

# Optimal Bid Sequences for Multiple-Object Auctions with Unequal Budgets\*

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## Abstract

In a multiple-object auction, every bidder tries to win as many objects as possible with a bidding algorithm. This paper studies *position-randomized auctions*, which form a special class of multiple-object auctions where a bidding algorithm consists of an initial bid sequence and an algorithm for randomly permuting the sequence. We are especially concerned with situations where some bidders know the bidding algorithms of others. For the case of only two bidders, we give an optimal bidding algorithm for the disadvantaged bidder. Our result generalizes previous work by allowing the bidders to have unequal budgets. One might naturally anticipate that the optimal expected numbers of objects won by the bidders would be proportional to their budgets. Surprisingly, this is not true. Our new algorithm runs in optimal  $O(n)$  time in a straightforward manner. The case with more than two bidders is open.

## 1 Introduction

Economists have long recognized the usefulness of auction as a means of price determination without intermediary market makers. As a result, there already exists an enormous Economics literature on auction theory and practice (see, e.g., [12, 16–18, 20, 27]). Relatively recently, computer scientists have become aware of the potential efficiency of auction as a general method of resource allocation [7]. For instance, Gagliano, Fraser, and Schaefer [11] applied auction techniques to allocating decentralized network resources. Bertsekas [2] designed an auction-type algorithm for the classical maximum flow problem.

With the advent of the World Wide Web, Internet-based auction is rapidly becoming an essential buying and selling medium for both individuals and organizations. It is projected that most of the future Internet auctions will necessarily be conducted by software agents instead of human bidders and auctioneers [13, 21, 22, 28]. Consequently, there is an increasing need for highly efficient and sophisticated auction mechanisms and bidding algorithms. To meet this need, Computer Science is witnessing heightened research efforts on such mechanisms and algorithms.

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Among the several basic research themes that have emerged from these efforts, the following three are particularly relevant to this paper.

The first theme is multiple-object auction [1, 8, 9, 15, 19, 23], where each bidder may bid on several objects simultaneously instead of one at a time. The second theme is the informational security of auction. For instance, Cachin [6] and Stajano and Anderson [26] were concerned with the privacy of bidders. Sako [24] discussed how to hide information about losing bids. The third theme is the computational complexity of auction [9, 10, 15, 19, 23]. For example, Sandholm and Suri [25] and Akcoglu, Aspnes, DasGupta, and Kao [1] proposed general frameworks for tackling the computational hardness of the winner determination problem for combinatorial auction, which is a special form of multiple-object auction.

Along these three themes, Kao, Qi, and Tan [14] considered the *position-randomized* multiple-object auction model specified as follows:

- M1 There are  $m$  bidders competing for  $n$  objects, where  $m \geq 2$  and  $n \geq 1$ . Each bidder has a positive budget and aims to win as many objects as possible.
- M2 Each bidder submits to the auction (1) an initial sequence of  $n$  bids whose total may not exceed the bidder's budget and (2) a randomized algorithm for permuting the bids. Each bid must be positive or zero. The final bid sequence that a bidder actually uses in the auction is obtained by permuting her initial bid sequence with her bid-permuting algorithm. The  $i$ -th bid of each final sequence is for the  $i$ -th object. If an object has  $m'$  highest bids, then each of these  $m'$  bidders wins this object with probability  $\frac{1}{m'}$ .
- M3 Before submitting their initial bid sequences and bid-permuting algorithms, all bidders know  $n$ ,  $m$ , and the budget of each bidder. Furthermore, some bidders may also know the initial bid sequences and bid-permuting algorithms of others, but not the final bid sequences.

The assumption M3 addresses the extreme case about informational security where electronically transmitted information about bids may be legitimately or illegitimately revealed against the wishes of their bidders. To enforce this assumption, the model can be implemented in an Internet auction as follows. Before the auction starts, each bidder submits her initial bid sequence and bid-permuting algorithm to the trusted auctioneer. After the auction stops accepting any new bid, the auctioneer will execute the bid-permuting algorithm publicly. In such an implementation, while a bidder's initial bid sequence and bid-permuting algorithm may be leaked to others, her final bid sequence is not known to anyone including herself and the auctioneer, until the auction commences.

Kao et al. [14] also considered an assumption M3' alternative to M3. Under M3', each bidder may submit any bidding algorithm which generates a final bid sequence without necessarily specifying an initial bid sequence. Therefore, less information may be revealed under M3' than under M3; in other words, M3' is a weaker security assumption. Moreover, it is not even clear that under M3', a bidder's optimal probability distribution of all possible bids can be computed in finite time. For these two reasons, this paper does not use M3'.

Under the above model, Kao et al. [14] gave optimal bidding algorithms for the case where (1) all bidders have equal budget, (2) every bid must have a positive dollar amount, and (3) the number of bidders is two or is an integral divisor of the number of objects. In this paper, we resolve only the case of two bidders where the *adversary* bidder  $\mathcal{A}$  knows the *disadvantaged* bidder  $\mathcal{D}$ 's initial bid sequence and bid-permuting algorithm, but not vice versa. We give a new optimal bidding algorithm for  $\mathcal{D}$  which improves upon the previous results with two generalizations: (1) the bidders may have unequal budgets, and (2) bids with zero dollar amounts are allowed. These

two seemingly minor relaxations make the design and analysis of the new algorithm considerably more difficult than those of the previous algorithms [14]. For one thing, one might naturally anticipate that the optimal expected numbers of objects won by  $\mathcal{A}$  and  $\mathcal{D}$  would be proportional to their budgets. Surprisingly, this is not true (Corollary 3.8). Our new algorithm runs in optimal  $O(n)$  time in a straightforward manner. The case with more than two bidders is open.

To outline the organization of the rest of the paper, we give some technical definitions first. The bid set of a bidder refers to the multiset formed by the bids in her initial bid sequence. For convenience, we refer to an initial sequence and its corresponding bid set interchangeably. Let  $B_{\mathcal{A}}$  (respectively,  $B_{\mathcal{D}}$ ) be the bid set of  $\mathcal{A}$  (respectively,  $\mathcal{D}$ ). Let  $\pi_{\mathcal{A}}$  (respectively,  $\pi_{\mathcal{D}}$ ) be the bid-permuting algorithm of  $\mathcal{A}$  (respectively,  $\mathcal{D}$ ).  $\mathcal{A}$  may know  $\pi_{\mathcal{D}}$  and  $B_{\mathcal{D}}$ , while  $\mathcal{D}$  does not know  $\pi_{\mathcal{A}}$  and  $B_{\mathcal{A}}$ . We assume that  $\mathcal{A}$  is *oblivious* in the sense that  $\mathcal{A}$  does not know in advance the outcome of permuting  $B_{\mathcal{D}}$  with  $\pi_{\mathcal{D}}$ . Note that bidding against a non-oblivious adversary is trivial.

Let  $w(\pi_{\mathcal{A}}, \pi_{\mathcal{D}}, B_{\mathcal{A}}, B_{\mathcal{D}})$  be the expected number of objects that  $\mathcal{A}$  wins. Since an auction in our model is a zero-sum game over the objects, the expected number of objects that  $\mathcal{D}$  wins is exactly  $n - w(\pi_{\mathcal{A}}, \pi_{\mathcal{D}}, B_{\mathcal{A}}, B_{\mathcal{D}})$ . Let  $w^*(\pi_{\mathcal{D}}, B_{\mathcal{D}})$  be the maximum of  $w(\pi_{\mathcal{A}}, \pi_{\mathcal{D}}, B_{\mathcal{A}}, B_{\mathcal{D}})$  over all  $\pi_{\mathcal{A}}$  and  $B_{\mathcal{A}}$ . We give a bidding algorithm  $(\pi_{\mathcal{D}}^*, B_{\mathcal{D}}^*)$  which is optimal for  $\mathcal{D}$ , i.e.,

$$w^*(\pi_{\mathcal{D}}^*, B_{\mathcal{D}}^*) = \min_{\pi_{\mathcal{D}}, B_{\mathcal{D}}} w^*(\pi_{\mathcal{D}}, B_{\mathcal{D}}). \quad (1)$$

Note that the game has an infinite pure strategy space, so it is not immediately clear that von Neumann's min-max theorem is applicable [3–5].

