ω-Searchlight Obedient Graph Drawings

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Abstract. A drawing of a graph in the plane is *ω-searchlight obedient* if every vertex of the graph is located on the centerline of some strip of width ω, which does not contain any other vertex of the graph. We estimate the maximum possible value $\omega(n)$ of an ω-searchlight obedient drawing of a graph with n vertices, which is contained in the unit square. We show a lower bound and an upper bound on $\omega(n)$, namely, $\omega(n) = \Omega(\log n/n)$ and $\omega(n) = O(1/n^{4/7-\varepsilon})$, for an arbitrarily small $\varepsilon > 0$. Any improvement for either bound will also carry on to the famous Heilbronn's triangle problem.

Keywords: Geometric optimization, Heilbronn's triangle problem.

1 Introduction

In this paper we represent a graph-drawing problem as an optimization problem in combinatorial geometry, and establish a lower and an upper bound for the solution of this problem. Such "Erdős-type" problems attracted much attention throughout the last century. The interested reader is referred to the rich literature on combinatorial geometry; see, e.g., [2,3,6]. Specifically, we would like to draw a graph within the unit square, such that each vertex of the graph is covered by a strip associated with it, and no vertex is covered by a strip associated with another vertex. Our goal is to maximize the width of the strips.

A *searchlight* is a strip in $\mathbb{R}^2$, that is characterized by three parameters:

1. A point, the *source* of the searchlight, which lies on the centerline of the strip.
2. A direction which defines the orientation of the strip.
3. The width of the strip.

We focus on searchlights whose source points are located in the unit square. A valid collection of searchlights has the property that no searchlight source point is covered by any other searchlight. We are interested in the following problem:

> Given a valid collection of n searchlights of width ω in the unit square, what is the maximum possible width $\omega(n)$ of the searchlights?

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The quantity $\omega(n)$ is the maximum possible value of ω for an *ω-searchlight obedient* drawing within the unit square of a graph with n vertices. It is possible to superimpose such a graph with n strips of width ω, such that every strip contains one vertex on its centerline, but it does not contain any other vertex of the graph; see Figure 1. The edges of the graph in the figure are drawn entirely

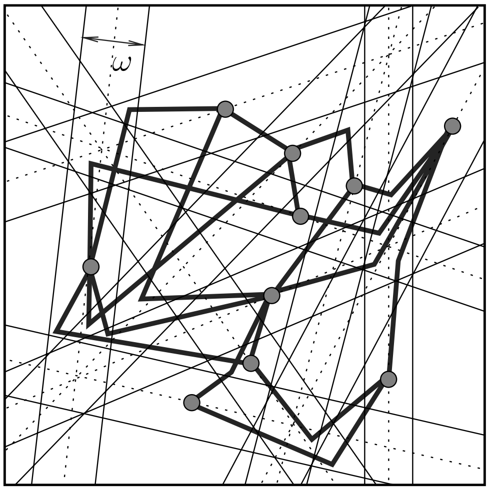


Fig. 1. An ω-searchlight obedient graph drawing

within the searchlights (bounded by the unit square), but requiring this does not make the problem harder. Any ω-searchlight obedient drawing can be modified so as to make all the clipped searchlights connected. This is done by rotating sufficiently the searchlights around their source points; before a strip hits another vertex, it must have a nonempty intersection (within the unit square) with the strip associated with the vertex. Such a drawing solves a routing problem in electronic chip design, where the vertices are ports and the edges are wires of the chip, and where all the wires pass through long skinny containers, where every container is associated with one port. (This is somewhat similar, but unrelated, to the notion of a *joint box* used in some graph-drawing algorithms; see, e.g., [1].)

We derive lower and upper bounds for the searchlights problem from its relation to the famous Heilbronn's triangle problem [7]:

> Given n points in the unit square, what is $\mathcal{H}(n)$, the maximum possible area of the *smallest* triangle defined by some three of these points?

Heilbronn posed the triangle problem about 50 years ago, and since then it intrigued the imagination of some of the best mathematicians. Yet there is still a large gap between the best currently-known lower and upper bounds for $\mathcal{H}(n)$,

$\Omega(\log n/n^2)$ [5] and $O(1/n^{8/7-\varepsilon})$ (for any $\varepsilon > 0$) [4].[1] A comprehensive survey of the history of this problem (excluding the results of Komlós et al.) is given by Roth in [9]. The relation between the triangle problem and the searchlights problem imposes a similar gap between the bounds shown in this paper for the latter problem. Improving either the lower or upper bound for one of the problems will also improve the respective bound for the other problem.

