# Quantum Multi-Prover Interactive Proof Systems and Quantum Characterizations of NEXP 

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#### Abstract

This paper gives the first formal treatment of a quantum analogue of multi-prover interactive proof systems. In quantum multi-prover interactive proof systems there can be two natural situations: one is with prior entanglement among provers, and the other does not allow prior entanglement among provers. This paper focuses on the latter situation and proves that, if provers do not share any prior entanglement each other, the class of languages that have quantum multi-prover interactive proof systems is equal to NEXP. It implies that the quantum multi-prover interactive proof systems without prior entanglement have no gain to the classical ones. This result can be extended to the following statement of the cases with prior entanglement: if a language $L$ has a quantum multi-prover interactive proof system allowing at most polynomially many prior entangled qubits among provers, $L$ is necessarily in NEXP. Another interesting result shown in this paper is that, in the case the prover does not have his private qubits, the class of languages that have single-prover quantum interactive proof systems is also equal to NEXP. Our results are also of importance in the sense of giving exact correspondances between quantum and classical complexity classes, because there have been known only a few results giving such correspondances.


## 1 Introduction

### 1.1 Motivation

After Deutsch [12] gave the first formal treatment of quantum computation, a number of papers have provided evidence that quantum computation has much more power than classical computation for solving certain computational tasks, including notable Shor's integer factoring algorithm [31. Watrous [36] showed that it might be also the case for single-prover interactive proof systems, by constructing a constant-round quantum interactive protocol for a PSPACE-complete language, which is impossible for classical interactive proof systems unless the polynomial-time hierarchy collapses to AM [4, 20. A natural question to ask is how strong a quantum analogue of multi-prover interactive proof systems is.

For the quantum multi-prover ones one can consider two natural models: one is with provers sharing prior entanglement with each other, and the other is without prior entanglement among provers. This corresponds to the fact that there have been considered two types of models of classical multi-prover interactive proof systems. In the original model by Ben-Or, Goldwasser, Kilian, and Wigderson [\$], provers are allowed to share randomness with each other. Later a number of papers [18, 5, 19, 13, 25, 14] considered the model without shared randomness among provers, because the

[^0]computational power of the model does not change unless zero-knowledge protocols are taken into account. However, once zero-knowledge protocols are taken into account, these two types of the classical multi-prover interactive proof systems make a significant difference in computational power [6]. This gives a good reason that we should consider quantum multi-prover interactive proof systems without prior entanglement among provers (a counterpart of the model without shared randomness) as well as the one allowing prior entanglement (a counterpart of the model with shared randomness). The former cases are relatively easier to treat than the latter ones, but the authors believe that it is a very important step for investigating quantum multi-prover interactive proof systems to analyze the power of the former model. Therefore this paper mainly focuses on the former model and proves that, if provers do not share any prior entanglement each other, the class of languages that have quantum multi-prover interactive proof systems is equal to NEXP. It implies that, in contrast to the singleprover case, the quantum multi-prover interactive proof systems without prior entanglement have no gain to the classical ones.

### 1.2 Related Works

Interactive proof systems were introduced by Babai [0] and Goldwasser, Micali, and Rackoff [19]. An interactive proof system consists of an interaction between a computationally unbounded prover and a polynomial-time probabilistic verifier. The prover attempts to convince the verifier that a given input string satisfies some property, while the verifier tries to verify the validity of the assertion of the prover. It is well-known that the class of languages that have interactive proof systems, denoted by IP, is equal to PSPACE, shown by Shamir 30 based on the work of Lund, Fortnow, Karloff, and Nisan [26], and on the result of Papadimitriou [29].

Quantum interactive proof systems were introduced by Watrous [36] in terms of quantum circuits. He showed that every PSPACE language has a quantum interactive protocol, with exponentially small one-sided error, in which the prover and the verifier exchange only three messages. A consecutive work of Kitaev and Watrous [23] showed that any quantum interactive protocol, even with two-sided bounded error, can be parallelized to a three-message quantum protocol with exponentially small one-sided error. They also showed that the class of languages that have quantum interactive proof systems is necessarily contained in deterministic exponential time (EXP).

A multi-prover interactive proof system, introduced by Ben-Or, Goldwasser, Kilian, and Wigderson [8], is an extension of the (single-prover) interactive proof system in which a verifier communicates with not only one but multiple provers, while provers cannot communicate with each other prover and cannot know messages exchanged between the verifier and other provers. A language $L$ is said to have a multi-prover interactive proof system if, for some $k$ denoting the number of provers, there exists a verifier $V$ such that (i) in case the input is in $L$, there exist provers $P_{1}, \ldots, P_{k}$ that can convince $V$ with probability 1 , and (ii) in case the input is not in $L$, any set of provers $P_{1}^{\prime}, \ldots, P_{k}^{\prime}$ cannot convince $V$ with probability more than $1 / 2$. Babai, Fortnow, and Lund [5], combining the result by Fortnow, Rompel, and Sipser [18], showed that the class of languages that have multi-prover interactive proof systems, denoted by MIP, is equal to non-deterministic exponential time (NEXP). A sequence of papers by Cai, Condon, and Lipton [10], Feige [13], and Lapidot and Shamir [25] led to a result of Feige and Lovász [14] that any language in NEXP has a two-prover interactive proof system with just one round (i.e. two messages) of communication (meaning that the verifier sends one question to each of the provers in parallel, then receives their responses), with exponentially small one-sided error.

### 1.3 Main Results

In this paper we first define quantum multi-prover interactive proof systems by naturally extending the quantum single-prover model, that is, our definition properly contains the quantum single-prover model discussed in [36, 23] in the sense that ours is just the same as their single-prover model if we fix the number of provers to one. As mentioned before, this paper mainly deals with the cases of no prior
entanglement among provers. However our model can be easily extended to the cases allowing prior entanglement by only setting the initial state to be entangled. Hereafter we simply call our model of no prior entanglement a quantum multi-prover interactive proof system. Then we show that the class of languages that have quantum multi-prover interactive proof systems, denoted by QMIP, is equal to NEXP.

To prove QMIP $\subseteq$ NEXP, a key idea is to bound the number of private qubits of provers without diminishing the computational power of them. Suppose that each prover has only polynomially many private qubits during the protocol. Then the total number of qubits of the quantum multi-prover interactive proof system is polynomially bounded, and we can show that it can be simulated in classical non-deterministic exponential time. Now the point is whether the space-bounded quantum provers (i.e. the provers can apply any unitary transformations on their spaces, but the number of qubits in their spaces is bounded polynomial with respect to the input length) are as powerful as the space-unbounded quantum provers or not. We show that, even with only polynomially many private qubits, each prover can do everything that he could with as many qubits as he pleases, in the sense that the verifier cannot distinguish the difference at all. For this, we also prove one fundamental property on quantum information theory utilizing the entanglement measure introduced by Nielsen [27. Apart from quantum interactive proof systems, this property itself is also of interest and worth while stating.

The other side of inclusion, NEXP $\subseteq$ QMIP, is rather easy to show, because in our model of quantum multi-prover interactive proof systems the quantum verifier can successfully simulate any classical multi-prover protocol, in particular, the two-prover one-round classical protocol for NEXP with exponentially small one-sided error. Recall that our model does not allow any prior entanglement among private qubits of the quantum provers, for it is reported by Cleve [1] that a pair of provers with entanglement can in some sense cheat a classical verifier.

Since our proof of QMIP $\subseteq$ NEXP still holds even if we allow protocols with two-sided bounded error, our result implies an important property of quantum multi-prover interactive proof systems, that is, if a language $L$ has a quantum multi-prover interactive proof system even with two-sided bounded error, then $L$ has a two-message quantum two-prover interactive proof system with exponentially small one-sided error.

Another interesting result shown in this paper is for a special case of single-prover quantum interactive proof systems, in which the prover does not have his private qubits. We call this model a quantum oracle circuit, since it can be regarded as a quantum counterpart of a probabilistic oracle machine [18, 16, 5] in the sense that there is no private space for the prover during the protocol. It is proved that the class of languages accepted by quantum oracle circuits, denoted by QOC , is also equal to NEXP, or in other words, in the case the prover does not have his private qubits, the class of languages that have single-prover quantum interactive proof systems is equal to NEXP. In this paper, instead of proving QMIP = NEXP and QOC = NEXP independently, we combine these two proofs, that is, we first show $\mathrm{QMIP} \subseteq$ QOC, then $\mathrm{QOC} \subseteq$ NEXP, and finally NEXP $\subseteq$ QMIP.

Although this paper mainly treats the model without prior entanglement, our results can be extended to the following statement of the cases with prior entanglement: if a language $L$ has a quantum multi-prover interactive proof system allowing at most polynomially many prior entangled qubits among provers, $L$ is necessarily in NEXP. Thus, even if provers are allowed to share polynomially many prior entangled qubits, quantum multi-prover interactive proof systems cannot be stronger than the classical counterpart.

Apart from the theory of interactive proof systems, our results are of importance in the sense that they give exact correspondances between quantum and classical complexity classes, for there have been known only a few results giving such correspondances, including NQP $=$ co- $\mathrm{C}_{=} \mathrm{P}$ result by [17, 15, 37], and the characterization of PSPACE by BQSPACE(poly), or "quantum polynomial space", by [34, 35].

The remainder of this paper is organized as follows. In Section 2 we briefly review basic notations
and definitions in quantum computation and quantum information theory. In Section 3 we state two key properties on quantum information theory, which play important roles in our proofs. In Section 4 we give a formal definition of quantum multi-prover interactive proof systems and quantum oracle circuits. In Section 5 we show our main result, $\mathrm{QMIP}=\mathrm{QOC}=$ NEXP. In Section 6 we mention how our results relate to the cases with prior entanglement. Finally we conclude with Section 7, which summarizes our results and mentions a number of open problems related to our work.

## 2 Quantum Fundamentals

Here we briefly review basic notations and definitions in quantum computation and quantum information theory. Detailed descriptions are, for instance, in 21, 28.

A pure state is described by a unit vector in some Hilbert space. In particular, an $n$-dimensional pure state is a unit vector $|\psi\rangle$ in $\mathbb{C}^{n}$. Let $\left\{\left|e_{1}\right\rangle, \ldots,\left|e_{n}\right\rangle\right\}$ be an orthonormal basis for $\mathbb{C}^{n}$. Then any pure state in $\mathbb{C}^{n}$ can be described as $\sum_{i=1}^{n} \alpha_{i}\left|e_{i}\right\rangle$ for some $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{C}, \sum_{i=1}^{n}\left|\alpha_{i}\right|^{2}=1$.

A mixed state is a classical probability distribution $\left(p_{i},\left|\psi_{i}\right\rangle\right), 0 \leq p_{i} \leq 1, \sum_{i} p_{i}=1$ over pure states $\left|\psi_{i}\right\rangle$. This can be interpreted as being in the pure state $\left|\psi_{i}\right\rangle$ with probability $p_{i}$. A mixed state is often described in the form of a density matrix $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$. Any density matrix is positive semidefinite and has trace 1.

If a unitary transformation $U$ is applied to a state $|\psi\rangle$, the state becomes $U|\psi\rangle$, or in the form of density matrices, a state $\rho$ changes to $U \rho U^{\dagger}$ after $U$ is applied.

One of the important operations to density matrices is the trace-out operation. Given a density matrix $\rho$ over $\mathcal{H} \otimes \mathcal{K}$, the state after tracing out $\mathcal{K}$ is a density matrix over $\mathcal{H}$ described by

$$
\operatorname{tr}_{\mathcal{K}} \rho=\sum_{i=1}^{n}\left(I_{\mathcal{H}} \otimes\left\langle e_{i}\right|\right) \rho\left(I_{\mathcal{H}} \otimes\left|e_{i}\right\rangle\right)
$$

for any orthonormal basis $\left\{\left|e_{1}\right\rangle, \ldots,\left|e_{n}\right\rangle\right\}$ of $\mathcal{K}$, where $n$ is the dimension of $\mathcal{K}$ and $I_{\mathcal{H}}$ is the identity operator over $\mathcal{H}$. To perform this operation on some part of a quantum system gives a partial view of the quantum system with respect to the remaining part.