It has been shown [14] that without loss of generality, (1)  $\mathcal{D}$  always uses the *uniform* bid-permuting algorithm  $\pi_{\text{unif}}$  which permutes a sequence  $x_1, \dots, x_n$  with equal probability for every permutation of the indices  $1, \dots, n$  and (2) thus,  $\mathcal{A}$  uses the *identity* bid-permuting algorithm  $\pi_{\text{id}}$  which leaves a sequence unchanged (see Fact 1). Therefore, our main task is to design an initial bid sequence for  $\mathcal{D}$ . A sequence  $x_1, x_2, \dots, x_\ell$  of bids is *proportional* if  $\frac{x_i}{x_j} = \frac{i}{j}$  holds for all  $1 \leq i, j \leq \ell$ . A bid is *unbeatable* if it is greater than the budget of  $\mathcal{A}$ . In this paper, we give a  $B_{\mathcal{D}}^*$  that consists of (i) a sequence of zero bids, (ii) a sequence of proportional bids, and (iii) a sequence of unbeatable bids. The length of each sequence, which could be zero, depends on the ratio  $R$  of the budget of  $\mathcal{A}$  over that of  $\mathcal{D}$ .

Section 2 details  $B_{\mathcal{D}}^*$ . Section 3 proves its optimality for  $\mathcal{D}$  by showing that Equation (1) holds. Section 4 concludes the paper with open problems.

## 2 The bidding algorithm of the disadvantaged bidder

This section gives an optimal bidding algorithm  $(\pi_{\mathcal{D}}^*, B_{\mathcal{D}}^*)$  for  $\mathcal{D}$ . All sets in this paper are multisets. Let  $|X|$  be the number of elements in  $X$  counting multiplicity. Let  $X^d = \bigcup_{i=1}^d X$ , for each positive integer  $d$ . Let  $X^0 = \emptyset$ . Let  $\text{sum}(X) = \sum_{x \in X} x$ . Let  $\beta$  be the budget of  $\mathcal{D}$ . Hence, the budget of  $\mathcal{A}$  is  $\beta R$ .

We discuss the case  $\frac{1}{n} \leq R < n$  first. Let  $\Psi = \{0\}^{\ell_0} \cup \left\{ \frac{\beta}{n - \ell_0} \right\}^{n - \ell_0}$ , where

$$\ell_0 = \begin{cases} n & R < \frac{1}{n}; \\ \left\lceil \frac{n-1}{n} \right\rceil & R = \frac{1}{n}; \\ 0 & R > n. \end{cases}$$

One can easily verify that  $(\pi_{\text{unif}}, \Psi)$  is an optimal bidding algorithm for  $\mathcal{D}$ , where  $w^*(\pi_{\text{unif}}, \Psi)$  equals  $\min \left\{ \frac{1}{2}, \frac{1}{n} \right\}$  for  $R = \frac{1}{n}$  and equals  $n - \ell_0$  for  $R < \frac{1}{n}$  or  $R > n$ .

Hence, the rest of the paper assumes  $\frac{1}{n} < R \leq n$ . In §2.1, we give a bid set  $\Psi$  for  $\mathcal{D}$ . In §2.2, we prove an upper bound on the number of objects that  $\mathcal{A}$  can win against  $\Psi$ . In §3, we prove a matching lower bound, thereby proving the optimality of  $\Psi$ .

## 2.1 An optimal bid set for the disadvantaged bidder

The next fact simplifies our analysis.

**Fact 1 (See [14])**

1. If  $B_{\mathcal{A}} \cap B_{\mathcal{D}} = \emptyset$ , then  $w(\pi_{\mathcal{A}}, \pi_{\text{unif}}, B_{\mathcal{A}}, B_{\mathcal{D}}) = w(\pi_{\text{id}}, \pi_{\text{unif}}, B_{\mathcal{A}}, B_{\mathcal{D}}) \leq w(\pi_{\text{unif}}, \pi_{\mathcal{D}}, B_{\mathcal{A}}, B_{\mathcal{D}})$  for any bid-permuting algorithms  $\pi_{\mathcal{A}}$  and  $\pi_{\mathcal{D}}$ .
2. If  $\pi_{\mathcal{D}} = \pi_{\text{unif}}$ , then  $\mathcal{A}$  has an optimal bidding algorithm with  $B_{\mathcal{A}} \cap B_{\mathcal{D}} = \emptyset$ .

By Fact 1, the rest of the paper may assume  $\pi_{\mathcal{D}} = \pi_{\text{unif}}$  without loss of generality. Thus, let  $\pi_{\mathcal{D}}^* = \pi_{\text{unif}}$ . Moreover, as long as  $B_{\mathcal{A}}$  and  $B_{\mathcal{D}}$  are disjoint, we may assume  $\pi_{\mathcal{A}} = \pi_{\text{id}}$ .

For any positive real numbers  $x$  and  $y$ , define  $\phi(x, y) = y \cdot \left( \left\lceil \frac{x}{y} \right\rceil - 1 \right)$ , which is the largest integral multiple of  $y$  that is less than  $x$ . Let  $\phi(x) = \phi(x, 1)$ . Clearly,  $\phi(x) = \frac{1}{y} \cdot \phi(xy, y)$ . Define

$$\Psi = \begin{cases} \{0\}^{\ell_1} \cup \left\{ \frac{\beta}{n-\ell_1} \right\}^{n-\ell_1} & \text{if } \frac{1}{n} < R \leq \frac{2}{n+1}; \\ \left\{ \frac{2i}{\ell_2(\ell_2+1)} \cdot \beta \mid i = 1, 2, \dots, \ell_2 \right\} \cup \{0\}^{n-\ell_2} & \text{if } \frac{2}{n+1} < R \leq n, \end{cases}$$

where

$$\begin{aligned} \ell_1 &= \phi\left(2n - \frac{2}{R} + 1\right); \\ \ell_2 &= \min\left\{n, \left\lfloor \frac{n}{R} \right\rfloor\right\}. \end{aligned}$$

Note that  $\frac{1}{n} < R \leq \frac{2}{n+1}$  implies  $0 < n - \frac{1}{R} < \ell_1 < n$ . Also,  $\frac{2}{n+1} < R \leq n$  implies  $1 \leq \ell_2 \leq n$ . Therefore,  $\Psi$  is well defined. Clearly,  $\text{sum}(\Psi) = \beta$ .

## 2.2 An upper bound on $\mathcal{A}$ 's winning

For each  $\ell = 1, 2, \dots, n$ , let

$$\begin{aligned} R_{\ell} &= \phi\left(R, \frac{2}{\ell(\ell+1)}\right); \\ f(\ell) &= n - \ell + \frac{\ell(\ell+1)R_{\ell}}{2n}. \end{aligned}$$

Define

$$\text{equilibrium}(n, R) = \begin{cases} \frac{\ell_1}{n} & \text{if } \frac{1}{n} < R \leq \frac{2}{n+1}; \\ f(\ell_2) & \text{if } \frac{2}{n+1} < R \leq n. \end{cases}$$

The next lemma provides an upper bound for  $w^*(\pi_{\text{unif}}, \Psi)$ .

**Lemma 2.1**  $w^*(\pi_{\text{unif}}, \Psi) \leq \text{equilibrium}(n, R)$ .

*Proof.*

Case 1:  $\frac{1}{n} < R \leq \frac{2}{n+1}$ . By  $\ell_1 > n - \frac{1}{R}$ , we know  $\frac{\beta}{n-\ell_1} > \beta R$ . Since  $\Psi$  contains  $n - \ell_1$  unbeatable bids, the lemma is proved.

Case 2:  $\frac{2}{n+1} < R \leq n$ . Let  $\Psi'$  consist of the nonzero bids in  $\Psi$ . It suffices to show that  $\mathcal{A}$  wins no more than  $\frac{\ell_2(\ell_2+1)R\ell_2}{2n}$  bids in  $\Psi'$  on average. By Fact 1(2),  $\mathcal{A}$  has an optimal algorithm  $(\pi_{\text{id}}, B_{\mathcal{A}})$  with  $B_{\mathcal{A}} \cap \Psi' = \emptyset$ . Clearly, for each bid  $x \in B_{\mathcal{A}}$ , if  $i$  is the largest index with  $\frac{2i\beta}{\ell_2(\ell_2+1)} < x$ , then  $x$  wins  $\frac{i}{n}$  bids in  $\Psi'$  on average. Hence, the unit price for  $\mathcal{A}$  to win a bid in  $\Psi'$  is greater than  $\frac{2n\beta}{\ell_2(\ell_2+1)}$ . By  $\pi_{\mathcal{D}} = \pi_{\text{unif}}$  and  $B_{\mathcal{A}} \cap \Psi' = \emptyset$ , the expected number of bids in  $\Psi'$  that  $B_{\mathcal{A}}$  wins is an integral multiple of  $\frac{1}{n}$ . Since the budget of  $\mathcal{A}$  is  $\beta R$ , the expected number of bids in  $\Psi'$  that  $\mathcal{A}$  wins is at most  $\phi\left(\frac{\ell_2(\ell_2+1)\beta R}{2n\beta}, \frac{1}{n}\right) = \frac{1}{n} \cdot \phi\left(\frac{\ell_2(\ell_2+1)R}{2}\right) = \frac{\ell_2(\ell_2+1)}{2n} \cdot \phi\left(R, \frac{2}{\ell_2(\ell_2+1)}\right) = \frac{\ell_2(\ell_2+1)R\ell_2}{2n}$ .  $\square$

