Unavoidable Configurations in Complete Topological Graphs

János Pach[1] and Géza Tóth[2]

[1] Courant Institute, NYU and Hungarian Academy of Sciences
[2] Massachusetts Institute of Technology and Hungarian Academy of Sciences

Abstract. A *topological graph* is a graph drawn in the plane so that its vertices are represented by points, and its edges are represented by Jordan curves connecting the corresponding points, with the property that any two curves have at most one point in common. We define two canonical classes of topological complete graphs, and prove that every topological complete graph with n vertices has a canonical subgraph of size at least $c \log \log n$, which belongs to one of these classes. We also show that every complete topological graph with n vertices has a non-crossing subgraph isomorphic to any fixed tree with at most $c \log^{1/6} n$ vertices.

1 Introduction, Results

A *topological graph* G is a graph drawn in the plane by Jordan curves, any two of which have at most one point in common. That is, it is defined as a pair $\{V(G), E(G)\}$, where $V(G)$ is a set of points in the plane and $E(G)$ is a set of simple continuous arcs connecting them so that they satisfy the following conditions:

1. no arc passes through any other element of $V(G)$ different from its endpoints;
2. any two arcs have at most one point in common, which is either a common endpoint or a proper crossing.

$V(G)$ and $E(G)$ are the *vertex-set* and *edge set* of G, respectively. We say that H is a *(topological) subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two topological graphs, G and H, are called *weakly isomorphic* if there is an incidence preserving one-to-one correspondence between $\{V(G), E(G)\}$ and $\{V(H), E(H)\}$ such that two edges of G intersect if and only if the corresponding edges of H do (see [C81]). If all edges of a topological graph are straight-line segments, then it is called a *geometric graph*. A geometric graph, whose vertices are in convex position, is called *convex*. Obviously, any two complete convex geometric graphs with m vertices are weakly isomorphic to each other, and to the convex geometric graph C_m, whose edge set consists of all sides and chords of a regular m-gon. (See Fig. 1.)

J. Marks (Ed.): GD 2000, LNCS 1984, pp. 328–337, 2001.

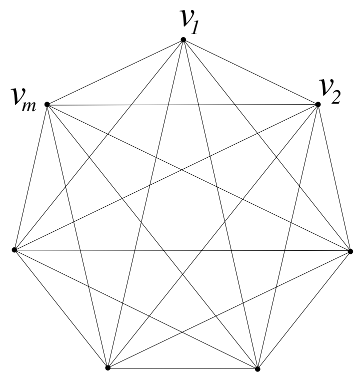

Figure 1: *The convex geometric graph C_m.*

The fairly extensive literature on topological graphs focuses on very few special questions, and there is no standard terminology. For topological graphs, Erdős and Guy [EG73] (see also [AR88]) use the term "good drawings," while Gronau, Harborth, Mengersen, and Thürmann [GH90], [HM74], [HM90], [HT94] simply call them "drawings." For a complete topological graph, Ringel [R64] and Mengersen [M78] use the term "immersion." The most popular problems in this field are Turán's Brick Factory Problem [T77] (Zarankiewicz's Conjecture [G69] and other problems about *crossing numbers*, i.e., about the *minimum* number of crossings in certain drawings of a graph [PT98]) and Conway's Thrackle Conjecture [W71], [LPS97], [CN00] (and other problems about the *maximum* number of crossings in certain drawings of a graph [HM92]).

The systematic study of geometric graphs was initiated by Erdős, Avital–Hanani [AH66], Kupitz [K79], and Perles. (See [P99] and [PA95], Chapter 14, for the most recent surveys on the subject.) It is not hard to see that every *complete geometric graph K_n* of n vertices has a non-crossing subgraph isomorphic to any triangulation of a cycle of length n (cf. [GMPP91]). Consequently, K_n has a non-crossing subtree isomorphic to any fixed tree of n vertices. In particular, K_n has a non-crossing path of n vertices and a non-crossing matching of size $\lfloor n/2 \rfloor$.

On the other hand, it is known that K_n has at least constant times $\sqrt{n}$ pairwise crossing edges.

Our aim is to establish analogous results for topological graphs.