One of the important concepts in quantum physics is a measurement. Any collection of linear operators $\left\{A_{1}, \ldots, A_{k}\right\}$ satisfying $\sum_{i=1}^{k} A_{i}^{\dagger} A_{i}=I$ defines a measurement. If a system is in a pure state $|\psi\rangle$, such a measurement results in $i$ with probability $\| A_{i}|\psi\rangle \|^{2}$, and the state becomes $A_{i}|\psi\rangle / \| A_{i}|\psi\rangle \|$. If a system is in a mixed state with density matrix $\rho$, the result $i$ is observed with probability $\operatorname{tr}\left(A_{i} \rho A_{i}^{\dagger}\right)$, and the state after the measurement is with density matrix $A_{i} \rho A_{i}^{\dagger} / \operatorname{tr}\left(A_{i} \rho A_{i}^{\dagger}\right)$. A special class of measurements are projection or von Neumann measurements where $\left\{A_{1}, \ldots, A_{k}\right\}$ is a collection of orthonormal projections. In this scheme, an observable is a decomposition of $\mathcal{H}$ into orthogonal subspaces $\mathcal{H}_{1}, \ldots, \mathcal{H}_{k}$, that is, $\mathcal{H}=\mathcal{H}_{1} \oplus \cdots \oplus \mathcal{H}_{k}$. It is important to note that two mixed states having same density matrix cannot be distinguished at all by any measurement.

For any linear operator $A$ over $\mathcal{H}$, the $l_{2}$-norm of $A$ is defined by

$$
\|A\|=\sup _{|\psi\rangle \in \mathcal{H} \backslash\{0\}} \frac{\| A|\psi\rangle \|}{\||\psi\rangle \|} .
$$

## 3 Useful Properties on Quantum Fundamentals

Here we state two useful properties on quantum information theory, which play key roles in the proof of Lemma 8 in Section 4.

The first property we state is a well-known property, while the second one is a key property first shown in this paper. Although the proof may not be so difficult for those who are familiar with
quantum information theory, the authors believe it is worth while stating the second property itself. Furthermore, an interesting and important point in this paper is how to combine and apply these two to the theory of quantum multi-prover interactive proof systems (see the proof of Lemma 8 in Section 4).

Theorem 1 ([33, 22]) Let $|\phi\rangle,|\psi\rangle \in \mathcal{H}_{1} \otimes \mathcal{H}_{2}$ satisfy $\operatorname{tr}_{\mathcal{H}_{2}}|\phi\rangle\langle\phi|=\operatorname{tr}_{\mathcal{H}_{2}}|\psi\rangle\langle\psi|$. Then there is a unitary transformation $U$ over $\mathcal{H}_{2}$ such that $\left(I_{\mathcal{H}_{1}} \otimes U\right)|\phi\rangle=|\psi\rangle$, where $I_{\mathcal{H}_{1}}$ is the identity operator over $\mathcal{H}_{1}$.

Theorem 2 Fix a state $|\phi\rangle$ in $\mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \mathcal{H}_{3}$ and a unitary transformation $U$ over $\mathcal{H}_{2} \otimes \mathcal{H}_{3}$ arbitrarily, and let $|\psi\rangle$ denote $\left(I_{\mathcal{H}_{1}} \otimes U\right)|\phi\rangle$. Then, for any Hilbert space $\mathcal{H}_{3}^{\prime}$ of $\operatorname{dim}\left(\mathcal{H}_{3}^{\prime}\right) \leq \operatorname{dim}\left(\mathcal{H}_{3}\right)$ such that there is a state $\left|\phi^{\prime}\right\rangle$ in $\mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \mathcal{H}_{3}^{\prime}$ satisfying $\operatorname{tr}_{\mathcal{H}_{3}^{\prime}}\left|\phi^{\prime}\right\rangle\left\langle\phi^{\prime}\right|=\operatorname{tr}_{\mathcal{H}_{3}}|\phi\rangle\langle\phi|$, there exist a Hilbert space $\mathcal{H}_{3}^{\prime \prime}$ of $\operatorname{dim}\left(\mathcal{H}_{3}^{\prime \prime}\right)=\left(\operatorname{dim}\left(\mathcal{H}_{2}\right)\right)^{2} \cdot \operatorname{dim}\left(\mathcal{H}_{3}^{\prime}\right)$ and a state $\left|\psi^{\prime}\right\rangle$ in $\mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \mathcal{H}_{3}^{\prime \prime}$ such that $\operatorname{tr}_{\mathcal{H}_{3}^{\prime \prime}}\left|\psi^{\prime}\right\rangle\left\langle\psi^{\prime}\right|=\operatorname{tr}_{\mathcal{H}_{3}}|\psi\rangle\langle\psi|$.

For the proof of Theorem 2, we use the entanglement measure introduced by Nielsen [27]. Let us decompose a vector $|\xi\rangle \in \mathcal{H}_{1} \otimes \mathcal{H}_{2}$ into

$$
\begin{equation*}
|\xi\rangle=\sum_{i, j} \alpha_{i j}\left|e_{i}^{1}\right\rangle \otimes\left|e_{j}^{2}\right\rangle, \tag{1}
\end{equation*}
$$

where $\left\{\left|e_{i}^{1}\right\rangle\right\}$ and $\left\{\left|e_{i}^{2}\right\rangle\right\}$ are orthonormal bases of $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$, respectively. Then the entanglement measure $\operatorname{ent}_{2}\left(|\xi\rangle, \mathcal{H}_{1}, \mathcal{H}_{2}\right)$ is defined by the minimum number of non-zero terms in the right hand side of ( $\mathbb{1})$, where the minimum is taken over all the possible choices of the bases $\left\{\left|e_{i}^{1}\right\rangle\right\},\left\{\left|e_{i}^{2}\right\rangle\right\}$. The decomposition with the minimum number of non-zero terms is given by the Schmidt decomposition (33),

$$
|\xi\rangle=\sum_{i} \beta_{i}\left|e_{i}^{1}\right\rangle \otimes\left|e_{i}^{2}\right\rangle,
$$

where each $\left|e_{i}^{1}\right\rangle$ and $\left|e_{i}^{2}\right\rangle$ are normalized eigenvectors of $\operatorname{tr}_{\mathcal{H}_{1}}|\xi\rangle\langle\xi|$ and $\operatorname{tr}_{\mathcal{H}_{2}}|\xi\rangle\langle\xi|$, respectively. Therefore, the entanglement measure $\operatorname{ent}_{2}\left(|\xi\rangle, \mathcal{H}_{1}, \mathcal{H}_{2}\right)$ is nothing but the minimum dimension of the Hilbert space $\mathcal{H}_{2}^{\prime}$ such that there is a vector $\left|\xi^{\prime}\right\rangle \in \mathcal{H}_{1} \otimes \mathcal{H}_{2}^{\prime}$ which satisfies $\operatorname{tr}_{\mathcal{H}_{2}}|\xi\rangle\langle\xi|=\operatorname{tr}_{\mathcal{H}_{2}^{\prime}}\left|\xi^{\prime}\right\rangle\left\langle\xi^{\prime}\right|$.

We extend the definition of ent ${ }_{2}$ to three party cases. For a vector $|\zeta\rangle \in \mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \mathcal{H}_{3}$ we define the three party entanglement measure ent $_{3}\left(|\zeta\rangle, \mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}\right)$ by the minimum number of non-zero terms in the decomposition

$$
|\zeta\rangle=\sum_{i, j, k} \gamma_{i j k}\left|e_{i}^{1}\right\rangle \otimes\left|e_{j}^{2}\right\rangle \otimes\left|e_{k}^{3}\right\rangle,
$$

where $\left\{\left|e_{i}^{j}\right\rangle\right\}$ denotes an orthonormal basis of the space $\in \mathcal{H}_{j}$ for each $j=1,2,3$.
Proof of Theorem 因. Since $\operatorname{ent}_{2}\left(|\psi\rangle, \mathcal{H}_{1} \otimes \mathcal{H}_{2}, \mathcal{H}_{3}\right)$ gives the minimum dimension of $\mathcal{H}_{3}^{\prime \prime}$ such that there is a state $\left|\psi^{\prime}\right\rangle \in \mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \mathcal{H}_{3}^{\prime \prime}$ satisfying $\operatorname{tr}_{\mathcal{H}_{3}^{\prime \prime}}\left|\psi^{\prime}\right\rangle\left\langle\psi^{\prime}\right|=\operatorname{tr}_{\mathcal{H}_{3}}|\psi\rangle\langle\psi|$, it is sufficient to show that $\operatorname{ent}_{2}\left(|\psi\rangle, \mathcal{H}_{1} \otimes \mathcal{H}_{2}, \mathcal{H}_{3}\right) \leq \operatorname{dim}\left(\mathcal{H}_{3}^{\prime}\right) \cdot\left(\operatorname{dim}\left(\mathcal{H}_{2}\right)\right)^{2}$. This can be proved as follows:

$$
\begin{aligned}
\operatorname{ent}_{2}\left(|\psi\rangle, \mathcal{H}_{1} \otimes \mathcal{H}_{2}, \mathcal{H}_{3}\right) & \leq \operatorname{ent}_{3}\left(|\psi\rangle, \mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}\right) \\
& \leq \operatorname{ent}_{3}\left(|\phi\rangle, \mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}\right) \cdot \operatorname{dim}\left(\mathcal{H}_{2}\right) \\
& \leq \operatorname{ent}_{2}\left(|\phi\rangle, \mathcal{H}_{1} \otimes \mathcal{H}_{2}, \mathcal{H}_{3}\right) \cdot\left(\operatorname{dim}\left(\mathcal{H}_{2}\right)\right)^{2} \\
& \leq \operatorname{dim}\left(\mathcal{H}_{3}^{\prime}\right) \cdot\left(\operatorname{dim}\left(\mathcal{H}_{2}\right)\right)^{2} .
\end{aligned}
$$

The first inequality directly comes from the definition of the entanglement measure. To prove the second and third inequalities, let $|\phi\rangle=\sum_{i, j, k} \gamma_{i j k}\left|e_{i}^{1}\right\rangle \otimes\left|e_{j}^{2}\right\rangle \otimes\left|e_{k}^{3}\right\rangle$ be the decomposition of $|\phi\rangle$ with respect to the orthonormal bases $\left\{\left|e_{i}^{1}\right\rangle\right\},\left\{\left|e_{i}^{2}\right\rangle\right\},\left\{\left|e_{i}^{3}\right\rangle\right\}$ of $\mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}$, and let $|\phi\rangle=\sum_{i} \beta_{i}\left|f_{i}^{1,2}\right\rangle \otimes\left|f_{i}^{3}\right\rangle$ be that of $|\phi\rangle$ with respect to the orthonormal bases $\left\{\left|f_{i}^{1,2}\right\rangle\right\},\left\{\left|f_{i}^{3}\right\rangle\right\}$ of $\mathcal{H}_{1} \otimes \mathcal{H}_{2}, \mathcal{H}_{3}$. The second and third inequalities are the consequences of the equality

$$
|\psi\rangle=\sum_{i, j, k} \gamma_{i j k}\left|e_{i}^{1}\right\rangle \otimes U\left(\left|e_{j}^{2}\right\rangle \otimes\left|e_{k}^{3}\right\rangle\right)=\sum_{i, j, k} \gamma_{i j k}\left|e_{i}^{1}\right\rangle \otimes\left(\sum_{l=1}^{\operatorname{dim}\left(\mathcal{H}_{2}\right)} \beta_{j k l}^{\prime}\left|e_{j k l}^{2}\right\rangle \otimes\left|e_{j k l}^{3}\right\rangle\right)
$$

and the equality

$$
|\phi\rangle=\sum_{i} \beta_{i}\left|f_{i}^{1,2}\right\rangle \otimes\left|f_{i}^{3}\right\rangle=\sum_{i} \beta_{i}\left(\sum_{j=1}^{\operatorname{dim}\left(\mathcal{H}_{2}\right)} \beta_{i j}^{\prime \prime}\left|f_{i j}^{1}\right\rangle \otimes\left|f_{i j}^{2}\right\rangle\right) \otimes\left|f_{j}^{3}\right\rangle,
$$

respectively, where $\sum_{l=1}^{\operatorname{dim}\left(\mathcal{H}_{2}\right)} \beta_{j k l}^{\prime}\left|e_{j k l}^{2}\right\rangle \otimes\left|e_{j k l}^{3}\right\rangle$ and $\sum_{j=1}^{\operatorname{dim}\left(\mathcal{H}_{2}\right)} \beta_{i j}^{\prime \prime}\left|f_{i j}^{1}\right\rangle \otimes\left|f_{i j}^{2}\right\rangle$ are the Schmidt decompositions of $U\left(\left|e_{j}^{2}\right\rangle \otimes\left|e_{k}^{3}\right\rangle\right)$ and $\left|f_{i}^{1,2}\right\rangle$, respectively. The fourth inequality is from the definition of the entanglement measure, which ensures $\operatorname{ent}_{2}\left(|\phi\rangle, \mathcal{H}_{1} \otimes \mathcal{H}_{2}, \mathcal{H}_{3}\right) \leq \operatorname{dim}\left(\mathcal{H}_{3}^{\prime}\right)$.