### 3 The optimality of the bid set $\Psi$

The main result of this section is Theorem 3.7, which shows the optimality of  $\Psi$  by proving

$$w^*(\pi_{\text{unif}}, \Psi) = \min_{\pi_{\mathcal{D}}, B_{\mathcal{D}}} w^*(\pi_{\mathcal{D}}, B_{\mathcal{D}}). \quad (2)$$

Suppose  $B_{\mathcal{D}} = \{\beta_1, \beta_2, \dots, \beta_n\}$ , where  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$ . Without loss of generality, we may assume  $\text{sum}(B_{\mathcal{D}}) = \beta$ . Let  $B_\ell = \bigcup_{i=1}^\ell \{\beta_{n-\ell+i}\}$  and  $t_\ell = \text{sum}(B_\ell)$  for each  $\ell = 1, 2, \dots, n$ . For technical reason, define  $\beta_0 = 0$ .

#### 3.1 Technical lemmas

For each  $\ell = 1, 2, \dots, n$ , an  $\ell$ -set is a multiset over  $\{0, 1, \dots, \ell\}$ . For any  $\ell$ -set  $I$ , let  $\text{bsum}(I, \ell) = \sum_{i \in I} \beta_{n-\ell+i}$ . An  $\ell$ -set  $I$  satisfies *Property P* if the following conditions hold:

P1.  $|I| \leq n$ .

P2.  $\text{sum}(I) \geq \frac{R_\ell \ell(\ell+1)}{2}$ .

P3.  $\text{bsum}(I, \ell) + (n - |I|)\beta_{n-\ell} < \beta R$ .

For any positive real number  $q$ , an  $\ell$ -set  $I$  is an  $(\ell, q)$ -set if  $\text{sum}(I) \geq \frac{q\ell(\ell+1)}{2}$  and  $\text{bsum}(I, \ell) \leq qt_\ell$ . Clearly, the union of an  $(\ell, q_1)$ -set and an  $(\ell, q_2)$ -set is an  $(\ell, q_1 + q_2)$ -set.

**Lemma 3.1** *If there is an  $\ell$ -set set that satisfies Property P, then  $w^*(\pi_{\text{unif}}, B_{\mathcal{D}}) \geq f(\ell)$ .*

*Proof.* Let  $I$  be the  $\ell$ -set that satisfies Property P. By Property P1, the  $n$ -element set  $X = \{\beta_{n-\ell}\}^{n-|I|} \cup \{\beta_{n-\ell+i} \mid i \in I\}$  is well defined. By Property P3,  $\text{sum}(X) = \text{bsum}(I, \ell) + (n - |I|)\beta_{n-\ell} < \beta R$ . Therefore, there exists a positive number  $\delta$  such that  $B_{\mathcal{A}} = \bigcup_{x \in X} \{x + \delta\}$  satisfies  $\text{sum}(B_{\mathcal{A}}) \leq \beta R$  and  $B_{\mathcal{A}} \cap B_{\mathcal{D}} = \emptyset$ . Since each bid in  $B_{\mathcal{A}}$  is greater than  $\beta_{n-\ell}$ ,  $\mathcal{A}$  wins all  $n - \ell$  bids in  $B_{\mathcal{D}} - B_\ell$ . By Property P2, the expected number of bids in  $B_\ell$  that  $\mathcal{A}$  wins with  $B_{\mathcal{A}}$  is at least  $\frac{\text{sum}(I)}{n} \geq \frac{R_\ell \ell(\ell+1)}{2n}$ . Thus,  $w^*(\pi_{\text{unif}}, B_{\mathcal{D}}) \geq n - \ell + \frac{R_\ell \ell(\ell+1)}{2n} = f(\ell)$ .  $\square$

Roughly speaking, an  $(\ell, q)$ -set specifies a good bid set for  $\mathcal{A}$  that spends the budget effectively. For example, if  $I$  is an  $(n, R_n)$ -set with  $|I| \leq n$ , then, by  $\beta_0 = 0$  and  $R_n < R$ , one can easily verify that  $I$  satisfies Property P. The next lemma is crucial in designing cost-effective bid sets.

**Lemma 3.2** *For each  $\ell = 1, 2, \dots, n$ , the following statements hold.*

1. For each integer  $d \geq 0$ , there is an  $(\ell, \frac{2d}{\ell})$ -set  $I_1(\ell, d)$  with  $|I_1(\ell, d)| = 2d$ .
2. For each integer  $h$  with  $0 \leq h \leq \frac{\ell+1}{2}$ , there is an  $(\ell, 1 - \frac{2h}{\ell(\ell+1)})$ -set  $I_2(\ell, h)$  with  $\ell - 1 \leq |I_2(\ell, h)| \leq \ell$ .
3. For each integer  $k \geq 1$  and each  $h = 0, 1, \dots, \ell$ , there is an  $(\ell, k + \frac{2h}{\ell(\ell+1)})$ -set  $I_3(\ell, k, h)$  with  $k\ell + \lfloor \frac{2h}{\ell+1} \rfloor \leq |I_3(\ell, k, h)| \leq k\ell + \lceil \frac{2h}{\ell+1} \rceil$ .
4. For each integer  $d \geq 1$  and each  $h = 0, 1, \dots, \ell$ , there is an  $(\ell, \frac{2d}{\ell} + \frac{2h}{\ell(\ell+1)})$ -set  $I_4(\ell, d, h)$  with  $|I_4(\ell, d, h)| \leq 2d + 2$ .
5. If  $\ell \leq n - 1$ , then for each integer  $d \geq 0$ , there is an  $\ell$ -set  $I_5(\ell, d)$  with  $|I_5(\ell, d)| = 2d$ ,  $\text{sum}(I_5(\ell, d)) \geq \ell d$ , and  $\text{bsum}(I_5(\ell, d), \ell) \leq \frac{2d\ell+1}{\ell+1}$ .

*Proof.* Let  $L = \{1, 2, \dots, \ell\}$ . For each  $i = 0, 1, \dots, \ell$ , let  $x_i = \beta_{n-\ell+i}$ . Define  $y(i) = \frac{2it_\ell}{\ell(\ell+1)}$ , for any integer  $i$ . Let  $i_0 = \arg \max_{i \in L} x_i - y(i)$ . Clearly,  $x_i - x_{i_0} \leq y(i - i_0)$  holds for each  $i \in L$ . By  $\sum_{i \in L} x_i - y(i) = 0$ , we know  $x_{i_0} \geq y(i_0)$ . Let

$$\begin{aligned} i_1 &= \arg \min_{i \in L} x_i + x_{\ell-i+1}; \\ i_2 &= \arg \max_{i \in L} x_i + x_{\ell-i+1}; \\ i_3 &= \arg \max_{i \in L - \{\ell\}} x_i + x_{\ell-i}. \end{aligned}$$

Statement 1. Clearly, the inequality  $x_{i_1} + x_{\ell-i_1+1} \leq x_j + x_{\ell-j+1}$  holds for each  $j \in L$ . By averaging this inequality over all  $\ell$  values of  $j$ , we have  $x_{i_1} + x_{\ell+1-i_1} \leq \frac{2t_\ell}{\ell}$ . One can easily verify that the statement holds with  $I_1(\ell, d) = \{i_1, \ell - i_1 + 1\}^d$ .