The problem discussed in this paper was also motivated by the following. Suppose that the actual value of $\mathcal{H}(n)$ is Δ. Consider the set S that realizes Δ. Every pair of points $p, q \in S$ defines a strip of width $4\Delta/d(p,q)$ which contains no other points of S, where $d(p,q)$ is the distance between p and q. Thus, the width of the "forbidden strip" defined by p, q is inversely proportional to $d(p,q)$. Roth showed in [8] that $\mathcal{H}(n) = O(\frac{1}{n^{1.117\ldots}})$. In this work he made the distinction between 'bad' and 'good' strips according to a relation between their width to the number of points of S they contain. We would like to demonstrate here that the strip-width effect (caused by the distances between points) on Heilbronn's triangle problem is minor. We do that by showing a tight relation between Heilbronn's problem to the searchlights problem, where all the searchlights are strips of the same width.

2 Easy Bounds on $\omega(n)$

We first establish easy lower and upper bounds on $\omega(n)$.

2.1 Lower Bound

Theorem 1. *$\omega(n) = \Omega(1/n)$.*

Proof. We establish the lower bound from a simple example. Put n vertical searchlights of width $1/n$ so that their interiors do not intersect. The source points may be located anywhere along the searchlight centerlines, and obviously no source is covered by any other searchlight.[2]

2.2 Upper Bound

Theorem 2. *$\omega(n) = O(1/\sqrt{n})$.*

Proof. Let $k = \lfloor \sqrt{n-1} \rfloor$. Partition the unit square into a full grid of $k \times k$ small squares each of sidelength $1/k$. Now locate n searchlights in the unit square. Since the grid contains at most $(n-1)$ small squares, there must exist one such square that contains two searchlight source points. The distance between these two sources is at most $\sqrt{2}/k$, therefore the searchlight width cannot exceed $2\sqrt{2}/k = 2\sqrt{2}/\lfloor \sqrt{n-1} \rfloor = O(1/\sqrt{n})$.

[1] Actually, Komlós et al. showed in this work that $\mathcal{H}(n) = O(e^{c\sqrt{\log n}}/n^{8/7})$ for some constant $c > 0$.

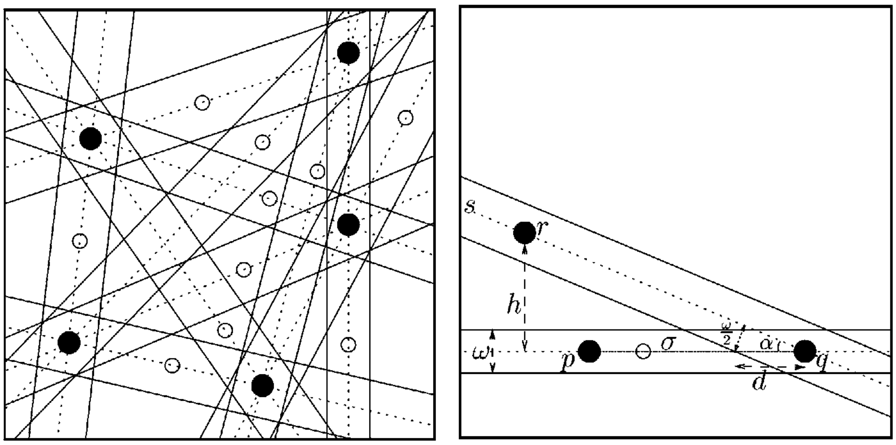
[2] In fact, it is rather easy, as an anonymous referee noted, to improve (increase) the constant of proportionality in this bound.

3 Improved Bounds on $\omega(n)$

In this section we improve the bounds on $\omega(n)$ obtained in the previous section. We begin with stating the relation between Heilbronn's triangle problem and the searchlights problem.