Theorem 1. *Every topological complete graph of n vertices has a non-crossing subgraph isomorphic to any fixed tree T with at most $c\log^{1/6} n$ vertices. In particular, it contains a non-crossing path with at least $c\log^{1/6} n$ vertices.*

According to a wellknown theorem of Erdős and Szekeres [ES35],[ES60], any set of n points in general position in the plane contains a subset with at least $c\log n$ elements which form the vertex set of a convex polygon. (Throughout this note, the letter c appearing in different assertions denote unrelated positive constants. The best known bound in the last statement is due Tóth and Valtr [TV98].) The Erdős-Szekeres Theorem can be reformulated, as follows.

Erdős-Szekeres Theorem. *Every complete geometric graph with* n *vertices has a complete geometric subgraph, weakly isomorphic to a convex complete graph* C_m *with* $m \geq c \log n$ *vertices.*

The situation is more complicated for *topological* graphs. In their study of topological complete graphs with m vertices and with the *maximum* possible number, $\binom{m}{4}$, of edge crossings, Harborth and Mengersen [HM92] found a drawing which contains no subgraph weakly isomorphic to C_5. We call this drawing, depicted in Figure 2, *twisted*, and denote it by T_m.

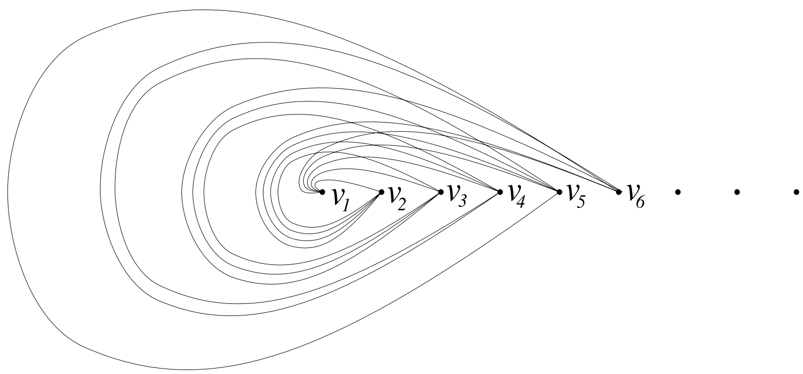

Figure 2: *The twisted drawing* T_m.

We show that one cannot avoid *both* C_m and T_m in a sufficiently large complete topological graph.

Theorem 2. *Every complete topological graph with* n *vertices has a complete topological subgraph with* $m \geq c \log\log n$ *vertices, which is weakly isomorphic either to a convex complete graph* C_m *or to a twisted complete graph* T_m.

2 An Erdős–Szekeres-Type Theorem

Before we turn to the proof of Theorem 2, we rephrase the definitions of convex and twisted complete topological graphs.

Definition 2.1. Let K_m be a complete topological graph on m vertices. If there is an enumeration of the vertices, $\{u_1, u_2, \ldots, u_m\}$, such that

(i) two edges, u_iu_j $(i < j)$ and u_ku_l $(k < l)$, cross each other if and only if $i < k < j < l$ or $k < i < l < j$, then K_m is called *convex*;

(ii) two edges, u_iu_j $(i < j)$ and u_ku_l $(k < l)$, cross each other if and only if $i < k < l < j$ or $k < i < j < l$, then K_m is called *twisted*.

Let K be a fixed complete topological graph with $n+1$ vertices. The edges of K divide the plane into several cells, precisely one of which is unbounded. Without loss of generality, we can assume that there is a vertex $v_0 \in V(K)$ on the boundary of the unbounded cell. Otherwise, we can apply a stereographic projection to transform K into a drawing on a sphere, and then, by another projection, we can turn it into a topological graph weakly isomorphic to K, which satisfies the required property.

Consider all edges emanating from v_0, and denote their other endpoints by $v_1, v_2, \ldots, v_n$, in clockwise order.

Color the triples $v_i v_j v_k$, $1 \le i < j < k \le n$ with eight different colors, according to the following rules. Each color is represented by a zero-one sequence abc of length 3. For any $i < j < k$,

1. set $a = 0$ if the edges $v_i v_j, v_0 v_k \in E(K)$ do not cross, and let $a = 1$ otherwise;
2. set $b = 0$ if the edges $v_i v_k, v_0 v_j \in E(K)$ do not cross, and let $b = 1$ otherwise;
3. set $c = 0$ if the edges $v_j v_k, v_0 v_i \in E(K)$ do not cross, and let $c = 1$ otherwise.