Statement 2. If  $h = 0$  (respectively,  $h = \frac{\ell+1}{2}$ ), then one can easily verify that the statement holds with  $I_2(\ell, h) = L$  (respectively,  $I_2(\ell, h) = I_1(\ell, \frac{\ell-1}{2})$ ). If  $i_0 = h$ , then  $x_h \geq y(h)$ , and thus the statement holds with  $I_2(\ell, h) = L - \{h\}$ . If  $i_0 > h$ , then  $i_0 - h \in L$ , and thus the statement holds with  $I_2(\ell, h) = L \cup \{i_0 - h\} - \{i_0\}$ . It remains to prove the statement for the case  $1 \leq i_0 < h < \frac{\ell+1}{2}$ . Clearly,  $i_0 \notin \{\ell - h, \ell + 1 - h\}$  and  $\{\ell - h, \ell + 1 - h, i_0 - 2h + \ell, i_0 - 2h + \ell + 1\} \subseteq L$ . If  $x_{\ell-h} \geq y(\ell - h)$ , then the statement holds with  $I_2(\ell, h) = L \cup \{i_0 - 2h + \ell\} - \{i_0, \ell - h\}$ . If  $x_{\ell+1-h} \geq y(\ell + 1 - h)$ , then the statement holds with  $I_2(\ell, h) = L \cup \{i_0 - 2h + \ell + 1\} - \{i_0, \ell + 1 - h\}$ . Now we assume  $x_{\ell-h} < y(\ell - h)$  and  $x_{\ell+1-h} < y(\ell + 1 - h)$ . If  $\ell$  is even, then clearly  $i_2 \neq \ell + 1 - i_2$ . One can verify that the statement holds with  $I_2(\ell, h) = L \cup \{\ell + 1 - h\} - \{i_2, \ell + 1 - i_2\}$ . If  $\ell$  is odd, then clearly  $i_3 \neq \ell - i_3$ . If  $x_{i_3} + x_{\ell-i_3} \geq x_\ell$ , then let  $J = \{i_3, \ell - i_3\}$ ; otherwise, let  $J = \{\ell\}$ . Clearly,  $\text{bsum}(J, \ell) \geq \frac{2t_\ell}{\ell+1}$  and  $\text{sum}(J) = \ell$ . One can verify that the statement holds with  $I_2(\ell, h) = L \cup \{\ell - h\} - J$ .

Statement 3. If  $h = 0$ , then the statement holds with  $I_3(\ell, k, h) = L^k$ . If  $\frac{\ell+1}{2} \leq h \leq \ell$ , then the statement holds with  $I_3(\ell, k, h) = L^{k-1} \cup I_2(\ell, \ell - h)$ . If  $x_h \leq y(h)$ , then the statement holds with  $I_3(\ell, k, h) = L^k \cup \{h\}$ . It remains to consider the case that both  $1 \leq h \leq \frac{\ell}{2}$  and  $x_h > y(h)$  hold. If  $i_0 + 2h - \ell - 1 \in L$  and  $i_0 \neq h$ , then, by  $x_h > y(h)$ , the statement holds with  $I_3(\ell, k, h) = L^k \cup I_1(\ell, 1) \cup \{i_0 + 2h - \ell - 1\} - \{i_0, h\}$ . When either  $i_0 + 2h - \ell - 1 \notin L$  or  $i_0 = h$  holds, we show  $i_0 + h \in L$ , which implies that the statement holds with  $I_3(\ell, k, h) = L^k \cup \{i_0 + h\} - \{i_0\}$ . If  $i_0 = h$ , then  $i_0 + h \in L$  holds trivially. If  $i_0 \neq h$ , then, by  $2h \leq \ell$ , we know  $i_0 + 2h - \ell - 1 < i_0$ . By  $i_0 \in L$  and  $i_0 + 2h - \ell - 1 \notin L$ , we have  $i_0 + 2h \leq \ell + 1$ , and thus  $i_0 + h \in L$ .

Statement 4. If there is an  $i_4 \in \{0, 1, \dots, h\}$  such that  $x_{i_4} + x_{h-i_4} \leq y(h)$ , then the statement holds with  $I_4(\ell, d, h) = I_1(\ell, d) \cup \{i_4, h - i_4\}$ . If there is an  $i_5 \in \{1, \dots, \ell - h\}$  such that  $x_{h+i_5} + x_{\ell+1-i_5} \leq y(\ell + 1 + h)$ , then, by  $d \geq 1$ , the statement holds with  $I_4(\ell, d, h) = I_1(\ell, d - 1) \cup \{h + i_5, \ell + 1 - i_5\}$ . If no such  $i_4$  or  $i_5$  exists, then we have  $2t_\ell = \sum_{0 \leq i \leq h} (x_i + x_{h-i}) + \sum_{1 \leq i \leq \ell-h} (x_{h+i} + x_{\ell+1-i}) > (h+1)y(h) + (\ell-h)y(\ell+h+1) = 2t_\ell$ , a contradiction.

Statement 5. By  $\ell + 1 \leq n$  and Statement 1, there is an  $(\ell + 1, \frac{2d}{\ell+1})$ -set  $I_1(\ell + 1, d)$  with  $|I_1(\ell + 1, d)| = 2d$ . We show that the statement holds with  $I_5(\ell, d) = \{j - 1 \mid j \in I_1(\ell + 1, d)\}$ . By the proof for Statement 1,  $I_1(\ell + 1, d)$  is an  $(\ell + 1)$ -set not containing 0. Thus  $I_5(\ell, d)$  is an  $\ell$ -set. Clearly,  $|I_5(\ell, d)| = |I_1(\ell + 1, d)| = 2d$ ,  $\text{sum}(I_5(\ell, d)) = \text{sum}(I_1(\ell + 1, d)) - 2d \geq (\ell + 2)d - 2d = \ell d$ , and  $\text{bsum}(I_5(\ell, d), \ell) = \text{bsum}(I_1(\ell + 1, d), \ell + 1) \leq \frac{2d\ell+1}{\ell+1}$ .  $\square$

For each  $\ell = 1, 2, \dots, n$ , let  $\delta_\ell = (R - R_\ell) \frac{\ell(\ell+1)}{2}$ . Clearly,

$$R = R_\ell + \frac{2\delta_\ell}{\ell(\ell+1)}. \quad (3)$$

By  $R_\ell = \phi\left(R, \frac{2}{\ell(\ell+1)}\right)$ , we know  $0 < \delta_\ell \leq 1$ . Let  $k_\ell = \lfloor R_\ell \rfloor$ ,  $d_\ell = \lfloor (R_\ell - k_\ell) \frac{\ell}{2} \rfloor$ ,  $d'_\ell = \lfloor (R_\ell - k_\ell) \frac{\ell+1}{2} \rfloor$ ,  $h_\ell = \left(R_\ell - k_\ell - \frac{2d_\ell}{\ell}\right) \frac{\ell(\ell+1)}{2}$ , and  $h'_\ell = \left(R_\ell - k_\ell - \frac{2d'_\ell}{\ell}\right) \frac{\ell(\ell+1)}{2}$ . Since  $R_\ell$  is an integral multiple of  $\frac{2}{\ell(\ell+1)}$ , we know that  $k_\ell$ ,  $d_\ell$ ,  $d'_\ell$ ,  $h_\ell$ , and  $h'_\ell$  are integers with  $k_\ell = \lfloor R_\ell \rfloor$ ,  $0 \leq d_\ell < \frac{\ell}{2}$ ,  $0 \leq d'_\ell < \frac{\ell+1}{2}$ ,  $0 \leq h_\ell < \ell + 1$ ,  $0 \leq h'_\ell < \ell$ , and

$$R_\ell = k_\ell + \frac{2d_\ell}{\ell} + \frac{2h_\ell}{\ell(\ell+1)} \quad (4)$$

$$= k_\ell + \frac{2d'_\ell}{\ell+1} + \frac{2h'_\ell}{\ell(\ell+1)}. \quad (5)$$

One can easily verify that either  $d'_\ell = d_\ell$  or  $d'_\ell = d_\ell + 1$  holds. Moreover, if  $d'_\ell = d_\ell$ , then  $h'_\ell = d_\ell + h_\ell$ . If  $d'_\ell = d_\ell + 1$ , then  $h'_\ell = d_\ell + h_\ell - \ell < \frac{\ell}{2}$ .

**Lemma 3.3** *For each  $\ell = 1, 2, \dots, n - 1$ , we have*

1.  $k_{\ell+1} = k_\ell$  and

2.  $d_{\ell+1} = d'_\ell$ .

*Proof.*

Statement 1. Assume for a contradiction that  $k_i < k_j$  holds for some  $1 \leq i \neq j \leq n$ . By  $k_j \leq R_j < R$ , we know  $k_i \leq k_j - 1 \leq \lceil R \rceil - 2$ . It suffices to show  $\lceil R \rceil - k_i \leq 1$  as follows. If  $i$  is even, then, by  $d_i < \frac{i}{2}$ , we know  $d_i \leq \frac{i-2}{2}$ . By Equations (3) and (4),  $\delta_i \leq 1$ , and  $h_i < i + 1$ , we have  $\lceil R \rceil - k_i = \left\lceil \frac{2d_i}{i} + \frac{2(h_i + \delta_i)}{i(i+1)} \right\rceil \leq \left\lceil \frac{i-2}{i} + \frac{2(i+1)}{i(i+1)} \right\rceil = 1$ . If  $i$  is odd, then, by  $d'_i < \frac{i+1}{2}$ , we know  $d'_i \leq \frac{i-1}{2}$ . By Equations (3) and (5),  $\delta_i \leq 1$ , and  $h'_i < i$ , we have  $\lceil R \rceil - k_i = \left\lceil \frac{2d'_i}{i+1} + \frac{2(h'_i + \delta_i)}{i(i+1)} \right\rceil \leq \left\lceil \frac{i-1}{i+1} + \frac{2i}{i(i+1)} \right\rceil = 1$ .

