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Systems of Reductions

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Introduction

Systems of reductions (or rewrite rules as they are often called) enjoy a growing popularity in theoretical computer science. They have also become a useful tool in computational algebra; these areas are anyway not very well separated and have much in common. An important branch of this common background is "equational logic". This is, strictly speaking, the fragment of predicate logic, where equality is the only predicate. In practice, equality logic is concerned with a class of problems which are more restricted as well as more general. Typical are implications of the form

$$\Sigma \Rightarrow P.$$

Here Σ is a universally quantified set of equations; P, however, need not to be an equation and not even a formula of first order predicate logic.

An example of the latter type of problems is:

"Is each algebra in which Σ holds finite?"

or equivalently

"Is the free algebra defined by Σ finite?"

The answer one expects in equality logic to such questions is an algorithmic one. That means, one does not consider single questions but whole classes of questions. These may arise in different ways:

1) Σ is fixed and P varies.
   An example is the word problem, where Σ defines the algebra and P varies over all equations $s = t$.

2) Σ varies and P is fixed.
   An example is the infinity problem for a class of algebras (which are defined by the axiom system Σ under consideration).

3) Σ and P both vary.
An example is the uniform word problem for a class of algebras.

Algorithmic decision problems arise in computer science mainly in the context of abstract data types and automated theorem proving. The algorithmic tradition in mathematics, in particular in algebra, is much older. A hundred years ago, at the time of Kronecker, solutions to algebraic problems were more or less automatically expected to be computable. In the first half of the twentieth century abstract and non-constructive methods became more dominant. There was still an enormous interest in principal aspects of computability and decision problems, but constructive methods did not play such a big role in the every-day-work of mathematicians. This has radically changed under the influence of the development of computer science. Today constructive methods are not only regarded as being useful in certain applied situations; their main purpose is to provide relevant structural and combinatorial insights.

Among the defining properties of equality reflexivity is the only trivial one. The basic idea of a reduction is to give up symmetry and allow only replacements of equals by equals in one direction. This idea is as old as the word problem for groups which was considered by M. Dehn and A.Thue in 1912-14 and which was one of the first decision problems to be formulated. In fact, Dehn's algorithm is the application of a specific system of reductions and its properties were studied for more than fifty years. Giving up symmetry looses, of course, some of the power of equality. The research on Dehn's algorithm was concerned with the question which aspects of full equality are preserved by its one-sided use.The idea of regaining part of the power provided by symmetry by systematically adding new reductions came up much later. This leads to concepts of systems of reductions like the Church-Rosser and the weak Church-Rosser property (which are known under various names) and the finite termination property; in connection with the latter there is growing interest in partial and total well-founded orderings on the terms.

The most useful property is certainly completeness, it ensures that each term t reduces to a uniquely determined irreducible term $t^{\#}$; $t^{\#}$ is the canonical form of t. The aim of the completion algorithm is to enlarge a given system of reductions in order to obtain a complete one. A complete system (if it exists) can be regarded as a link between the finite system of equations and the algebra defined by these equations which is a set-theoretic, often infinite, object.

There are two main lines of research here. On the one hand one studies the completion algorithm and searches for criteria which ensure its termination. As the completion algorithm in many (one is tempted to say "most") cases fails to terminate this leads to the investigation of infinite systems. In many cases these can be finitely described and are as useful as finite systems.

The other type of investigations is concerned with the use of complete systems. A complete system certainly provides an answer to the word problem but unravels much more of the structure of the algebra under investigation. This turns out to be most apparent in the case of groups.

Most of the material in these notes was obtained in the years after 1978 at the Technische Hochschule Aachen; it is partially contained in the dissertations of Hans Bücken, Patrick Horster and Susanne Kemmerich; Patrick Horster also wrote section IV.2.Very useful for computer experiments was an implementation of the completion algorithm for groups and semigroups as well as the forward-backward algorithm. This implementation was done by Klaus Dittrich in Pascal on a Cyber 175. The last part of these notes was written by Friedrich Otto; the material is part of his Habilitationsschrift at the University of Kaiserslautern.

The main interest of the authors is in general principles. Most concrete applications are in group and semigroup theory, however. There are several reasons for this. One is that these are familiar structures and one has a better feeling for difficulties and importance of results than in general universal algebras. Another reason is that in these areas computational methods are well established. This gives possibilities for interesting connections and comparisons and is also useful for a fruitful competition.

In order to make the volume somewhat self-contained much general material is included. The idea was to provide the reader with an at least almost complete introduction. Here completeness is meant in the sense that suffices for an understanding of the rest of the material. It is natural that many aspects had to be left out.

The authors have also been very reluctant with historical remarks. On the one hand, many results seem to be obtained independently by different authors. On the other hand, the situation was not so clear to us that we dared to make statements on priority questions.

There are several people whom the authors are indebted for useful help and discussions over the years. Among the former students of Aachen we would name Hans Bücken, Klaus Dittrich, Petra Zimmermann and Tom Beske. One of the authors wants to mention Dallas Lankford in particular; he was also early influenced by Woody Bledsoe. Later on useful discussions took place with Mike Ballantyne, Bruno Buchberger, Richard Göbel, Deepak Kapur, Wolfgang Küchlin and J. Neubüser.

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