## 4 Definitions

### 4.1 Polynomial-time Uniformly Generated Families of Quantum Circuits

Our model of quantum multi-prover interactive proof systems is a natural extension of that of quantum single-prover ones discussed in [36, 23], which is defined in terms of quantum circuits. Before proceeding to the definition of quantum multi-prover interactive proof systems, we review the concept of polynomial-time uniformly generated families of quantum circuits according to [23].

A family $\left\{Q_{x}\right\}$ of quantum circuits is said to be polynomial-time uniformly generated if there exists a deterministic procedure that, on each input $x$, outputs a description of $Q_{x}$ and runs in time polynomial in $n=|x|$. For simplicity, we assume all input strings are over the alphabet $\Sigma=\{0,1\}$. It is assumed that the circuits in such a family are composed of gates in some reasonable, universal, finite set of quantum gates such as the Shor basis [32, 9]: Hadamard gates, $\sqrt{\sigma_{z}}$ gates, and Toffoli gates. Furthermore, it is assumed that the number of gates in any circuit is not more than the length of the description of that circuit, therefore $Q_{x}$ must have size polynomial in $n$. For convenience, we may identify a circuit $Q_{x}$ with the unitary operator it induces.

As Kitaev and Watrous [23] noticed, to permit non-unitary quantum circuits, in particular, to permit measurements at any timing in the computation does not change the computational power of the model. See [1] for a detailed description of the equivalence of the unitary and non-unitary quantum circuit models.

### 4.2 Quantum Multi-Prover Interactive Proof Systems

Here we give the definition of quantum multi-prover interactive proof systems which is a natural extension of quantum single-prover ones defined by Watrous [36]. In fact, the model of quantum single-prover interactive proof systems discussed in [36, 23] is a special case of our quantum multiprover model with the restriction of the number of provers to one. Although the model to be defined here is for the cases of no prior entanglement among provers, it can be easily extended to the cases with prior entanglement by setting the initial state to be entangled.

Similar to the quantum single-prover case, we define quantum multi-prover interactive proof systems in terms of quantum circuits.

Let $k$ be the number of provers. For each input $x \in \Sigma^{*}$ of length $n=|x|$, the whole system of quantum $k$-prover interactive proof system consists of $q(n)=q_{\mathcal{V}}(n)+\sum_{i=1}^{k}\left(q_{\mathcal{M}_{i}}(n)+q_{\mathcal{P}_{i}}(n)\right)$ qubits, where $q_{\mathcal{V}}(n)$ is the number of qubits that are private to the verifier $V$, each $q_{\mathcal{P}_{i}}(n)$ is the number of qubits that are private to the prover $P_{i}$, and each $q_{\mathcal{M}_{i}}(n)$ is the number of message qubits used for communication between $V$ and $P_{i}$. Note that no communication is allowed between different provers $P_{i}$ and $P_{j}$. It is assumed that $q_{\mathcal{\nu}}$ and each $q_{\mathcal{M}_{i}}$ are polynomially bounded functions. Moreover, without loss of generality, we may assume that $q_{\mathcal{M}_{1}}=\cdots=q_{\mathcal{M}_{k}}=q_{\mathcal{M}}$ and $q_{\mathcal{P}_{1}}=\cdots=q_{\mathcal{P}_{k}}=q_{\mathcal{P}}$, accordingly, the whole system consists of $q(n)=q_{\mathcal{V}}(n)+k\left(q_{\mathcal{M}}(n)+q_{\mathcal{P}}(n)\right)$ qubits.

Given a polynomially bounded function $m: \mathbb{Z}^{+} \rightarrow \mathbb{N}$, an $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}\right)$-restricted quantum verifier $V$ is a polynomial-time computable mapping of the form $V: \Sigma^{*} \rightarrow \Sigma^{*}$, where $\Sigma=\{0,1\}$ is the alphabet set. $V$ uses at most $q_{\mathcal{V}}(n)$ qubits for his private space and at most $q_{\mathcal{M}}(n)$ qubits for communication with each prover. For each input $x \in \Sigma^{*}$ of length $n=|x|, V(x)$ is interpreted as a $\lfloor m(n) / 2+1\rfloor$-tuple $\left(V(x)_{1}, \ldots, V(x)_{\lfloor m(n) / 2+1\rfloor}\right)$, with each $V(x)_{j}$ a description of a polynomial-time uniformly generated quantum circuit acting on $q_{\mathcal{V}}(n)+k q_{\mathcal{M}}(n)$ qubits. One of the private qubits of the verifier is designated as the output qubit.

An $m$-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum prover $P_{i}$ for each $i=1, \ldots, k$ is a mapping of the form $P_{i}: \Sigma^{*} \rightarrow \Sigma^{*}$. Each $P_{i}$ uses at most $q_{\mathcal{P}}(n)$ qubits for his private space and at most $q_{\mathcal{M}}(n)$ qubits for communication with the verifier. For each input $x \in \Sigma^{*},|x|=n, P_{i}(x)$ is interpreted as a $\lfloor m(n) / 2+1 / 2\rfloor$-tuple $\left(P_{i}(x)_{1}, \ldots, P_{i}(x)_{\lfloor m(n) / 2+1 / 2\rfloor}\right)$, with each $P_{i}(x)_{j}$ a description of a quantum circuit acting on $q_{\mathcal{M}}(n)+q_{\mathcal{P}}(n)$ qubits. No restrictions are placed on the complexity of the mapping $P_{i}$ (i.e., each $P_{i}(x)_{j}$ can be an arbitrary unitary transformation).

An $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum $k$-prover interactive proof system consists of an $m$ message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}\right)$-restricted quantum verifier $V$ and $m$-message ( $q_{\mathcal{M}}, q_{\mathcal{P}}$ )-restricted quantum provers $P_{1}, \ldots, P_{k}$. Let $\mathcal{V}=l_{2}\left(\Sigma^{q \mathcal{V}}\right)$, each $\mathcal{M}_{i}=l_{2}\left(\Sigma^{q_{\mathcal{M}}}\right)$, and each $\mathcal{P}_{i}=l_{2}\left(\Sigma^{q_{\mathcal{P}}}\right)$ denote the Hilbert spaces corresponding to the private qubits of the verifier, the message qubits between the verifier and the $i$ th prover, and the private qubits of the $i$ th prover, respectively. Given a verifier $V$, provers $P_{1}, \ldots, P_{k}$, and an input $x$ of length $n$, we define a circuit $\left(P_{1}(x), \ldots, P_{k}(x), V(x)\right)$ acting on $q(n)$ qubits as follows. If $m(n)$ is odd, circuits

$$
P_{1}(x)_{1}, \ldots, P_{k}(x)_{1}, V(x)_{1}, \ldots, P_{1}(x)_{(m(n)+1) / 2}, \ldots, P_{k}(x)_{(m(n)+1) / 2}, V(x)_{(m(n)+1) / 2}
$$

are applied in sequence, each $P_{i}(x)_{j}$ to $\mathcal{M}_{i} \otimes \mathcal{P}_{i}$, and each $V(x)_{j}$ to $\mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k}$. If $m(n)$ is even, circuits

$$
V(x)_{1}, P_{1}(x)_{1}, \ldots, P_{k}(x)_{1}, \ldots, V(x)_{m(n) / 2}, P_{1}(x)_{m(n) / 2}, \ldots, P_{k}(x)_{m(n) / 2}, V(x)_{m(n) / 2+1}
$$

are applied in sequence. Figure [] illustrates the situation for the case $k=2$ and $m=3$. Note that the order of the applications of the circuits of the provers at each round has actually no sense since the space $\mathcal{M}_{i} \otimes \mathcal{P}_{i}$ on which the circuits of the $i$ th prover act is separated from each other prover.

At any given instant, the state of the whole system is a unit vector in the space $\mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes$ $\mathcal{M}_{k} \otimes \mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}$. For instance, in the case $m=3$, given input $x$, the state of the system after all of the circuits of the provers and the verifier have been applied is

$$
V_{2} P_{k, 2} \cdots P_{1,2} V_{1} P_{k, 1} \cdots P_{1,1}\left|\psi_{\text {init }}\right\rangle
$$

where each $V_{j}, P_{i, j}$ denotes the extension of $V(x)_{j}, P_{i}(x)_{j}$, respectively, to the space $\mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes$ $\mathcal{M}_{k} \otimes \mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}$ by tensoring with the identity, and $\left|\psi_{\text {init }}\right\rangle \in \mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k} \otimes \mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}$ denotes the initial state in which all $q(n)$ qubits are the $|0\rangle$-states.

For each input $x$, the probability that the $(k+1)$-tuple $\left(P_{1}, \ldots, P_{k}, V\right)$ accepts $x$ is defined to be the probability that an observation of the output qubit in the basis of $\{|0\rangle,|1\rangle\}$ yields $|1\rangle$, after the circuit $\left(P_{1}(x), \ldots, P_{k}(x), V(x)\right)$ is applied to the initial state $\left|\psi_{\text {init }}\right\rangle$.


Figure 1: Quantum circuit for a three-message quantum two-prover interactive proof system

Although $k$, the number of provers, has been treated to be constant so far, the above definition can be naturally extended to the case that $k: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ is a function of $n$. Thus, hereafter, we treat $k$ as a function. Note that the number of provers possible to communicate with the verifier must be bounded polynomial in $n=|x|$.

Definition 3 Given polynomially bounded functions $k, m, q_{\mathcal{V}}, q_{\mathcal{M}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$, a function $q_{\mathcal{P}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$, and functions $a, b: \mathbb{Z}^{+} \rightarrow[0,1]$, let $\operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}, a, b\right)$ denote the class of languages $L$ for which there exists an m-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}\right)$-restricted quantum verifier $V$ such that, for every input $x$, $|x|=n$,
(i) if $x \in L$, there exists a set of $k$ provers $P_{1}, \ldots, P_{k}$, each $P_{i}$ is an m-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum prover, such that $\left(P_{1}, \ldots, P_{k}, V\right)$ accepts $x$ with probability at least $a(n)$,
(ii) if $x \notin L$, for all sets of $k$ provers $P_{1}^{\prime}, \ldots, P_{k}^{\prime}$, each $P_{i}^{\prime}$ is an m-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum prover, $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ accepts $x$ with probability at most $b(n)$.

We write $\operatorname{QMIP}(k, m, a, b)$ in short if there exist some polynomially bounded functions $q_{\mathcal{V}}, q_{\mathcal{M}}$ such that $L$ is in $\operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}, a, b\right)$ for any function $q_{\mathcal{P}}$. Let $\operatorname{QMIP}(k, p o l y, a, b)$ denote the union of the classes $\operatorname{QMIP}(k, m, a, b)$ over all polynomially bounded functions $m$, and let $\operatorname{QMIP}($ poly poly $, a, b)$ denote the union of the classes $\operatorname{QMIP}(k$, poly $, a, b)$ over all polynomially bounded functions $k$.

Definition 4 A language $L$ is said to have a quantum $k$-prover interactive proof system iff $L \in$ QMIP(k,poly, 1, 1/2).

Definition 5 A language $L$ is said to have a quantum multi-prover interactive proof system iff $L \in$ QMIP(poly, poly, 1, 1/2).