Statement 2. By Equations (3), (4), and (5) and Statement 1, we have  $\frac{2d_{\ell+1}}{\ell+1} + \frac{2h_{\ell+1} + \delta_{\ell+1}}{(\ell+1)(\ell+2)} = \frac{2d'_\ell}{\ell+1} + \frac{2(h'_\ell + \delta_\ell)}{\ell(\ell+1)}$ . Therefore,  $d_{\ell+1} + \frac{h_{\ell+1} + \delta_{\ell+1}}{\ell+2} = d'_\ell + \frac{h'_\ell + \delta_\ell}{\ell}$ . By  $h'_\ell < \ell$ ,  $h_{\ell+1} < \ell + 2$ , and  $0 < \delta_\ell, \delta_{\ell+1} \leq 1$ , we have  $|d_{\ell+1} - d'_\ell| < 1$ , and thus  $d_{\ell+1} = d'_\ell$ .  $\square$

### 3.2 Matching lower bounds on $\mathcal{A}$ 's winning

Lemmas 3.4, 3.5, and 3.6 analyze the cases (1)  $\frac{1}{n} < R \leq \frac{2}{n+1}$ , (2)  $\frac{2}{n+1} < R \leq 1$ , and (3)  $1 < R \leq n$ , respectively. By Lemma 3.3(1), the rest of the section omits the subscript of  $k_\ell$ .

**Lemma 3.4** *If  $\frac{1}{n} < R \leq \frac{2}{n+1}$ , then  $w^*(\pi_{\text{unif}}, B_{\mathcal{D}}) \geq \frac{\ell_1}{n}$ .*

*Proof.* Let  $\ell$  be the number of bids in  $B_{\mathcal{D}}$  that are less than  $\beta R$ . If  $\ell \geq \ell_1$ , then the expected number of bids that  $\mathcal{A}$  wins with  $B_{\mathcal{A}} = \{0\}^{n-1} \cup \{\beta R\}$  is at least  $\frac{\ell}{n}$ , ensuring  $w^*(\pi_{\text{unif}}, B_{\mathcal{D}}) \geq \frac{\ell_1}{n}$ . The rest of the proof assumes  $\ell < \ell_1$ . By  $(n - \ell)\beta R \leq \sum_{j=\ell+1}^n \beta_j$ , we have  $\sum_{j=1}^{\ell} \beta_j \leq \beta R \left( \ell - n + \frac{1}{R} \right)$ . By  $(n - \ell)\beta R \leq \beta$ , we have  $\ell \geq n - \frac{1}{R}$ . By  $\ell_1 < 2n - \frac{2}{R} + 1$ , we know  $2\ell + 1 > \ell_1$ , which implies  $2\ell \geq \ell_1$ . Let  $i^* = \arg \min_{0 \leq i \leq 2\ell - \ell_1} \beta_{\ell-i} + \beta_{\ell_1 - \ell + i}$ . Let  $X = \{0\}^{n-2} \cup \{\beta_{\ell-i^*}, \beta_{\ell_1 - \ell + i^*}\}$ . Clearly,  $\text{sum}(X) \leq \frac{2 \sum_{j=\ell_1 - \ell}^{\ell} \beta_j}{2\ell - \ell_1 + 1} < \frac{\sum_{j=1}^{\ell} \beta_j}{\ell - n + R^{-1}} \leq \beta R$ . Let  $B_{\mathcal{A}} = \bigcup_{x \in X} \{x + \delta\}$ , where  $\delta$  is a number such that  $0 < \delta \leq \frac{\beta R - \text{sum}(X)}{n}$  and  $B_{\mathcal{A}} \cap B_{\mathcal{D}} = \emptyset$ . Since  $\text{sum}(B_{\mathcal{A}}) \leq \beta R$ ,  $|B_{\mathcal{A}}| = n$ , and the expected number of bids that  $\mathcal{A}$  wins with  $B_{\mathcal{A}}$  is at least  $\frac{\ell_1}{n}$ , the lemma is proved.  $\square$

**Lemma 3.5** *If  $\frac{2}{n+1} < R \leq 1$ , then  $w^*(\pi_{\text{unif}}, B_{\mathcal{D}}) \geq f(\ell_2)$ .*

*Proof.* By  $\frac{2}{n+1} < R \leq 1$ , we know  $\ell_2 = n \geq 2$  and  $f(\ell_2) = \frac{(n+1)R_n}{2}$ . By Lemma 3.1 and  $\beta_0 = 0$ , it suffices to show an  $(n, R_n)$ -set with at most  $n$  elements. If  $R_n = \frac{2}{n+1}$ , then, by  $n \geq 2$ ,  $\{i^*, n - i^*\}$  is a required  $(n, R_n)$ -set, where  $i^* = \arg \min_{1 \leq i \leq n} \beta_i + \beta_{n-i}$ . The rest of the proof assumes  $R_n > \frac{2}{n+1}$ . Since  $R_n$  is an integral multiple of  $\frac{2}{n(n+1)}$ , we know  $R_n \geq \frac{2}{n}$ . By  $R_n < R \leq 1$  and Equation (4), we know  $R_n = \frac{2d_n}{n} + \frac{2h_n}{n(n+1)}$ , where  $d_n \geq 1$  and  $0 \leq h_n \leq n$ . By Lemma 3.2(4), we know that  $I_4(n, d_n, h_n)$  is an  $(n, R_n)$ -set with  $|I_4(n, d_n, h_n)| \leq 2d_n + 2$ . It remains to consider the case  $2d_n + 2 > n$ . By  $d_n < \frac{n}{2}$ , we have  $d_n = \frac{n-1}{2}$ , and thus  $R_n = \frac{n-1}{n} + \frac{2h_n}{n(n+1)}$ . By  $R_n < 1$ , we know  $h_n < \frac{n+1}{2}$ . It follows that  $R_n = 1 - \frac{2h}{n(n+1)}$ , where  $0 < h = \frac{n+1}{2} - h_n \leq \frac{n+1}{2}$ . By Lemma 3.2(2),  $I_2(n, h)$  is an  $(n, R_n)$ -set with  $|I_2(n, h)| \leq n$ .  $\square$

**Lemma 3.6** *If  $1 < R \leq n$ , then  $w^*(\pi_{\text{unif}}, B_{\mathcal{D}}) \geq f(\ell_2)$ .*

*Proof.* For notational brevity, the proof omits the subscript of  $\ell_2$ . By  $1 < R \leq n$ , we know  $\ell = \lfloor \frac{n}{R} \rfloor$ ,  $1 \leq \ell \leq n - 1$ ,  $k \geq 1$ , and  $R\ell \leq n < R(\ell + 1)$ . We first show  $f(\ell) \leq f(\ell + 1)$  as follows. Let  $\Delta = f(\ell) - f(\ell + 1)$ . Clearly,  $\Delta = 1 + \frac{1}{n} \left( \left\lceil \frac{R\ell(\ell+1)}{2} \right\rceil - \left\lceil \frac{R(\ell+1)(\ell+2)}{2} \right\rceil \right)$ , and thus  $\Delta$  is an integral multiple of  $\frac{1}{n}$ . Therefore, it suffices to show  $\Delta < 1 + \frac{R}{2n} (\ell(\ell + 1) - (\ell + 1)(\ell + 2)) + \frac{1}{n} = 1 - \frac{R(\ell+1)}{n} + \frac{1}{n} < \frac{1}{n}$ .

By  $f(\ell) \leq f(\ell + 1)$  and Lemma 3.1, it suffices to show an  $\ell$ -set or an  $(\ell + 1)$ -set that satisfies Property P for each of the following cases.