It is easy to see that the complete topological subgraph of K induced by the vertices v_0, v_i, v_j, v_k (as any other complete topological graph with 4 vertices) has at most one pair of crossing edges. Therefore, we have

Claim 2.2. None of the colors 011, 101, 110, or 111 can occur.

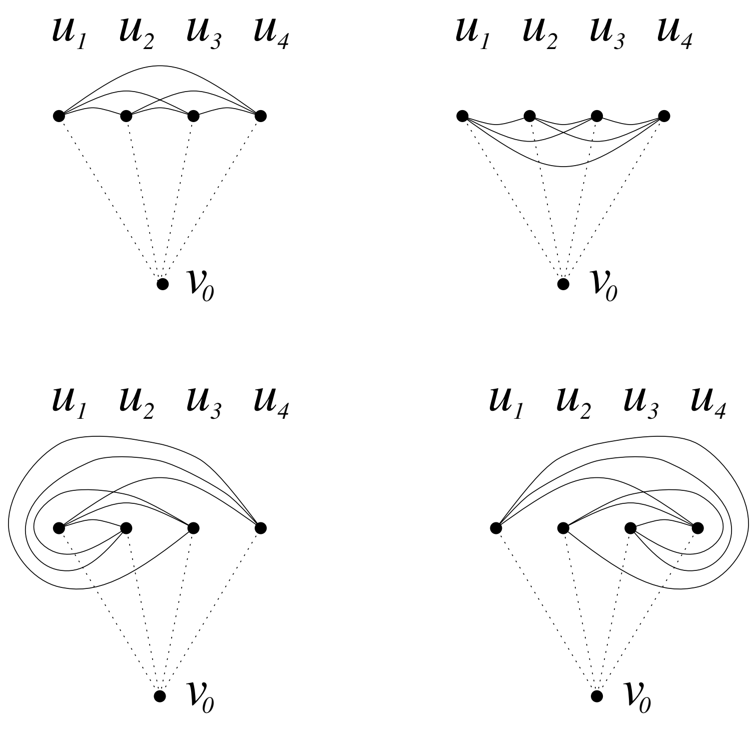

Figure 3: *All triples are of type* 000, 010, 001, *and* 100, *respectively.*

Proof of Theorem 2: By Ramsey's Theorem, there is an m-element subsequence, $(u_1, u_2, \ldots, u_m) \subseteq (v_1, v_2, \ldots, v_n)$, $m \ge c \log \log n$ such that all triples $u_i u_j u_k$ are of the same color (c is a positive constant).

Suppose first that all triples $u_i u_j u_k$ are of color 000 (see Fig. 3). Then, two edges, $u_i u_j$ $(i < j)$ and $u_k u_l$ $(k < l)$, cross each other if and only if $i < k < j < l$ or $k < i < l < j$. That is, $u_1, u_2, \ldots, u_m$ induce a *convex* complete topological subgraph in K.

We obtain exactly the same crossing pattern between the edges induced by $\{u_1, u_2, \ldots, u_m\}$ if all triples are of color 010 (see Fig. 3).

Suppose next that all triples in $\{u_1, u_2, \ldots, u_m\}$ are colored 001. Then two edges, $u_i u_j$ $(i < j)$ and $u_k u_l$ $(k < l)$, cross each other if and only if $i < k < l < j$ or $k < i < j < l$. In this case, $u_1, u_2, \ldots, u_m$ induce a *twisted* complete topological subgraph.

Finally, the case when all triples in $\{u_1, u_2, \ldots, u_m\}$ are of color 100, is isomorphic to the previous one, under a reflection reversing the orientation of the plane (and hence the numbering of the vertices v_i $(1 \leq i \leq n)$). □

It is very easy to check that both C_m and T_m contain non-crossing copies of every tree with m vertices. Thus, a weaker version of Theorem 1, with $c \log\log n$ instead of $c \log^{1/6} n$, readily follows from Theorem 2. In the next section, we apply a somewhat more delicate argument to improve this bound.

3 Proof of Theorem 1

Let G be a topological complete graph with an $(n+1)$-element vertex set V. Use the same numbering, $v_0, v_1, \ldots, v_n$, of the vertices as in the previous section. For any $0 < i < j$, we say that v_i *precedes* v_j (in notation, $v_i \prec v_j$). As before, color the triples $v_i v_j v_k$ $(1 \leq i < j < k \leq n)$ with *four* colors, 000, 100, 010, and 001.