For simplicity, let $\operatorname{QMIP}(k)=\operatorname{QMIP}(k$, poly, $1,1 / 2)$ and $\operatorname{QMIP}=\operatorname{QMIP}($ poly,poly, $1,1 / 2)$.

output qubit

Figure 2: Quantum circuit for a two-oracle-call quantum oracle circuit

### 4.3 Quantum Oracle Circuits

Consider a situation in which a verifier can communicate with only one prover, but the prover does not have his private qubits. We call this model a quantum oracle circuit, since it can be regarded as a quantum counterpart of a probabilistic oracle machine [18, 16, 5] in the sense that there is no private space for the prover during the computation.

For each input $x \in \Sigma^{*}$ of length $n=|x|$, the whole system of a quantum oracle circuit consists of $q(n)=q_{\mathcal{V}}(n)+q_{\mathcal{O}}(n)$ qubits, where $q_{\mathcal{V}}(n)$ is the number of qubits that are private to the verifier $V$, and $q_{\mathcal{O}}(n)$ is the number of qubits used for oracle calls. It is assumed that $q_{\mathcal{V}}$ and $q_{\mathcal{O}}$ are polynomially bounded functions.

Given a polynomially bounded function $m: \mathbb{Z}^{+} \rightarrow \mathbb{N}$, an $m$-oracle-call $\left(q \mathcal{V}, q_{\mathcal{O}}\right)$-restricted quantum verifier $V$ is a polynomial-time computable mapping of the form $V: \Sigma^{*} \rightarrow \Sigma^{*}$, where $\Sigma=\{0,1\}$ is the alphabet set. $V$ uses at most $q_{\mathcal{V}}(n)$ qubits for private space and at most $q_{\mathcal{O}}(n)$ qubits for oracle calls. For each input $x \in \Sigma^{*}$ of length $n=|x|, V(x)$ is interpreted as an ( $m(n)+1$ )-tuple $\left(V(x)_{1}, \ldots, V(x)_{m(n)+1}\right)$, with each $V(x)_{j}$ a description of a polynomial-time uniformly generated quantum circuit acting on $q_{\mathcal{V}}(n)+q_{\mathcal{O}}(n)$ qubits. One of the private qubits of the verifier is designated as the output qubit.

A $q_{\mathcal{O}}$-restricted quantum oracle $O$ for an $m$-oracle-call $\left(q_{\mathcal{V}}, q_{\mathcal{O}}\right)$-restricted verifier is an $m$-tuple $\left(O_{1}, \ldots, O_{m(n)}\right)$, with each $O_{j}$ a description of a quantum circuit corresponding to an arbitrary unitary transformation that acts on $q_{\mathcal{O}}(n)$ qubits. Note that our definition of a quantum oracle completely differs from that by Bennett, Bernstein, Brassard, and Vazirani $[7]$ where a quantum oracle is restricted to a unitary transformation that maps $|y, z\rangle$ to $|y, z \oplus f(y)\rangle$ in one step for an arbitrary function $f:\{0,1\}^{*} \rightarrow\{0,1\}$. In our definition, we may consider a quantum oracle as a quantum prover without his private qubits.

Let $\mathcal{V}=l_{2}\left(\Sigma^{q \mathcal{V}}\right)$ and $\mathcal{O}=l_{2}\left(\Sigma^{q \mathcal{O}}\right)$ denote the Hilbert spaces corresponding to the private qubits of the verifier and the qubits for oracle calls, respectively. Given a verifier $V$, an oracle $O$, and an input $x$ of length $n$, we define a circuit $(V(x), O)$ acting on $q(n)$ qubits as follows. Circuits

$$
V(x)_{1}, O_{1}, V(x)_{2}, O_{2}, \ldots, V(x)_{m(n)}, O_{m(n)}, V(x)_{m(n)+1}
$$

are applied in sequence, each $V(x)_{j}$ to $\mathcal{V} \otimes \mathcal{O}$, and each $O_{j}$ to $\mathcal{O}$. Figure 2 illustrates the situation for the case $m=2$.

At any given instant, the state of the whole system is a unit vector in the space $\mathcal{V} \otimes \mathcal{O}$. For instance, in the case $m=2$, the state of the system after all of the verifier's and the oracle's circuits have been applied, given input $x$, is

$$
V(x)_{3}\left(I_{\mathcal{V}} \otimes O_{2}\right) V(x)_{2}\left(I_{\mathcal{V}} \otimes O_{1}\right) V(x)_{1}\left|\psi_{\mathrm{init}}\right\rangle
$$

where $I_{\mathcal{V}}$ is the identity matrix on $\mathcal{V}$, and $\left|\psi_{\text {init }}\right\rangle \in \mathcal{V} \otimes \mathcal{O}$ denotes the initial state in which all $q(n)$ qubits are the $|0\rangle$-states. For convenience, we also write $V_{3} O_{2} V_{2} O_{1} V_{1}\left|\psi_{\text {init }}\right\rangle$ to denote this state.

For each input $x$, the probability that $V$ with access to $O$ accepts $x$ is defined to be the probability that an observation of the output qubit in the basis of $\{|0\rangle,|1\rangle\}$ yields $|1\rangle$, after the circuit $(V(x), O)$ is applied to the initial state $\left|\psi_{\text {init }}\right\rangle$.

Definition 6 Given polynomially bounded functions $m, q_{\mathcal{V}}, q_{\mathcal{O}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ and functions $a, b: \mathbb{Z}^{+} \rightarrow$ $[0,1]$, let $\operatorname{QOC}\left(m, q_{\mathcal{V}}, q_{\mathcal{O}}, a, b\right)$ denote the class of languages $L$ for which there exists an $m$-oracle-call $\left(q_{\mathcal{V}}, q_{\mathcal{O}}\right)$-restricted quantum verifier $V$ such that, for every input $x,|x|=n$,
(i) if $x \in L$, there exists a $q_{\mathcal{O}}$-restricted quantum oracle $O$ for $V$ such that $V$ with access to $O$ accepts $x$ with probability at least $a(n)$,
(ii) if $x \notin L$, for all $q_{\mathcal{O}}$-restricted quantum oracles $O^{\prime}$ for $V$, $V$ with access to $O^{\prime}$ accepts $x$ with probability at most b(n).

We write $\operatorname{QOC}(m, a, b)$ in short to denote the union of the classes $\operatorname{QOC}\left(m, q_{\mathcal{V}}, q_{\mathcal{O}}, a, b\right)$ over all polynomially bounded functions $q_{\mathcal{V}}, q_{\mathcal{O}}$. We also let $\mathrm{QOC}($ poly, $a, b)$ denote the union of the classes $\mathrm{QOC}(m, a, b)$ over all polynomially bounded functions $m$.

Definition 7 A language $L$ is said to be accepted by a quantum oracle circuit iff $L \in$ QOC(poly, 1, 1/2).

For simplicity, let $\mathrm{QOC}=\mathrm{QOC}($ poly $, 1,1 / 2)$.

## $5 \quad$ QMIP $=$ NEXP

Now we show that the class of languages that have quantum multi-prover interactive proof systems is equal to NEXP. As we have mentioned before, we utilize the concept of quantum oracle circuits and we actually show that QMIP $=$ QOC $=$ NEXP. For simplicity, in this section and after, we often drop the argument $x$ and $n$ in the various functions defined in the previous section. We also assume that operators acting on subsystems of a given system are extended to the entire system by tensoring with the identity, when it is clear from context upon what part of a system a given operator acts.

## 5.1 $\mathrm{QMIP} \subseteq \mathrm{QOC}$

First we show that every language that has a quantum multi-prover interactive proof system is accepted by a quantum oracle circuit. For this, it is useful to show that, for any protocol of quantum multiprover interactive proof systems, there exists a quantum multi-prover interactive protocol with the same number of provers and with the same number of messages, in which each prover uses only polynomially many qubits for his private space with respect to the input length, and the probability of acceptance is exactly same as that of the original one. In the proof of the following lemma, Theorem 11 and Theorem 2 play very important roles. A point of our proof is how to combine and apply these two to the theory of quantum multi-prover interactive proof systems.

Lemma 8 Let $k, m, q_{\mathcal{V}}, q_{\mathcal{M}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be polynomially bounded functions and $q_{\mathcal{P}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be a function. For any protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ of an m-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum $k$-prover interactive proof system, there exists a protocol $\left(Q_{1}, \ldots, Q_{k}, W\right)$ of an m-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}\right)$-restricted quantum $k$-prover interactive proof system such that, for every input $x,\left(Q_{1}, \ldots, Q_{k}, W\right)$ accepts $x$ with just the same probability as $\left(P_{1}, \ldots, P_{k}, V\right)$ does.

Proof. We assume that $q_{\mathcal{P}} \geq 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}$, since there is nothing to show in case $q_{\mathcal{P}}<2\lfloor m / 2+$ $1 / 2\rfloor q_{\mathcal{M}}$. We also assume that the values of $m$ are even (odd cases can be dealt with a similar argument).

Given a protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ of an $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum $k$-prover interactive proof system, we first show that $P_{1}$ can be replaced by such $P_{1}^{\prime}$ that each transformation of $P_{1}^{\prime}$ acts on at most $m q_{\mathcal{M}}\left(=2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}\right)$ qubits, and $\left(P_{1}^{\prime}, P_{2}, \ldots, P_{k}, V\right)$ accepts the input with the same probability as $\left(P_{1}, \ldots, P_{k}, V\right)$ does. Having shown this, we repeat the same process for each of provers to construct a protocol $\left(P_{1}^{\prime}, P_{2}^{\prime}, P_{3}, \ldots, P_{k}, V\right)$ from $\left(P_{1}^{\prime}, P_{2}, P_{3}, \ldots, P_{k}, V\right)$ and so on, and finally we obtain a protocol $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ in which all $k$ provers use at most $m q_{\mathcal{M}}\left(=2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}\right)$ qubits in their private spaces. We construct $P_{1}^{\prime}$ by showing, for every input $x$, how to construct each $P_{1, j}^{\prime}(x)$ based on the original $P_{1, j}(x)$. In the following proof, each $P_{i, j}(x), P_{i, j}^{\prime}(x)$ will be denoted by $P_{i, j}, P_{i, j}^{\prime}$, respectively.

Let each $\left|\psi_{j}\right\rangle,\left|\phi_{j}\right\rangle \in \mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k} \otimes \mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}$, for $0 \leq j \leq m / 2$, denote a state of the original $m$-message ( $q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}$ )-restricted quantum $k$-prover interactive proof system defined in a recursive manner by

$$
\begin{array}{ll}
\left|\phi_{1}\right\rangle=V_{1}\left|\psi_{\text {init }}\right\rangle, & \\
\left|\phi_{j}\right\rangle=V_{j} P_{k, j-1} \cdots P_{1, j-1}\left|\phi_{j-1}\right\rangle, & 2 \leq j \leq m / 2, \\
\left|\psi_{j}\right\rangle=P_{1, j}\left|\phi_{j}\right\rangle, & 1 \leq j \leq m / 2 .
\end{array}
$$

Note that $\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right|=\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\phi_{j}\right\rangle\left\langle\phi_{j}\right|$, for each $1 \leq j \leq m / 2$. We define each $P_{1, j}^{\prime}$ recursively. To define $P_{1,1}^{\prime}$, consider the states $\left|\phi_{1}\right\rangle$ and $\left|\psi_{1}\right\rangle$. Let $\left|\phi_{1}^{\prime}\right\rangle=\left|\phi_{1}\right\rangle$. Since all the qubits in $\mathcal{P}_{1}$ in the state $\left|\phi_{1}\right\rangle$ are the $|0\rangle$-states and $\left|\psi_{1}\right\rangle=P_{1,1}\left|\phi_{1}\right\rangle$, by Theorem 2, there exists a state $\left|\psi_{1}^{\prime}\right\rangle$ such that

$$
\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{1}^{\prime}\right\rangle\left\langle\psi_{1}^{\prime}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|
$$

and all the qubits but the first $2 q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\psi_{1}^{\prime}\right\rangle$. Then we have

$$
\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\psi_{1}^{\prime}\right\rangle\left\langle\psi_{1}^{\prime}\right|=\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|=\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\phi_{1}\right\rangle\left\langle\phi_{1}\right|=\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\phi_{1}^{\prime}\right\rangle\left\langle\phi_{1}^{\prime}\right|,
$$

therefore, by Theorem [1] , there exists a unitary transformation $P_{1,1}^{\prime}$ acting on $\mathcal{M}_{1} \otimes \mathcal{P}_{1}$ such that $P_{1,1}^{\prime}\left|\phi_{1}^{\prime}\right\rangle=\left|\psi_{1}^{\prime}\right\rangle$ and $P_{1,1}^{\prime}$ is of the form $P_{1,1}^{\prime \prime} \otimes I_{q_{\mathcal{P}}-2 q_{\mathcal{M}}}$, where $P_{1,1}^{\prime \prime}$ is a unitary transformation acting on $\mathcal{M}_{1}$ and first $2 q_{\mathcal{M}}$ qubits of $\mathcal{P}_{1}$, and $I_{q_{\mathcal{P}}-2 q_{\mathcal{M}}}$ is $\left(q_{\mathcal{P}}-2 q_{\mathcal{M}}\right)$-dimensional identity matrix.