Case 1:  $R\ell \leq n \leq \lfloor \ell R_\ell \rfloor + k$ . Let  $I = I_1(\ell, d_\ell) \cup I_3(\ell, k, h_\ell)$ . By Equation (4) and Lemmas 3.2(1) and 3.2(3), we know that  $I$  is an  $(\ell, R_\ell)$ -set with  $\lfloor \ell R_\ell \rfloor \leq |I| \leq \lceil \ell R_\ell \rceil \leq \lceil R\ell \rceil \leq n$ , proving Property P1. Being an  $(\ell, R_\ell)$ -set,  $I$  satisfies Property P2 and  $\text{bsum}(I, \ell) \leq R_\ell t_\ell$ . By  $|I| \geq \lfloor \ell R_\ell \rfloor \geq n - k$ ,  $k \leq R_\ell < R$ , and  $\beta_{n-\ell} + t_\ell \leq \beta$ , we know  $(n - |I|)\beta_{n-\ell} + \text{bsum}(I, \ell) \leq k\beta_{n-\ell} + R_\ell t_\ell < R(t_\ell + \beta_{n-\ell}) \leq \beta R$ . Therefore,  $I$  satisfies Property P3.

Case 2:  $\lfloor \ell R_\ell \rfloor + k + 1 \leq n \leq k(\ell + 1) + 2d'_\ell + \left\lceil \frac{2h'_\ell}{\ell+1} \right\rceil$ . Let  $I = I_5(\ell, d'_\ell) \cup I_3(\ell, k, h'_\ell)$ . By Equation (5),  $k \geq 1$ , and  $2d'_\ell < \ell + 1$ , we have  $\lfloor \ell R_\ell \rfloor + k = k\ell + \left\lceil \frac{2d'_\ell \ell + 2h'_\ell}{\ell+1} \right\rceil + k \geq$



$k\ell + \left\lfloor \frac{2d'_\ell \ell + 2h'_\ell + \ell + 1}{\ell + 1} \right\rfloor \geq k\ell + 2d'_\ell + \left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor$ . By Lemmas 3.2(3) and 3.2(5), we have  $n - k \leq k\ell + 2d'_\ell + \left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor \leq |I| \leq k\ell + 2d'_\ell + \left\lceil \frac{2h'_\ell}{\ell + 1} \right\rceil \leq \lfloor \ell R_\ell \rfloor + k + 1 \leq n$ , proving Property P1. By Lemmas 3.2(3) and 3.2(5) and Equation (5), we have  $\text{sum}(I) \geq \ell d'_\ell + \left(k + \frac{2h'_\ell}{\ell(\ell + 1)}\right) \frac{\ell(\ell + 1)}{2} = \frac{\ell(\ell + 1)}{2} R_\ell$ , proving Property P2. By  $|I| \geq n - k$ ,  $\beta_{n-\ell} + t_\ell = t_{\ell+1} \leq \beta$ , and Equation (5), we know  $(n - |I|)\beta_{n-\ell} + \text{bsum}(I, \ell) \leq k\beta_{n-\ell} + \frac{2d'_\ell}{\ell+1}t_{\ell+1} + \left(k + \frac{2h'_\ell}{\ell(\ell+1)}\right)t_\ell \leq R_\ell\beta < \beta R$ , proving Property P3.

Case 3:  $k(\ell + 1) + 2d'_\ell + \left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor + 1 \leq n < R(\ell + 1)$ . By  $n < R(\ell + 1)$ , we have  $n \leq \lceil R(\ell + 1) \rceil - 1$ . By  $\ell + 1 \leq n$  and Equations (3) and (5), we have  $\lceil R(\ell + 1) \rceil = k(\ell + 1) + 2d'_\ell + \left\lceil \frac{2(h'_\ell + \delta_\ell)}{\ell} \right\rceil$ . By  $k(\ell + 1) + 2d'_\ell + \left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor + 1 \leq n \leq \lceil R(\ell + 1) \rceil - 1$ , we have  $\left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor + 2 \leq \left\lceil \frac{2(h'_\ell + \delta_\ell)}{\ell} \right\rceil$ . It follows from  $h'_\ell + \delta_\ell \leq \ell$  and  $h'_\ell \geq 0$  that  $\left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor = 0$  and  $\left\lceil \frac{2(h'_\ell + \delta_\ell)}{\ell} \right\rceil = 2$ . By  $k(\ell + 1) + 2d'_\ell + \left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor + 1 \leq n \leq k(\ell + 1) + 2d'_\ell + \left\lceil \frac{2(h'_\ell + \delta_\ell)}{\ell + 1} \right\rceil - 1$ , we know  $n = k(\ell + 1) + 2d'_\ell + 1$ . By Lemma 3.3(2) and Equations (3), (4), and (5), we have  $\left\lceil \frac{2(h_{\ell+1} + \delta_{\ell+1})}{\ell + 2} \right\rceil = R(\ell + 1) - (k(\ell + 1) + 2d_{\ell+1}) = R(\ell + 1) - (k(\ell + 1) + 2d'_\ell) = \left\lceil \frac{2(h'_\ell + \delta_\ell)}{\ell} \right\rceil = 2$ . Therefore  $1 < \frac{2(h_{\ell+1} + \delta_{\ell+1})}{\ell + 2} \leq 2$ . By  $0 < \delta_{\ell+1} \leq 1$  and  $\ell \geq 1$ , we have  $0 < \frac{2h_{\ell+1}}{\ell + 2} < 2$ , and thus  $\left\lfloor \frac{2h_{\ell+1}}{\ell + 2} \right\rfloor \leq 1 \leq \left\lceil \frac{2h_{\ell+1}}{\ell + 2} \right\rceil$ . It follows from  $n = k(\ell + 1) + 2d'_\ell + 1$ , Equation (4), and Lemma 3.3(2) that  $\lfloor R_{\ell+1}(\ell + 1) \rfloor \leq n \leq \lceil R_{\ell+1}(\ell + 1) \rceil$ . We prove the statement for the following two sub-cases.

Case 3(a):  $n = \lceil R_{\ell+1}(\ell + 1) \rceil$ . Let  $I = I_1(\ell + 1, d_{\ell+1}) \cup I_3(\ell + 1, k, h_{\ell+1})$ . By Equation (4), Lemmas 3.2(1) and 3.2(3), we know that  $I$  is an  $(\ell + 1, R_{\ell+1})$ -set with  $\lfloor (\ell + 1)R_{\ell+1} \rfloor \leq |I| \leq \lceil (\ell + 1)R_{\ell+1} \rceil = n$ , satisfying Property P1. Being an  $(\ell + 1, R_{\ell+1})$ -set,  $I$  satisfies Property P2 and  $\text{bsum}(I, \ell + 1) \leq R_{\ell+1}t_{\ell+1}$ . By  $|I| \geq \lfloor (\ell + 1)R_{\ell+1} \rfloor \geq n - 1 \geq n - k$ ,  $k \leq R_{\ell+1} < R$ , and  $\beta_{n-\ell-1} + t_{\ell+1} \leq \beta$ , we know  $\text{bsum}(I, \ell + 1) + (n - |I|)\beta_{n-\ell-1} \leq R_{\ell+1}t_{\ell+1} + k\beta_{n-\ell-1} < R(t_{\ell+1} + \beta_{n-\ell-1}) \leq \beta R$ , satisfying Property P3.

Case 3(b):  $n = \lfloor R_{\ell+1}(\ell + 1) \rfloor$ . Let  $J_1 = I_3(\ell, k, 0) \cup I_5(\ell, d'_\ell) \cup \{h'_\ell\}$ . Let  $J_2 = I_3(\ell, k, 0) \cup I_5(\ell, d'_\ell + 1) - \{h'_\ell\}$ . By the proof of Lemma 3.2(3), we know  $h'_\ell \in \{1, 2, \dots, \ell\} \subseteq I_3(\ell, k, 0)$ . Therefore,  $|J_1| = |J_2| = k\ell + 2d'_\ell + 1 = n - k$ . By  $\left\lfloor \frac{2h'_\ell}{\ell + 1} \right\rfloor = 0$ , we know  $\ell - h'_\ell \geq h'_\ell$ . By  $\ell - h'_\ell \geq h'_\ell$  and Lemmas 3.2(1) and 3.2(3), one can verify that each of  $J_1$  and  $J_2$  satisfies Properties P1 and P2. It remains to show that either  $J_1$  or  $J_2$  satisfies Property P3 as follows. If  $\beta_{n-\ell+h'_\ell} < \frac{2(h'_\ell + \delta_\ell)\beta}{\ell(\ell + 1)}$ , then, by  $t_{\ell+1} = \beta_{n-\ell} + t_\ell \leq \beta$  and Equations (3) and (5), we know  $\text{bsum}(J_1, \ell) + (n - |J_1|)\beta_{n-\ell} < \frac{2d'_\ell}{\ell+1}t_{\ell+1} + kt_\ell + \frac{2(h'_\ell + \delta_\ell)\beta}{\ell(\ell+1)} + k\beta_{n-\ell} \leq \beta R$ . Thus  $J_1$  satisfies Property P3. Now we assume  $\beta_{n-\ell+h'_\ell} \geq \frac{2(h'_\ell + \delta_\ell)\beta}{\ell(\ell+1)}$ . By  $\left\lceil \frac{2(h'_\ell + \delta_\ell)}{\ell} \right\rceil = 2$ , we know  $h'_\ell + \delta_\ell > \frac{\ell}{2}$ , and thus  $\ell - (h'_\ell + \delta_\ell) < h'_\ell + \delta_\ell$ . It follows from  $t_\ell \leq t_{\ell+1} \leq \beta$  and Equations (3) and (5) that  $\text{bsum}(J_2, \ell) + (n - |J_2|)\beta_{n-\ell} < kt_\ell + \frac{2(d'_\ell + 1)t_{\ell+1}}{\ell+1} - \frac{2(h'_\ell + \delta_\ell)\beta}{\ell(\ell+1)} + k\beta_{n-\ell} \leq \left(k + \frac{2d'_\ell}{\ell+1} + \frac{2(h'_\ell + \delta_\ell)}{\ell(\ell+1)}\right)\beta = \beta R$ . Thus  $J_2$  satisfies Property P3.  $\square$

