda)$	coefficients (from $F[\lambda]$) of bilinear λ -algorithms	6
$f, f', f'', \tilde{f}, \tilde{f}', \tilde{f}''$		23
FFT	fast Fourier transform	Intr., 2, 38
$h(s)$	2^{1-s}	23
u^H, W^H	complex conjugate of number u , conjugate transpose of matrix W	19
I (also I_n)	identity matrix (of size $n \times n$)	19
$l_h(l_2, l_\infty)$	l_h -norm of a matrix or of a vector	24
$\log u$	logarithm to the base 2 of u	1

IX

L_q, L'_q		2
L''_q		10
M	rank of algorithm, λ -rank of λ -algorithm	2,4,6
MA,MS	matrix addition, subtraction	20
MI	matrix inversion	Part 2 (Summary); 19
MM	matrix multiplication	Intr., 1
(m,n,p) ; also $m \times n \times p$ MM	the problem of $m \times n$ by $n \times p$ MM	2
$O(g(s)), o(g(s))$	see Notation 18.1	Intr., 1, 18
O, O_n	null matrix	19, 20
PM	polynomial multiplication	2
Q	field of rational numbers	2
Q	unitary matrix (a QR-factor)	20
$Q(s)$	computed approximation to Q	26-30
QR, \tilde{QR}, QR^*		20
R	upper triangular matrix (a QR-factor)	20
$R(s)$	computed approximation to R	26-30
R	field of real numbers	2
\underline{R}	set of vectors in the proof of Theorem 7.2	9 only
SLE	the problem of solving a system of linear equations	Part 2 (Summary); 19
$sm(A)$	number of scalar multiplications in A	32, 33
T	trilinear form	10
TA	trilinear aggregating	Intr., 3, 11

$\text{Tr}(W)$	trace of a matrix W	10
TMI	triangular matrix inversion	21
t	tensor	2, 10
U, V, W, X, Y, Z	matrices	1, 2, 4, 6, 10
\mathbb{Z}	ring of integers	2, 5
$\mathbb{Z}(\mathbb{V})$	ring of integers modulo \mathbb{V}	2, 5
$Z(V)$	output matrix	24-30
$\delta(i, j)$	$\delta(i, j) = 0, i \neq j; \delta(i, i) = 1$	2
Δ, Δ'	error value, error matrix	23-30
λ	see λ -algorithms	4, 6
ρ, ρ_F	rank, rank over a ring F	2
$\rho(m, n, p)$	rank of $m \times n \times p$ MM	2
br	λ -rank	36
ω	exponent of MM	2
ω_F	exponent of MM over a ring F of constants	2
Σ, Π	symbols of sums, products	
\hat{Z}	diagonal matrix	20 only
$[u], [u]$	see Notation 18.1	18
\oplus	direct sum of disjoint problems	8
\odot	direct sum of identical disjoint problems	2, 5, 8
\otimes	(tensor) product of bilinear problems	2, 5, 8
\boxtimes	direct (Kronecker) product of vectors, matrices, tensors, and of linear, bilinear, or polylinear forms	10, 14, 16

$\#$	generalized MM	18 only
$ \underline{v} , W $	norms of vector \underline{v} , matrix W	24
$t \leftarrow t'$	mapping (algorithm)	5,8
$ S $	cardinality of a set S	
$ u $	absolute value (modulus) of a number u	
C, \subseteq	inclusion of one set into another	5
\in	inclusion of an element into a set	9
\cup	union of sets	5
■	end of clause, of proof, of algorithm	