Assume that $P_{1, j}^{\prime},\left|\phi_{j}^{\prime}\right\rangle$, and $\left|\psi_{j}^{\prime}\right\rangle$ have been defined for each $j, 1 \leq j \leq \xi \leq m / 2-1$, to satisfy

- $\left|\phi_{1}^{\prime}\right\rangle=V_{1}\left|\psi_{\text {init }}\right\rangle$,
$\left|\phi_{j}^{\prime}\right\rangle=V_{j} P_{k, j-1} \cdots P_{2, j-1} P_{1, j-1}^{\prime}\left|\phi_{j-1}^{\prime}\right\rangle, 2 \leq j \leq \xi$,
$\left|\psi_{j}^{\prime}\right\rangle=P_{1, j}^{\prime}\left|\phi_{j}^{\prime}\right\rangle, \quad 1 \leq j \leq \xi$,
- $\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{j}^{\prime}\right\rangle\left\langle\psi_{j}^{\prime}\right|, 1 \leq j \leq \xi$,
- all the qubits but the first $2(j-1) q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\phi_{j}^{\prime}\right\rangle$,
- all the qubits but the first $2 j q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\psi_{j}^{\prime}\right\rangle$.

Notice that $P_{1,1}^{\prime},\left|\phi_{1}^{\prime}\right\rangle$, and $\left|\psi_{1}^{\prime}\right\rangle$ defined above satisfy such conditions. Define $P_{1, \xi+1}^{\prime},\left|\phi_{\xi+1}^{\prime}\right\rangle,\left|\psi_{\xi+1}^{\prime}\right\rangle$ in the following way to satisfy the above four conditions for $j=\xi+1$.

Let $U_{\xi}=V_{\xi+1} P_{k, \xi} \cdots P_{2, \xi}$ and define $\left|\phi_{\xi+1}^{\prime}\right\rangle=U_{\xi}\left|\psi_{\xi}^{\prime}\right\rangle$. Then all the qubits but the first $2 \xi q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\phi_{\xi+1}^{\prime}\right\rangle$, since none of $P_{2, \xi}, \ldots, P_{k, \xi}, V_{\xi+1}$ acts on the space $\mathcal{P}_{1}$ and $\left|\psi_{\xi}^{\prime}\right\rangle$ satisfies the fourth condition. Since $\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{\xi}\right\rangle\left\langle\psi_{\xi}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{\xi}^{\prime}\right\rangle\left\langle\psi_{\xi}^{\prime}\right|$, by Theorem 11 there exists a unitary transformation $A_{\xi}$ acting on $\mathcal{P}_{1}$ which satisfies $A_{\xi}\left|\psi_{\xi}^{\prime}\right\rangle=\left|\psi_{\xi}\right\rangle$. Thus we have

$$
\begin{equation*}
\left|\psi_{\xi+1}\right\rangle=P_{1, \xi+1} U_{\xi}\left|\psi_{\xi}\right\rangle=P_{1, \xi+1} U_{\xi} A_{\xi}\left|\psi_{\xi}^{\prime}\right\rangle=P_{1, \xi+1} A_{\xi} U_{\xi}\left|\psi_{\xi}^{\prime}\right\rangle=P_{1, \xi+1} A_{\xi}\left|\phi_{\xi+1}^{\prime}\right\rangle . \tag{2}
\end{equation*}
$$

Hence, by Theorem 2 , there exists a state $\left|\psi_{\xi+1}^{\prime}\right\rangle$ such that

$$
\begin{equation*}
\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{\xi+1}^{\prime}\right\rangle\left\langle\psi_{\xi+1}^{\prime}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{\xi+1}\right\rangle\left\langle\psi_{\xi+1}\right| \tag{3}
\end{equation*}
$$

and all the qubits but the first $2(\xi+1) q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\psi_{\xi+1}^{\prime}\right\rangle$. From (24) and (3), we have

$$
\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\psi_{\xi+1}^{\prime}\right\rangle\left\langle\psi_{\xi+1}^{\prime}\right|=\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\psi_{\xi+1}\right\rangle\left\langle\psi_{\xi+1}\right|=\operatorname{tr}_{\mathcal{M}_{1} \otimes \mathcal{P}_{1}}\left|\phi_{\xi+1}^{\prime}\right\rangle\left\langle\phi_{\xi+1}^{\prime}\right|,
$$

since $P_{1, \xi+1}$ and $A_{\xi}$ act only on $\mathcal{M}_{1} \otimes \mathcal{P}_{1}$. Therefore, by Theorem 11, there exists a unitary transformation $P_{1, \xi+1}^{\prime}$ acting on $\mathcal{M}_{1} \otimes \mathcal{P}_{1}$ such that $P_{1, \xi+1}^{\prime}\left|\phi_{\xi+1}^{\prime}\right\rangle=\left|\psi_{\xi+1}^{\prime}\right\rangle$. It follows that $P_{1, \xi+1}^{\prime}$ is of the form $P_{1, \xi+1}^{\prime \prime} \otimes I_{q_{\mathcal{P}}-2(\xi+1) q_{\mathcal{M}}}$, where $P_{1, \xi+1}^{\prime \prime}$ is a unitary transformation acting on $\mathcal{M}_{1}$ and the first $2(\xi+1) q_{\mathcal{M}}$ qubits of $\mathcal{P}_{1}$, and $I_{q_{\mathcal{P}}-2(\xi+1) q_{\mathcal{M}}}$ is $\left(q_{\mathcal{P}}-2(\xi+1) q_{\mathcal{M}}\right)$-dimensional identity matrix, because all the qubits but the first $2(\xi+1) q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in both of the states $\left|\phi_{\xi+1}^{\prime}\right\rangle$ and $\left|\psi_{\xi+1}^{\prime}\right\rangle$. We can see that $P_{1, \xi+1}^{\prime},\left|\phi_{\xi+1}^{\prime}\right\rangle$, and $\left|\psi_{\xi+1}^{\prime}\right\rangle$ satisfy the four conditions above by their construction.

Having defined $P_{1, j}^{\prime},\left|\phi_{j}^{\prime}\right\rangle,\left|\psi_{j}^{\prime}\right\rangle$ for each $1 \leq j \leq m / 2$, compare the state just before the final measurement is performed in the original protocol and that in the modified protocol applying $P_{1, j}^{\prime}$ 's in stead of $P_{1, j}$ 's. For $U_{m / 2}=V_{m / 2+1} P_{k, m / 2} \cdots P_{2, m / 2}$, let $\left|\phi_{m / 2+1}\right\rangle=U_{m / 2}\left|\psi_{m / 2}\right\rangle$ and $\left|\phi_{m / 2+1}^{\prime}\right\rangle=$ $U_{m / 2}\left|\psi_{m / 2}^{\prime}\right\rangle$. These $\left|\phi_{m / 2+1}\right\rangle$ and $\left|\phi_{m / 2+1}^{\prime}\right\rangle$ are exactly the states we want to compare. Noticing that $\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{m / 2}\right\rangle\left\langle\psi_{m / 2}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\psi_{m / 2}^{\prime}\right\rangle\left\langle\psi_{m / 2}^{\prime}\right|$, we have $\operatorname{tr}_{\mathcal{P}_{1}}\left|\phi_{m / 2+1}\right\rangle\left\langle\phi_{m / 2+1}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\phi_{m / 2+1}^{\prime}\right\rangle\left\langle\phi_{m / 2+1}^{\prime}\right|$, since none of $V_{m / 2+1}, P_{k, m / 2}, \ldots, P_{2, m / 2}$ acts on $\mathcal{P}_{1}$. Thus we have

$$
\operatorname{tr}_{\mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}}\left|\phi_{m / 2+1}\right\rangle\left\langle\phi_{m / 2+1}\right|=\operatorname{tr}_{\mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}}\left|\phi_{m / 2+1}^{\prime}\right\rangle\left\langle\phi_{m / 2+1}^{\prime}\right|,
$$

which implies that the verifier $V$ cannot distinguish $\left|\phi_{m / 2+1}^{\prime}\right\rangle$ from $\left|\phi_{m / 2+1}\right\rangle$ at all. Hence, for every input $x$, the protocol $\left(P_{1}^{\prime}, P_{2}, \ldots, P_{k}, V\right)$ accepts $x$ with just the same probability as the original protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ does, and $P_{1}^{\prime}$ uses only $m q_{\mathcal{M}}=2 \cdot(m / 2) \cdot q_{\mathcal{M}}$ qubits in his private space.

Now we repeat the above process for each of provers to construct a protocol ( $\left.P_{1}^{\prime}, P_{2}^{\prime}, P_{3}, \ldots, P_{k}, V\right)$ from $\left(P_{1}^{\prime}, P_{2}, P_{3}, \ldots, P_{k}, V\right)$ and so on, and finally we obtain a protocol $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ in which all $k$ provers use only $m q_{\mathcal{M}}$ qubits in their private spaces. It is obvious that, for every input $x$, the protocol $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ accepts $x$ with just the same probability as the original protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ does.

In the protocol $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$, each $P_{i, j}^{\prime}$, the $j$ th transformation of $P_{i}^{\prime}$, can be described as $P_{i, j}^{\prime}=R_{i, j} \otimes I_{q_{\mathcal{P}}-m q_{\mathcal{M}}}$, where $I_{q_{\mathcal{P}}-m q_{\mathcal{M}}}$ is $\left(q_{\mathcal{P}}-m q_{\mathcal{M}}\right)$-dimensional identity matrix. Consequently, by constructing a protocol $\left(Q_{1}, \ldots, Q_{k}, W\right)$ of an $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, m q_{\mathcal{M}}\right)$-restricted quantum $k$-prover interactive proof system as

$$
\begin{aligned}
W_{j} & =V_{j}, \quad 1 \leq j \leq m / 2+1, \\
Q_{i, j} & =R_{i, j}, \quad 1 \leq i \leq k, 1 \leq j \leq m / 2,
\end{aligned}
$$

for every input $x$, the protocol $\left(Q_{1}, \ldots, Q_{k}, W\right)$ accepts $x$ with just the same probability as the original protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ does.

From Lemma 8, it is straightforward to show the following lemma.
Lemma 9 Let $k, m, q_{\mathcal{V}}, q_{\mathcal{M}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be polynomially bounded functions, $q_{\mathcal{P}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be a function such that $q_{\mathcal{P}} \geq 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}$, and $a, b: \mathbb{Z}^{+} \rightarrow[0,1]$ be functions such that $a \geq b$. Then $\operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}, a, b\right) \subseteq \operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}, a, b\right)$.

Proof. Let $L$ be a language in $\operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}, a, b\right)$, and consider an $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}\right)$ restricted quantum $k$-prover interactive proof system for $L$. Then there exists an $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}\right)$ restricted quantum verifier $V$ of this proof system such that, in case the input $x$ of length $n$ is in $L$, there exist $m$-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum provers $P_{1}, \ldots, P_{k}$ of this proof system and $\left(P_{1}, \ldots, P_{k}, V\right)$ accepts $x$ with probability at least $a(n)$, while in case the input $x$ of length $n$ is not in $L$, for any set of $m$-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum provers $P_{1}^{\prime}, \ldots, P_{k}^{\prime}$ of this proof system, $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ accepts $x$ with probability at most $b(n)$.