**Theorem 3.7**  $(\pi_{\text{unif}}, \Psi)$  is an optimal bidding algorithm for  $\mathcal{D}$ . Furthermore,  $w^*(\pi_{\text{unif}}, \Psi) = \text{equilibrium}(n, R)$ .

*Proof.* Clearly,  $w^*(\pi_{\text{unif}}, \Psi) \geq \min_{\pi_{\mathcal{D}}, B_{\mathcal{D}}} w^*(\pi_{\mathcal{D}}, B_{\mathcal{D}})$  holds trivially. By Lemmas 2.1, 3.4, 3.5, and 3.6, we know that  $w^*(\pi_{\text{unif}}, \Psi) \leq \text{equilibrium}(n, R) \leq w^*(\pi_{\text{unif}}, B_{\mathcal{D}})$  holds for any bid set  $B_{\mathcal{D}}$  of  $\mathcal{D}$ . Therefore, we have Equation (2), and thus the equality  $w^*(\pi_{\text{unif}}, \Psi) = \text{equilibrium}(n, R)$ .  $\square$

By Theorem 3.7, the optimal expected winning of  $\mathcal{A}$  (respectively,  $\mathcal{D}$ ) is  $\text{equilibrium}(n, R)$  (respectively,  $n - \text{equilibrium}(n, R)$ ). We define  $\mathcal{A}$ 's *effective winning ratio*  $E_{\mathcal{A}}(n, R)$  to be

$$\frac{\text{equilibrium}(n, R)}{\frac{nR}{R+1}}.$$

Similarly,  $\mathcal{D}$ 's *effective winning ratio*  $E_{\mathcal{D}}(n, R)$  is

$$\frac{n - \text{equilibrium}(n, R)}{\frac{n}{R+1}}.$$

Note that  $\frac{R}{R+1}$  (respectively,  $\frac{1}{R+1}$ ) is the fraction of  $\mathcal{A}$ 's (respectively,  $\mathcal{D}$ 's) budget in the total budget of  $\mathcal{A}$  and  $\mathcal{D}$ . One might intuitively expect that  $\mathcal{A}$  (respectively,  $\mathcal{D}$ ) would win  $\frac{nR}{R+1}$  (respectively,  $\frac{n}{R+1}$ ) objects optimally on average. In other words,  $E_{\mathcal{A}}(n, R) = E_{\mathcal{D}}(n, R) = 1$ . Surprisingly, these equalities are not true, as shown in the next corollary.

Figures 1 and 2 show  $E_{\mathcal{D}}(n, R)$  in 3D plots. Figure 3 shows  $E_{\mathcal{D}}(n, R)$  for some values of  $R$  in 2D plots.

### Corollary 3.8

1. If  $R \geq 1$ , then  $\lim_{n \rightarrow \infty} E_{\mathcal{A}}(n, R) = \frac{(2R-1)(R+1)}{2R^2}$  and  $\lim_{n \rightarrow \infty} E_{\mathcal{D}}(n, R) = \frac{R+1}{2R}$ .
2. If  $R \leq 1$ , then  $\lim_{n \rightarrow \infty} E_{\mathcal{A}}(n, R) = \frac{R+1}{2}$  and  $\lim_{n \rightarrow \infty} E_{\mathcal{D}}(n, R) = \frac{(2-R)(R+1)}{2}$ .

*Proof.* Straightforward.  $\square$

*Remark.* The formulas in Corollary 3.8 are symmetric in the sense that those in Statement 1 can be obtained from Statement 2 by replacing  $R$  with  $\frac{1}{R}$ .

## 4 Open problems

This paper solves the case with two bidders. The case with more than two bidders remains open. Another research direction is auction with *collusion*. Note that our model is equivalent to auction with colluding groups where the bidders all have equal budgets, and those in the same group pool their money. For example, if the budgets of two money-pooling bidders are \$100 and \$100, then either of them can make a bid of \$150. If pooling is not allowed, then neither can make a bid of \$150. It would be of interest to optimally or approximately achieve game-theoretic equilibria for auctions with non-pooling collusion.

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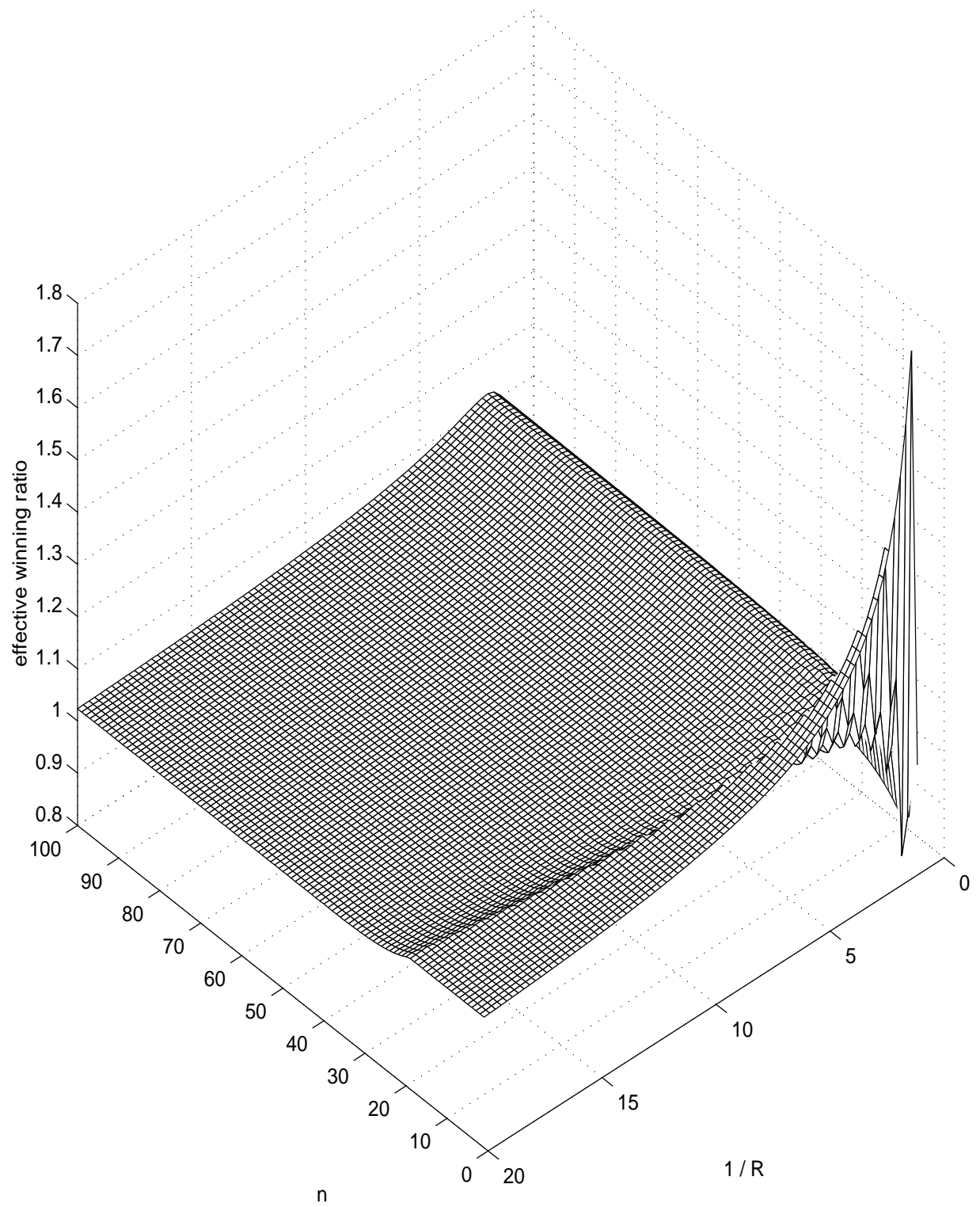


Figure 1: The values of  $E_{\mathcal{D}}$  for  $1 \leq n \leq 100$  and  $\frac{1}{20} \leq R \leq 1$ .