Theorem 3. *Assume that* $c_1 f_1(n) \le \mathcal{H}(n) \le c_2 f_2(n)$*, for some monotonically-growing functions* $f_1(n), f_2(n)$ *and constants* $c_1, c_2 > 0$*. Then* $c_3 f_1(\sqrt{2n}) \le \omega(n) \le c_4 \sqrt{f_2(2n)}$*, for some constants* $c_3, c_4 > 0$*.*

Proof. We first show the lower bound on $\omega(n)$. We put N points in the unit square such that the area of the smallest triangle defined by some three of these points assumes its maximum $\mathcal{H}(N)$. (See Figure 2(a); Heilbronn's points appear



(a) The construction (b) Avoiding source points

Fig. 2. Connecting Heilbronn's points (•) with searchlights

as black circles.) We then draw the $\binom{N}{2}$ lines defined by the N points. These will be searchlight centerlines; the locations of the source points (shown as white circles in Figure 2(a)) will be defined later. Every such line is now split by the other lines into $O(N^2)$ segments. We select one of these segments, $\sigma = pq$ (p and q are the endpoints of the segment σ), and locate on it a source point of a searchlight of width ω (see Figure 2(b)). (We repeat this for all the $\Theta(N^2)$ lines.) For ease of exposition we assume without loss of generality that σ is horizontal.

Refer to the leftmost searchlight s emanating upwards from q, the right endpoint of σ. This is the strip that overlaps the most of the right side of σ among all searchlights whose centerline passes through q. (The rightmost searchlight emanating upwards from q can also overlap a significant portion of the right side of σ, in which case we apply the same analysis for searchlights that emanate downwards from σ.) Let r be the other point which, together with q, defined the strip s. Without loss of generality we assume that r is above σ. By the assumption, the area of the triangle defined by the points p, q, r is at least $c_1 f_1(N)$. Thus, the altitude h of r relative to σ satisfies $|\sigma|h/2 \geq c_1 f_1(N)$, that is,

$$h \geq 2c_1 f_1(N)/|\sigma|. \tag{1}$$

We now compute d, the amount of overlap between s and σ. Obviously $|qr| \leq \sqrt{2}$ and $\sin\alpha = h/|qr| = \omega/(2d)$. Thus

$$d = \frac{\omega|qr|}{2h} \leq \frac{\sqrt{2}\omega}{2h}. \tag{2}$$

By substituting Equation (1) in Equation (2) we obtain

$$d \leq \frac{\sqrt{2}\omega|\sigma|}{4c_1 f_1(N)}. \tag{3}$$

A similar analysis shows the same upper bound on the length of the portion of σ overlapped by the rightmost (or leftmost) searchlight emanating upwards from p. Therefore, the total length of the two overlapped portions is at most $\sqrt{2}\omega|\sigma|/(2c_1 f_1(N))$. Hence, in order to preclude a searchlight source point on σ we must have $\sqrt{2}\omega|\sigma|/(2c_1 f_1(N)) \geq |\sigma|$, that is,

$$\omega \geq \sqrt{2}c_1 f_1(N). \tag{4}$$

Finally, we set $N = \sqrt{2n}$ in order to have $n(1 + o(1))$ searchlights, and obtain $\omega \geq \sqrt{2}c_1 f_1(\sqrt{2n})$. Setting $c_3 = \sqrt{2}c_1$ completes the argument.

A reduction in the opposite direction shows the upper bound on $\omega(n)$. Refer to Figure 3. Here we put n searchlights of the maximum possible width ω such that their source points are located in the unit square. The n centerlines of the searchlights intersect inside the unit square in at most $\binom{n}{2}$ points. On each centerline we position two "Heilbronn points" at the two intersection points which define the segment on which the source point of the respective searchlight lies. (For this purpose we also consider the intersections of the searchlight centerlines with the unit square.) In total we thus mark $N = 2n$ points.[3]

Refer to the smallest-area triangle defined by some triple p, q, r of these N points (see again Figure 2(b)). Here r is one of the points marked on the searchlight s whose intersection with the searchlight containing σ defines q. By the assumption, the area of this triangle is at most $c_2 f_2(N)$. Fix again the segment σ whose endpoints are p and q. The altitude h of r relative to σ must satisfy $|\sigma|h/2 \leq c_2 f_2(N)$, that is,

$$h \leq \frac{2c_2 f_2(N)}{|\sigma|}. \tag{5}$$

[3] Figure 3 is misleading in the sense that this example contains three or more collinear Heilbronn points, which obviously define a triangle of area 0. Note, however, that we use here an *upper* bound on the area of Heilbronn's triangles.