Define the $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}\right)$-restricted quantum verifier $W$ of the corresponding $m$-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}\right)$-restricted quantum $k$-prover interactive proof system as $W=V$.
(i) In case the input $x$ of length $n$ is in $L$ :

By Lemma 8 , we can construct $m$-message ( $q_{\mathcal{M}}, 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}$ )-restricted quantum provers $Q_{1}, \ldots, Q_{k}$ of this proof system from $P_{1}, \ldots, P_{k}$, and $\left(Q_{1}, \ldots, Q_{k}, W\right)$ accepts $x$ with probability at least $a(n)$.
(ii) In case the input $x$ of length $n$ is not in $L$ :

Suppose that there exist $m$-message $\left(q_{\mathcal{M}}, 2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}\right)$-restricted quantum provers $Q_{1}^{\prime}, \ldots, Q_{k}^{\prime}$ of this proof system and $\left(Q_{1}^{\prime}, \ldots, Q_{k}^{\prime}, W\right)$ accepts $x$ with probability more than $b(n)$. Then, obviously, we can construct $m$-message ( $q_{\mathcal{M}}, q_{\mathcal{P}}$ )-restricted quantum provers $P_{1}^{\prime}, \ldots, P_{k}^{\prime}$ of the original proof system by appropriately tensoring the identity matrix for each $Q_{i, j}^{\prime}$ to construct $P_{i, j}^{\prime}$, and $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ accepts $x$ with probability more than $b(n)$. This is a contradiction.

Now, in order to conclude QMIP $\subseteq$ QOC, it is sufficient to show the following lemma.
Lemma 10 Let $k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be polynomially bounded functions, and $a, b: \mathbb{Z}^{+} \rightarrow[0,1]$ be functions such that $a \geq b$. Then $\operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}, a, b\right) \subseteq Q O C\left(k\lfloor(m+1) / 2\rfloor, q \mathcal{V}+k\left(q_{\mathcal{M}}+\right.\right.$ $\left.\left.q_{\mathcal{P}}\right), q_{\mathcal{M}}+q_{\mathcal{P}}, a, b\right)$.

Proof. Let $L$ be a language in $\operatorname{QMIP}\left(k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}, a, b\right)$. For simplicity, we assume that the values of $m$ are even (odd cases can be proved with a similar argument).

We construct a $k m / 2$-oracle-call verifier $V^{\mathrm{QOC}}$ of the quantum oracle circuit as follows. Let us consider that quantum registers (collections of qubits upon which various transformations are performed) $\mathbf{W}, \mathbf{M}_{i}$, and $\mathbf{P}_{i}$ for $1 \leq i \leq k$ are prepared among the private qubits of the verifier $V^{\mathrm{QOC}}$, and quantum registers $\mathbf{M}$ and $\mathbf{P}$ are prepared among the qubits for oracle calls. $\mathbf{W}$ consists of $q \mathcal{\nu}$ qubits, each $\mathbf{M}_{i}$ and $\mathbf{M}$ consist of $q_{\mathcal{M}}$ qubits, and each $\mathbf{P}_{i}$ and $\mathbf{P}$ consist of $q_{\mathcal{P}}$ qubits. Let $\mathcal{W}^{\text {QOC }}$, each $\mathcal{M}_{i}^{\mathrm{QOC}}$, and each $\mathcal{P}_{i}^{\mathrm{QOC}}$ denote the Hilbert spaces corresponding to the register $\mathbf{W}$, the register $\mathbf{M}_{i}$, and the register $\mathbf{P}_{i}$, respectively. Take the Hilbert space $\mathcal{V}^{\mathrm{QOC}}$ corresponding to the qubits private to the verifier $V^{\mathrm{QOC}}$ as $\mathcal{V}^{\mathrm{QOC}}=\mathcal{W}^{\mathrm{QOC}} \otimes \mathcal{M}_{1}^{\mathrm{QOC}} \otimes \cdots \otimes \mathcal{M}_{k}^{\mathrm{QOC}} \otimes \mathcal{P}_{1}^{\mathrm{QOC}} \otimes \cdots \otimes \mathcal{P}_{k}^{\mathrm{QOC}}$. Hence the number of private qubits of $V^{\mathrm{QOC}}$ is $q_{\mathcal{V}}^{\mathrm{QOC}}=q_{\mathcal{V}}+k\left(q_{\mathcal{M}}+q_{\mathcal{P}}\right)$. Let $\mathcal{M}^{\mathrm{QOC}}$ and $\mathcal{P}^{\mathrm{QOC}}$ denote the Hilbert spaces corresponding to the registers $\mathbf{M}$ and $\mathbf{P}$, respectively. Take the Hilbert space $\mathcal{O}^{\text {QOC }}$ corresponding to the qubits for oracle calls as $\mathcal{O}^{\mathrm{QOC}}=\mathcal{M}^{\mathrm{QOC}} \otimes \mathcal{P}^{\mathrm{QOC}}$. Hence the number of qubits for oracle calls is $q_{\mathcal{O}}^{\mathrm{QOC}}=q_{\mathcal{M}}+q_{\mathcal{P}}$.

Consider each $V_{j}$, the $j$ th quantum circuit of the verifier $V$ of the original quantum $k$-prover interactive proof system, which acts on $\mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k}$. For each $j$, let $U_{j}^{\mathrm{QOC}}$ be just the same unitary transformation as $V_{j}$ and $U_{j}^{\mathrm{QOC}}$ acts on $\mathcal{W}^{\mathrm{QOC}} \otimes \mathcal{M}_{1}^{\mathrm{QOC}} \otimes \cdots \otimes \mathcal{M}_{k}^{\mathrm{QOC}}$, which corresponds to that $V_{j}$ acts on $\mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k}$. Define the verifier $V^{\mathrm{QOC}}$ of the corresponding quantum oracle circuit in the following way:

- At the first transformation of $V^{\mathrm{QOC}}, V^{\mathrm{QOC}}$ first applies $U_{1}^{\mathrm{QOC}}$, and then swaps the contents of $\mathbf{M}_{1}$ for those of $\mathbf{M}$.
- At the $((j-1) k+1)$-th transformation of $V^{\mathrm{QOC}}$ for each $2 \leq j \leq m / 2, V^{\mathrm{QOC}}$ first swaps the contents of $\mathbf{M}, \mathbf{P}$ for those of $\mathbf{M}_{k}, \mathbf{P}_{k}$, respectively, then applies $U_{j}^{\mathrm{QOC}}$, and finally swaps the contents of $\mathbf{M}_{1}, \mathbf{P}_{1}$ for those of $\mathbf{M}, \mathbf{P}$.
- At the $((j-1) k+i)$-th transformation of $V^{\mathrm{QOC}}$ for each $2 \leq i \leq k, 1 \leq j \leq m / 2, V^{\mathrm{QOC}}$ first swaps the contents of $\mathbf{M}, \mathbf{P}$ for those of $\mathbf{M}_{i-1}, \mathbf{P}_{i-1}$, respectively, then swaps the contents of $\mathbf{M}_{i}, \mathbf{P}_{i}$ for those of $\mathbf{M}, \mathbf{P}$.
(i) In case the input $x$ of length $n$ is in $L$ :

In the original $m$-message quantum $k$-prover interactive proof system, there exist $m$-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum provers $P_{1}, \ldots, P_{k}$ that cause $V$ to accept $x$ with probability at least $a(n)$. Hence, if we let $O_{(j-1) k+i}$ for each $1 \leq i \leq k, 1 \leq j \leq m / 2$ be just the same unitary transformation as $P_{i, j}\left(O_{(j-1) k+i}\right.$ acts on $\mathcal{O}^{\mathrm{QOC}}=\mathcal{M}^{\mathrm{QOC}} \otimes \mathcal{P}^{\mathrm{QOC}}$ corresponding to that $P_{i, j}$ acts on $\left.\mathcal{M}_{i} \otimes \mathcal{P}_{i}\right)$, it is obvious that $V^{\mathrm{QOC}}$ with access to $O$ accepts $x$ with just the same probability as the original $V$ does, which is at least $a(n)$.
(ii) In case the input $x$ of length $n$ is not in $L$ :

Suppose that there were an oracle $O^{\prime}$ that makes the verifier $V^{\mathrm{QOC}}$ accept $x$ with probability more than $b(n)$. Consider $m$-message $\left(q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted provers $P_{1}^{\prime}, \ldots, P_{k}^{\prime}$ of the original $m$-message quantum $k$-prover interactive proof system such that, for each $1 \leq i \leq k, 1 \leq j \leq m / 2, P_{i, j}^{\prime}$ is just the same transformation as $O_{(j-1) k+i}^{\prime}\left(P_{i, j}^{\prime}\right.$ acts on $\mathcal{M}_{i} \otimes \mathcal{P}_{i}$ corresponding to that $O_{(j-1) k+i}^{\prime}$ acts on $\left.\mathcal{M}^{\mathrm{QOC}} \otimes \mathcal{P}^{\mathrm{QOC}}\right)$. By their construction, it is obvious that these provers $P_{1}^{\prime}, \ldots, P_{k}^{\prime}$ can convince the verifier $V$ with the same probability that the oracle $O^{\prime}$ does, which is more than $b(n)$. This contradicts the assumption.

Thus we have the following theorem.
Theorem 11 QMIP $\subseteq Q O C$.

## 5.2 $\mathrm{QOC} \subseteq$ NEXP

Here we show that every language accepted by a quantum oracle circuit is accepted by a nondeterministic Turing machine which runs in exponential time with respect to the input length.

Theorem $12 Q O C \subseteq$ NEXP.
Proof. Let $L$ be a language in QOC. Then $L$ is accepted by an $m$-oracle-call $\left(q_{\mathcal{V}}, q_{\mathcal{O}}\right)$-restricted quantum oracle circuit for some polynomially bounded functions $m, q \mathcal{V}$, and $q_{\mathcal{O}}$. Let $V$ be the $m$ -oracle-call quantum verifier for this quantum oracle circuit.

For the input $x$ of length $n$, consider a classical simulation of this quantum oracle circuit by a non-deterministic Turing machine. Since each $V_{j}$ applied in the original quantum oracle circuit is polynomial-time uniformly generated and $q_{\mathcal{V}}$ and $q_{\mathcal{O}}$ are polynomially bounded in $n$, it is routine to show that an approximation $V_{j}^{\prime}$ of a matrix description of $V_{j}$ can be computed in time exponential in $n$ with accuracy of $\left\|V_{j}^{\prime}-V_{j}\right\|<2^{-p_{1}(n)}$ for any fixed polynomial $p_{1}$. Since $q_{\mathcal{O}}$ is polynomially bounded in $n$, for each oracle operation $O_{j}$ applied in the original quantum oracle circuit, an approximation $O_{j}^{\prime}$ of a matrix description of $O_{j}$ can be guessed in time non-deterministic exponential in $n$ with accuracy of $\left\|O_{j}^{\prime}-O_{j}\right\|<2^{-p_{1}(n)}$ for any fixed polynomial $p_{1}$. Thus, for the quantum state

$$
\left|\psi_{\text {final }}\right\rangle=V_{m+1} O_{m} V_{m} \cdots O_{1} V_{1}\left|\psi_{\text {init }}\right\rangle
$$

which is the state just before the final measurement in the original quantum oracle circuit, the approximation $\left|\psi_{\text {final }}^{\prime}\right\rangle$ of $\left|\psi_{\text {final }}\right\rangle$ can be computed in time non-deterministic exponential in $n$ with accuracy of $\|\left|\psi_{\text {final }}^{\prime}\right\rangle-\left|\psi_{\text {final }}\right\rangle \|<2^{-p_{2}(n)}$ for any fixed polynomial $p_{2}$ by appropriately choosing $p_{1}$.

Now, after having computed $\left|\psi_{\text {final }}^{\prime}\right\rangle$, a measurement of the output qubit is simulated by summing up squares of the computed amplitudes in the accepting states. The input $x$ is accepted if and only if this sum, the computed probability that the measurement results in $|1\rangle$, is more than $1-\varepsilon$. From the property of the original quantum oracle circuit, this computed probability is more than $1-2^{-2 p_{2}(n)}$ if $x$ is in $L$, while it is less than $1 / 2+2^{-2 p_{2}(n)}$ if $x$ is not in $L$. Thus, taking $p_{2}=n$ and $\varepsilon=2^{-2 n}$, the input $x$ is accepted if and only if $x$ is in $L$ and the whole computation is done in time non-deterministic exponential in $n$.