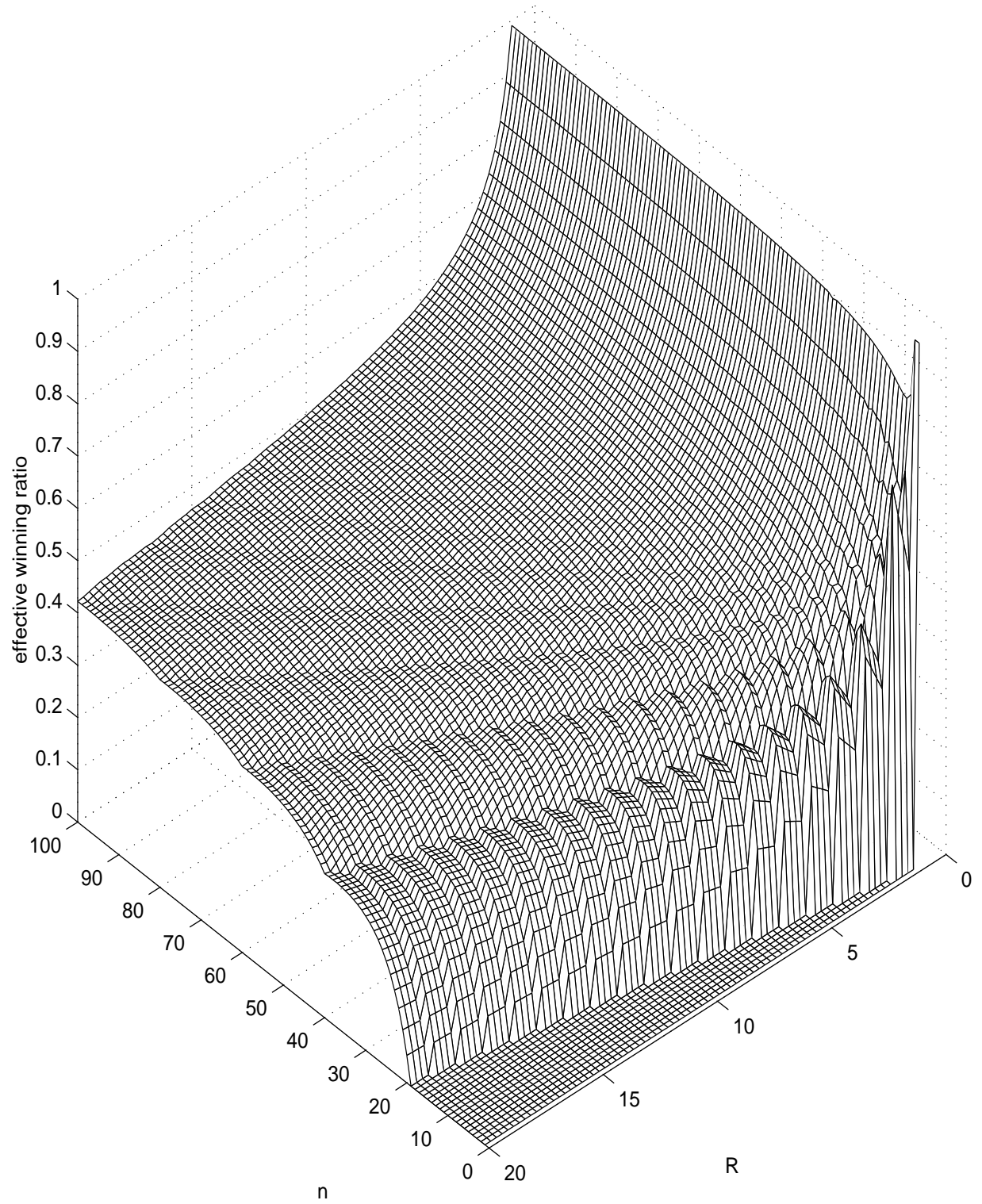


Figure 2: The values of  $E_{\mathcal{D}}$  for  $1 \leq n \leq 100$  and  $1 \leq R \leq 20$ .

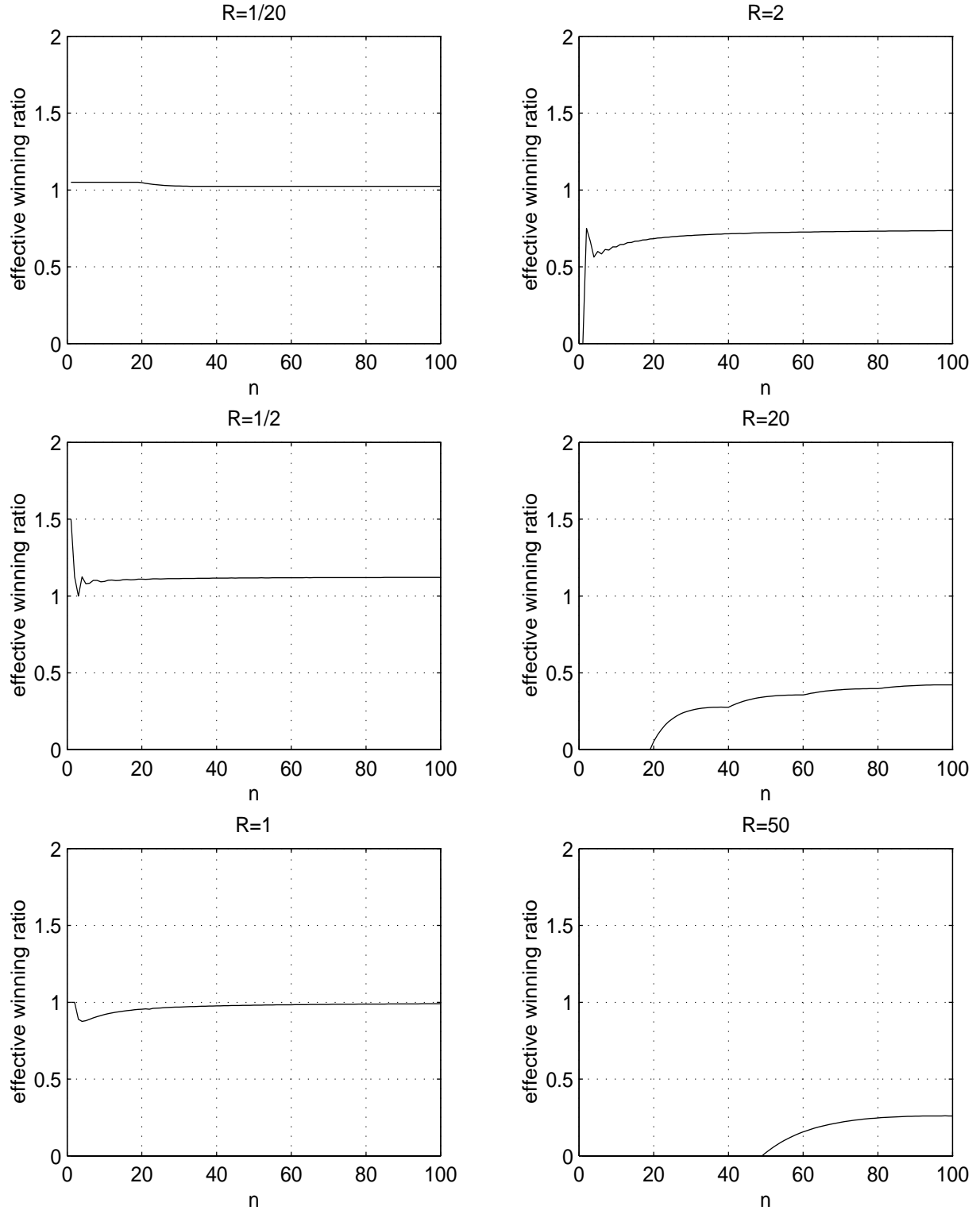


Figure 3: The values of  $E_{\mathcal{D}}$  for  $1 \leq n \leq 100$  and  $R \in \left\{ \frac{1}{20}, \frac{1}{2}, 1, 2, 20, 50 \right\}$ .

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