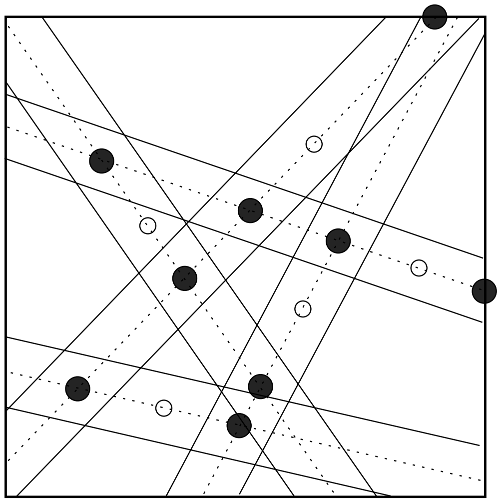


Fig. 3. Locating Heilbronn points on searchlights

We now compute the amount of overlap between s and σ. Again, $\sin\alpha = h/|qr| = \omega/(2d)$. This time we use the fact that $|qr| \geq \omega/2$ (otherwise the source point of the searchlight on σ would be covered by the searchlight centered either at pr or at qr).[4] Thus,

$$d = \frac{\omega|qr|}{2h} \geq \frac{\omega^2}{4h}. \tag{6}$$

By substituting Equation (5) in Equation (6) we obtain

$$d \geq \frac{\omega^2|\sigma|}{8c_2 f_2(N)}. \tag{7}$$

A similar analysis applies for the left side of σ. Therefore, the total length of the two overlapped portions of σ is at least $\omega^2|\sigma|/(4c_2 f_2(N))$. On the other hand, this total is at most $|\sigma|$ (since σ contains a searchlight source point), so we must have $\omega^2|\sigma|/(4c_2 f_2(N)) \leq |\sigma|$, that is,

$$\omega \leq 2\sqrt{c_2}\sqrt{f_2(N)} = 2\sqrt{c_2}\sqrt{f_2(2n)}. \tag{8}$$

Setting $c_4 = 2\sqrt{c_2}$ completes the argument.

Next we quote the best currently-known bounds for Heilbronn's triangle problem:

Theorem 4. [5] $\mathcal{H}(n) = \Omega(\log n/n^2)$.

[4] Note the difference between this lower bound on $|qr|$ and the upper bound of $\sqrt{2}$ used above.

Theorem 5. [4] *$\mathcal{H}(n) = O(1/n^{8/7-\varepsilon})$ for any $\varepsilon > 0$.*

Finally we combine Theorems 3, 4, and 5 to obtain our main result:

Theorem 6. *$\omega(n) = \Omega(\log n/n)$ and $\omega(n) = O(1/n^{4/7-\varepsilon})$ for any $\varepsilon > 0$.* □

We also have the opposite dependence:

Theorem 7. *Assume that $d_1 g_1(n) < \omega(n) < d_2 g_2(n)$, for some monotonically-growing functions $g_1(n), g_2(n)$ and constants $d_1, d_2 > 0$. Then $d_3 g_1^2(n/2) < \mathcal{H}(n) < d_4 g_2(n^2/2)$, for some constants $d_3, d_4 > 0$.*

Proof. First we show that there exists a constant d_3 for which $\mathcal{H}(n) > d_3 g_1^2(n/2)$. Assume to the contrary that no such constant exists, that is, for any $d_3 > 0$ and for sufficiently-large values of n, we have $\mathcal{H}(n) \leq d_3 g_1^2(n/2)$. Then by Theorem 3 there exists a constant $d_3' > 0$ for which $\omega(n) \leq d_3' \sqrt{g_1^2((2n)/2)} = d_3' g_1(n)$, which is a contradiction.

Similarly we show that there exists a constant d_4 for which $\mathcal{H}(n) < d_4 g_2 (n^2/2)$. Assume to the contrary that no such constant exists, that if, for any $d_4 > 0$ and for sufficiently-large values of n, we have $\mathcal{H}(n) \geq d_4 g_2(n^2/2)$. Then by Theorem 3 there exists a constant $d_4' > 0$ for which $\omega(n) \geq d_4' g_2(\sqrt{2(n^2/2)}) = d_4' g_2(n)$, which is also a contradiction.

Theorem 7 implies that any improvement of the lower or the upper bound for the searchlights problem will also carry on to Heilbronn's triangle problem, which, as mentioned in the introduction, has puzzled many mathematicians throughout the last half of century.

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