### 5.3 NEXP $\subseteq$ QMIP

It remains to show that every language in NEXP has a quantum multi-prover interactive proof system. However, it is a quite easy task since it is known that every language in NEXP has a (classical) multiprover interactive proof system, in particular, a one-round two-prover classical interactive proof system with exponentially small one-sided error [14]. Indeed, the quantum verifier has only to simulate this classical protocol and such a simulation works well, since our model does not allow any prior entanglement among private qubits of the quantum provers (cf. [11]).

Theorem 13 NEXP $\subseteq Q M I P$.
Proof. Consider such a quantum $k$-prover protocol that the quantum verifier performs measurements in $|0\rangle,|1\rangle$ basis on every qubit of his part at every time he sends questions to quantum provers and at every time he receives responses from them, and for the rest part of computation the quantum verifier behaves in the same manner as the classical verifier does. Such a protocol can be simulated without intermediate measurements by only using unitary transformations [1, 21]. Furthermore, since there is no prior entanglement among private qubits of the quantum provers, such a quantum protocol makes no difference from a classical protocol where the classical verifier chooses a set of $k$ classical provers probabilistically at the beginning of the protocol. Therefore, in such a quantum $k$-prover protocol, for every input, the quantum provers can be only as powerful as the classical provers, i.e., the quantum provers can behave just in the same way as the classical provers do, while no set of $k$ quantum provers can convince the quantum verifier with probability more than the maximum probability with which a set of $k$ classical provers can convince the classical verifier.

Now we explain in more detail. Let $L$ be a language in NEXP, then $L$ has a one-round two-prover interactive proof system. Let $V$ be the classical verifier of this one-round two-prover interactive proof system. We construct a two-message quantum two-prover interactive proof system by just simulating this classical protocol.

Assume that, just after the classical verifier $V$ has sent questions to the provers $P_{1}, P_{2}$, the contents of $V$ 's private tape, the question to $P_{1}$, and the question to $P_{2}$ are $v, q_{1}$, and $q_{2}$, respectively, with probability $p\left(v, q_{1}, q_{2}\right)$. Our two-message quantum verifier $V^{(Q)}$ prepares the quantum registers $\mathbf{V}$, $\mathbf{Q}_{1}, \mathbf{Q}_{2}, \mathbf{A}_{1}$, and $\mathbf{A}_{2}$ among his private qubits. $V^{(\mathrm{Q})}$ first stores $v, q_{1}, q_{2}$ in $\mathbf{V}, \mathbf{Q}_{1}, \mathbf{Q}_{2}$, respectively, then copies the contents of each $\mathbf{Q}_{i}$ to the message qubits shared with a quantum prover $P_{i}^{(\mathrm{Q})}$. That is, $V^{(\mathrm{Q})}$ prepares the superposition

$$
\sum_{v, q_{1}, q_{2}}(\sqrt{p\left(v, q_{1}, q_{2}\right)} \underbrace{|v\rangle}_{\mathbf{V}} \underbrace{\left|q_{1}\right\rangle}_{\mathbf{Q}_{1}} \underbrace{\left|q_{2}\right\rangle}_{\mathbf{Q}_{2}} \underbrace{|0\rangle}_{\mathbf{A}_{1}} \underbrace{|0\rangle}_{\mathbf{A}_{2}} \underbrace{\left|q_{1}\right\rangle}_{\mathbf{M}_{1}} \underbrace{|0\rangle}_{\mathbf{P}_{1}} \underbrace{\left|q_{2}\right\rangle}_{\mathbf{M}_{2}} \underbrace{|0\rangle}_{\mathbf{P}_{2}}),
$$

where, for each $i=1,2, \mathbf{M}_{i}$ denotes the quantum register that consists of the message qubits between $V^{(\mathrm{Q})}$ and $P_{i}^{(\mathrm{Q})}$, and $\mathbf{P}_{i}$ denotes the quantum register that consists of $P_{i}^{(\mathrm{Q})}$,s private qubits.

Next the quantum provers $P_{1}^{(\mathrm{Q})}$ and $P_{2}^{(\mathrm{Q})}$ apply some unitary transformations on their qubits. Now the state becomes

$$
\begin{aligned}
& \sum_{v, q_{1}, q_{2}}\{\sqrt{p\left(v, q_{1}, q_{2}\right)} \underbrace{|v\rangle}_{\mathbf{V}} \underbrace{\left|q_{1}\right\rangle}_{\mathbf{Q}_{1}} \underbrace{\left|q_{2}\right\rangle}_{\mathbf{Q}_{2}} \underbrace{|0\rangle}_{\mathbf{A}_{1}} \underbrace{|0\rangle}_{\mathbf{A}_{2}} \\
&\otimes(\sum_{a_{1}} \alpha_{1}\left(q_{1}, a_{1}\right) \underbrace{\left|a_{1}\right\rangle}_{\mathbf{M}_{1}} \underbrace{\left|\psi_{1}\left(q_{1}, a_{1}\right)\right\rangle}_{\mathbf{P}_{1}}) \otimes(\sum_{a_{2}} \alpha_{2}\left(q_{2}, a_{2}\right) \underbrace{\left|a_{2}\right\rangle}_{\mathbf{M}_{2}} \underbrace{\left|\psi_{2}\left(q_{2}, a_{2}\right)\right\rangle}_{\mathbf{P}_{2}})\} \\
&=\sum_{v, q_{1}, q_{2}, a_{1}, a_{2}}\left(\sqrt{p\left(v, q_{1}, q_{2}\right)} \alpha_{1}\left(q_{1}, a_{1}\right) \alpha_{2}\left(q_{2}, a_{2}\right)\right. \\
&\times \underbrace{|v\rangle}_{\mathbf{V}} \underbrace{\left|q_{1}\right\rangle}_{\mathbf{Q}_{1}} \underbrace{\left|q_{2}\right\rangle}_{\mathbf{Q}_{2}} \underbrace{|0\rangle}_{\mathbf{A}_{1}} \underbrace{|0\rangle}_{\mathbf{A}_{2}} \underbrace{\left|a_{1}\right\rangle}_{\mathbf{M}_{1}} \underbrace{\left|\psi_{1}\left(q_{1}, a_{1}\right)\right\rangle}_{\mathbf{P}_{1}} \underbrace{\left|a_{2}\right\rangle}_{\mathbf{M}_{2}} \underbrace{\left|\psi_{2}\left(q_{2}, a_{2}\right)\right\rangle}_{\mathbf{P}_{2}}),
\end{aligned}
$$

where each $\alpha_{i}\left(q_{i}, a_{i}\right)$ denotes the transition amplitude and each $\left|\psi_{i}\left(q_{i}, a_{i}\right)\right\rangle$ is a unit vector in the private space of $P_{i}^{(\mathrm{Q})}$.

Finally, $V^{(\mathrm{Q})}$ copies the contents of the message qubits shared with the quantum prover $P_{i}^{(\mathrm{Q})}$ to $\mathbf{A}_{i}$ to have the following state

$$
\sum_{v, q_{1}, q_{2}, a_{1}, a_{2}}(\sqrt{p\left(v, q_{1}, q_{2}\right)} \alpha_{1}\left(q_{1}, a_{1}\right) \alpha_{2}\left(q_{2}, a_{2}\right) \underbrace{|v\rangle}_{\mathbf{V}} \underbrace{\left|q_{1}\right\rangle}_{\mathbf{Q}_{1}} \underbrace{\left|q_{2}\right\rangle}_{\mathbf{Q}_{2}} \underbrace{\left|a_{1}\right\rangle}_{\mathbf{A}_{1}} \underbrace{\left|a_{2}\right\rangle}_{\mathbf{A}_{2}} \underbrace{\left|a_{1}\right\rangle}_{\mathbf{M}_{1}} \underbrace{\left|\psi_{1}\left(q_{1}, a_{1}\right)\right\rangle}_{\mathbf{P}_{1}} \underbrace{\left|a_{2}\right\rangle}_{\mathbf{M}_{2}} \underbrace{\left|\psi_{2}\left(q_{2}, a_{2}\right)\right\rangle}_{\mathbf{P}_{2}}),
$$

and does just the same computation as the classical verifier $V$ using $\mathbf{V}, \mathbf{M}_{1}$ and $\mathbf{M}_{2} . V^{(\mathrm{Q})}$ accepts the input if and only if $V$ accepts it.
(i) In case the input $x$ of length $n$ is in $L$ :

The quantum provers have only to answer in just the same way as the classical provers do, and $V^{(Q)}$ accepts $x$ with probability 1.
(ii) In case the input $x$ of length $n$ is not in $L$ :

Since no quantum interference occurs among the computational paths with different 4 -tuple ( $q_{1}, q_{2}, a_{1}, a_{2}$ ), and from the fact that any pair of classical provers cannot convince the classical verifier with probability more than $1 / 2$ (actually $1 / 2^{n}$ ), it is obvious that, for any pair of quantum provers, $V^{(\mathrm{Q})}$ accepts $x$ with probability at most $1 / 2$ (actually $1 / 2^{n}$ ).

As a result, from Theorem 11, Theorem 12, and Theorem 13 we have the main theorem.
Theorem $14 Q M I P=Q O C=N E X P$.
Since our proofs of Lemma 8 , Lemma 9 , and Lemma 10 do not depend on the accepting probabilities $a, b$, and the proof of Theorem 12 can be easily modified to two-sided bounded error cases, we have actually shown that the class of languages that have quantum multi-prover interactive proof systems (or quantum oracle circuits) with two-sided bounded error is equal to NEXP. From this fact and the proof of Theorem 13, we have the following corollary.

Corollary 15 If a language $L$ has a quantum multi-prover interactive proof system with two-sided bounded error, then L has a two-message quantum two-prover interactive proof system with exponentially small one-sided error.

## 6 Provers with Limited Prior Entanglement

In this section we mention the relation between our results and the model with prior entangled provers.
Here we consider the case in which each prover shares only polynomially many qubits prior entangled with other provers. Particular cases are the protocols with two provers sharing at most polynomially many EPR pairs. For the sake of generality, here we allow protocols with any number of provers and with any kind of prior entanglement, not limited to EPR-type ones. We show that if a language $L$ has a quantum multi-prover interactive proof system allowing at most polynomially many prior entangled qubits among provers, $L$ is necessarily in NEXP.

For every function $q_{\text {ent }}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$, let us say that a quantum multi-prover interactive proof system is with $q_{\text {ent }}$-prior-entanglement if, for every input $x$ of length $n$, each prover has at most $q_{\text {ent }}(n)$ private qubits prior entangled with other provers. Then Lemma 8 can be extended to the following corollary.

Corollary 16 Let $k, m, q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\text {ent }}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be polynomially bounded functions, and $q_{\mathcal{P}}: \mathbb{Z}^{+} \rightarrow \mathbb{N}$ be a function. For any protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ of an m-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\mathcal{P}}\right)$-restricted quantum $k$ prover interactive proof system with $q_{\text {ent }}$-prior-entanglement, there exists a protocol $\left(Q_{1}, \ldots, Q_{k}, W\right)$ of an m-message $\left(q_{\mathcal{V}}, q_{\mathcal{M}}, q_{\text {ent }}+2\lfloor m / 2+1 / 2\rfloor q_{\mathcal{M}}\right)$-restricted quantum $k$-prover interactive proof system with $q_{\text {ent }}$-prior-entanglement such that, for any input $x,\left(Q_{1}, \ldots, Q_{k}, W\right)$ accepts $x$ with just the same probability as $\left(P_{1}, \ldots, P_{k}, V\right)$ does.

Proof. The proof is almost same as the proof of Lemma 8. We assume that the values of $m$ are even (odd cases can be dealt with a similar argument). Instead of each $\left|\psi_{j}\right\rangle,\left|\phi_{j}\right\rangle$ in the proof of Lemma 8 , consider the following $\left|\Psi_{j}\right\rangle,\left|\Phi_{j}\right\rangle \in \mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k} \otimes \mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}$ for $0 \leq j \leq m / 2$ defined by

$$
\begin{array}{ll}
\left|\Phi_{1}\right\rangle=V_{1}\left|\Psi_{\mathrm{init}}\right\rangle, & \\
\left|\Phi_{j}\right\rangle=V_{j} P_{k, j-1} \cdots P_{1, j-1}\left|\Phi_{j-1}\right\rangle, & 2 \leq j \leq m / 2 \\
\left|\Psi_{j}\right\rangle=P_{1, j}\left|\Phi_{j}\right\rangle, & 1 \leq j \leq m / 2
\end{array}
$$

Here $\left|\Psi_{\text {init }}\right\rangle \in \mathcal{V} \otimes \mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{k} \otimes \mathcal{P}_{1} \otimes \cdots \otimes \mathcal{P}_{k}$ is the initial state where each qubit in $\in \mathcal{V} \otimes \mathcal{M}_{1} \otimes$ $\cdots \otimes \mathcal{M}_{k}$ is the $|0\rangle$ state, and the first $q_{\text {ent }}(n)$ qubits in each $\mathcal{P}_{k}$ may be entangled with private qubits of other provers than $P_{k}$. Without loss of generality, we can assume that all of the last $q_{\mathcal{P}}-q_{\text {ent }}$ qubits in each $\mathcal{P}_{k}$ are the $|0\rangle$-states in the state $\left|\Psi_{\text {init }}\right\rangle$.

Then with similar arguments to the proof of Lemma , we can construct each $P_{1, j}^{\prime},\left|\Phi_{j}^{\prime}\right\rangle,\left|\Psi_{j}^{\prime}\right\rangle, 1 \leq$ $j \leq m / 2$ to satisfy

- $P_{1, j}^{\prime}=P_{1, j}^{\prime \prime} \otimes I_{q_{\mathcal{P}}-q_{\mathrm{ent}}-2 j q_{\mathcal{M}}}$ where $P_{1, j}^{\prime \prime}$ is a unitary transformation acting on $\mathcal{M}_{1}$ and the first $q_{\text {ent }}+2 j q_{\mathcal{M}}$ qubits of $\mathcal{P}_{1}$, and $I_{q_{\mathcal{P}}-q_{\text {ent }}-2 j q_{\mathcal{M}}}$ is $\left(q_{\mathcal{P}}-q_{\text {ent }}-2 j q_{\mathcal{M}}\right)$-dimensional identity matrix,
- $\left|\Phi_{1}^{\prime}\right\rangle=V_{1}\left|\Psi_{\text {init }}\right\rangle$, $\left|\Phi_{j}^{\prime}\right\rangle=V_{j} P_{k, j-1} \cdots P_{2, j-1} P_{1, j-1}^{\prime}\left|\Phi_{j-1}^{\prime}\right\rangle$, $\left|\Psi_{j}^{\prime}\right\rangle=P_{1, j}^{\prime}\left|\Phi_{j}^{\prime}\right\rangle$,
- $\operatorname{tr}_{\mathcal{P}_{1}}\left|\Psi_{j}\right\rangle\left\langle\Psi_{j}\right|=\operatorname{tr}_{\mathcal{P}_{1}}\left|\Psi_{j}^{\prime}\right\rangle\left\langle\Psi_{j}^{\prime}\right|$
- all the qubits but the first $q_{\text {ent }}+2(j-1) q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\Phi_{j}^{\prime}\right\rangle$,
- all the qubits but the first $q_{\text {ent }}+2 j q_{\mathcal{M}}$ qubits in $\mathcal{P}_{1}$ are the $|0\rangle$-states in the state $\left|\Psi_{j}^{\prime}\right\rangle$.

Thus, for every input $x$, the protocol $\left(P_{1}^{\prime}, P_{2}, \ldots, P_{k}, V\right)$ accepts $x$ with just the same probability as the original protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ does, and $P_{1}^{\prime}$ uses only $q_{\text {ent }}+m q_{\mathcal{M}}$ qubits in his private space.

Repeating above process for each of provers gives a protocol $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ in which all $k$ provers use only $q_{\text {ent }}+m q_{\mathcal{M}}$ qubits in their private spaces, and the protocol $\left(P_{1}^{\prime}, \ldots, P_{k}^{\prime}, V\right)$ accepts $x$ with just the same probability as the original protocol $\left(P_{1}, \ldots, P_{k}, V\right)$ does. From this, it is easy to conclude that the corollary holds.

Now it is straightforward to conclude the following theorem.
Theorem 17 Let L be a language that has a quantum multi-prover interactive proof system allowing at most polynomially many prior entangled qubits among provers. Then $L$ is in NEXP.

## 7 Conclusions and Open Problems

This paper analyzed the power of quantum multi-prover interactive proof systems with prior unentangled provers, which gives the first quantum characterization of NEXP. In the proof we introduced the model of quantum oracle circuits, which are quantum single-prover interactive proof systems in which the prover does not have his private qubits, and this gives another quantum characterization of NEXP.

A number of interesting problems remain open regarding quantum interactive proof systems.

- If provers are allowed to share prior entanglement, how does the power of quantum multi-prover interactive proof systems change? Note that, if the number of prior entangled qubits is polynomially bounded, we have shown that the quantum multi-prover interactive proof systems cannot be stronger than NEXP. Furthermore, with prior entangled provers, as Cleve [11] discussed, the power of multi-prover interactive proof systems is unclear even if the verifier remains a classical one.
- Probabilistic oracle machines are closely related to the theory of probabilistic checkable proofs [3, 2]. How is the relation between the quantum oracle circuits introduced in this paper and possible quantum analogues of probabilistic checkable proofs?
- In the classical setting the power of one-message multi-prover interactive proof systems obviously remains same as that of one-message single-prover one. However, as Kobayashi, Matsumoto, and Yamakami [24] noticed, it might not be so in the quantum setting. How is the power of onemessage quantum multi-prover interactive proof systems (both in the cases with and without prior entanglement)?


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## References

[1] D. Aharonov, A. Kitaev, and N. Nisan. Quantum circuits with mixed states. In Proceedings of the 30th Annual ACM Symposium on Theory of Computing, pages 20-30, 1998.
[2] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and the hardness of approximation problems. Journal of the ACM, 45(3):501-555, 1998.
[3] S. Arora and S. Safra. Probabilistic checking of proofs: a new characterization of NP. Journal of the ACM, 45(1):70-122, 1998.
[4] L. Babai. Trading group theory for randomness. In Proceedings of the 17th Annual ACM Symposium on Theory of Computing, pages 421-429, 1985.
[5] L. Babai, L. Fortnow, and C. Lund. Non-deterministic exponential time has two-prover interactive protocols. Computational Complexity, 1(1):3-40, 1991.
[6] M. Bellare, U. Feige, and J. Kilian. On the role of shared randomness in two prover proof systems. In Proceedings of the 3rd Israel Symposium on the Theory of Computing and Systems, pages 199-208, 1995.
[7] C. H. Bennett, E. Bernstein, G. Brassard, and U. V. Vazirani. Strengths and weaknesses of quantum computing. SIAM Journal on Computing, 26(3):1510-1523, 1997.
[8] M. Ben-Or, S. Goldwasser, J. Kilian, and A. Wigderson. Multi-prover interactive proofs: how to remove the intractability assumptions. In Proceedings of the 20th Annual ACM Symposium on Theory of Computing, pages 113-131, 1988.
[9] P. O. Boykin, T. Mor, M. Pulver, V. P. Roychowdhury, and F. Vatan. On universal and faulttolerant quantum computing: a novel basis and a new constructive proof of universality for Shor's basis. In Proceedings of the 40th Annual Symposium on Foundations of Computer Science, pages 486-494, 1999.
[10] J. Cai, A. Condon, and R. Lipton. On bounded round multi-prover interactive proof systems. In Proceedings of the 5th Annual Conference on Structure in Complexity Theory, pages 45-54, 1990.
[11] R. Cleve. An entangled pair of provers can cheat. Talk at the Workshop on Quantum Computation and Information, California Institute of Technology, November 2000.
[12] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. Proceedings of the Royal Society London A, 400:97-117, 1985.
[13] U. Feige. On the success probability of two provers in one-round proof systems. In Proceedings of the 6th Annual Conference on Structure in Complexity Theory, pages 116-123, 1991.
[14] U. Feige and L. Lovász. Two-prover one-round proof systems: their power and their problems. In Proceedings of the 24th Annual ACM Symposium on Theory of Computing, pages 733-744, 1992.
[15] S. A. Fenner, F. Green, S. Homer, and R. Pruim. Determining acceptance possibility for a quantum computation is hard for the polynomial hierarchy. Proceedings of the Royal Society London A, 455:3953-3966, 1999.
[16] L. Fortnow. Complexity-theoretic aspects of interactive proof systems. Ph.D. Thesis, Department of Mathematics, Massachusetts Institute of Technology, May 1989. Technical Report MIT/LCS/TR-447.
[17] L. Fortnow and J. Rogers. Complexity limitations on quantum computation. Journal of Computer and System Sciences, 59(2):240-252, 1999.
[18] L. Fortnow, J. Rompel, and M. Sipser. On the power of multi-prover interactive protocols. Theoretical Computer Science, 134(2):545-557, 1994.
[19] S. Goldwasser, S. Micali, and C. Rackoff. The knowledge complexity of interactive proof systems. SIAM Journal on Computing, 18(1):186-208, 1989.
[20] S. Goldwasser and M. Sipser. Private coins versus public coins in interactive proof systems. In S. Micali, editor, Randomness and Computation, volume 5 of Advances in Computing Research, pages 73-90, JAI Press, 1989.
[21] J. Gruska. Quantum Computing. McGraw-Hill, 1999.
[22] L. Hughston, R. Jozsa, and W. Wootters. A complete classification of quantum ensembles having a given density matrix. Physics Letters A, 183:14-18, 1993.
[23] A. Kitaev and J. Watrous. Parallelization, amplification, and exponential time simulation of quantum interactive proof systems. In Proceedings of the 32nd Annual ACM Symposium on Theory of Computing, pages 608-617, 2000.
[24] H. Kobayashi, K. Matsumoto, and T. Yamakami. Quantum certificate verification: single versus multiple quantum certificates. Submitted. Los Alamos e-print archive, quant-ph/0110006, 2001.
[25] D. Lapidot and A. Shamir. Fully parallelized multi prover protocols for NEXP-time. In Proceedings of the 32nd Annual Symposium on Foundations of Computer Science, pages 13-18, 1991.
[26] C. Lund, L. Fortnow, H. Karloff, and N. Nisan. Algebraic methods for interactive proof systems. Journal of the ACM, 39(4):859-868, 1992.
[27] M. A. Nielsen. Entanglement and distributed quantum computation. Talk at the 4th Workshop on Quantum Information Processing, Amsterdam, January 2001.
[28] M. A. Nielsen and I. L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.
[29] C. H. Papadimitriou. Games against nature. Journal of Computer and System Sciences, 31(2):288-301, 1985.
[30] A. Shamir. IP = PSPACE. Journal of the ACM, 39(4):869-877, 1992.
[31] P. W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM Journal on Computing, 26(5):1484-1509, 1997.
[32] P. W. Shor. Fault-tolerant quantum computation. In Proceedings of the 37th Annual Symposium on Foundations of Computer Science, pages 56-65, 1996.
[33] A. Uhlmann. Parallel transport and "quantum holonomy" along density operators. Reports on Mathematical Physics, 24:229-240, 1986.
[34] J. Watrous. Space-bounded quantum computation. Ph.D. Thesis, University of Wisconsin Madison, 1998.
[35] J. Watrous. Space-bounded quantum complexity. Journal of Computer and System Sciences, 59(2):281-326, 1999.
[36] J. Watrous. PSPACE has constant-round quantum interactive proof systems. In Proceedings of the 40 th Annual Symposium on Foundations of Computer Science, pages 112-119, 1999.
[37] T. Yamakami and A. C. Yao. $\mathrm{NQP}_{\mathbb{C}}=\mathrm{co-C}=\mathrm{P}$. Information Processing Letters, 71(2):63-69, 1999.


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