Claim 3.1. There exists an m-element subset $U := \{u_1, u_2, \ldots, u_m\} \subset \{v_1, v_2, \ldots, v_n\}$, $m \geq \sqrt{\log_4 (n+1)}$ such that the triples $u_i u_j u_k$ and $u_i u_j u_l$ have the same color for any $i < j < k < l$.

Proof: The construction is recursive. Let $U_2 := \{v_1, v_2\}$ and $V_2 := V \setminus \{v_1, v_2\}$. Suppose that, for some $2 \leq p < m$, we have already found two subsets $U_p = \{u_1, u_2, \ldots, u_p\}$ and $V_p \subset V$ with the properties

1. $u_1 \prec u_2 \prec \cdots \prec u_p$,
2. every element of U_p precedes all elements of V_p,
3. $|V_p| \geq \frac{|V_{p-1}|-1}{4^p}$.

Let u_{p+1} be the smallest element of V_p with respect to the ordering '$\prec$.' Since we used *four* colors for coloring the triples, there is a subset $W \subset V_p \setminus \{u_{p+1}\}$ with $|W| \geq (|V_p| - 1)/4^p$ such that, for each $1 \leq i \leq p$, all triples $u_i u_{p+1} w$ $(w \in W)$ have the same color. Let $U_{p+1} := U_p \cup \{u_{p+1}\}$ and $V_{p+1} := W$. An easy computation shows that this procedure can be repeated at least $\lceil \sqrt{\log_4 (n+1)} \rceil$ times. □

Define the *type of an edge* $u_i u_j$ $(i < j < m)$ as the color of a triple $u_i u_j u_k$ for any $k > j$. The type of $u_i u_m$ can be defined arbitrarily.

Let $G(100)$ and $G(001)$ denote the topological subgraphs of G consisting of all edges of type 100 and 001, resp., whose both endpoints belong to $U = \{u_1, u_2, \ldots, u_m\}$. The topological subgraph consisting of all other edges of G induced by U (of types 000 and 010) is denoted by G'.

Claim 3.2. Let $i < j < k < m$.

(i) If u_iu_j and u_ju_k belong to $G(100)$, then so does u_iu_k.

(ii) If u_iu_j and u_iu_k belong to $G(001)$, then so does u_ju_k.

Proof: If u_iu_j is of type 100, it must cross both v_0u_k and v_0u_m. If the type of u_ju_k is also 100, it must cross v_0u_m, too. Using the assumption that two edges that share an endpoint cannot have any other point in common, we obtain that u_iu_k must cross v_0u_m, which implies that its type is also 100 (see Fig. 4). This proves part (i). Part (ii) can be established similarly. □

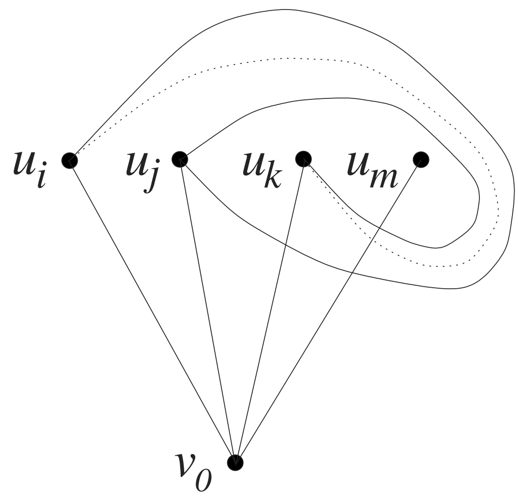

Figure 4: *u_iu_k must cross v_0u_m.*

Claim 3.3. If $G(100)$, $G(001)$, or G' contains a complete subgraph of size $r := \lceil m^{1/3} \rceil$, then G has a non-crossing subgraph isomorphic to any tree of r vertices.

Proof: Suppose that $w_1 \prec w_2 \prec \ldots \prec w_r$ induce a complete (topological) subgraph in $G(100)$. It is easy to see that this subgraph is *twisted*, i.e., it is weakly isomorphic to T_r. Take an arbitrary tree T with r vertices. Starting at any vertex $z_1 \in V(T)$, explore all other vertices of T using *breadth-first search*. Let $z_1, z_2, \ldots, z_r$ be a numbering of the elements of $V(T)$, in the order in which they are encountered by the algorithm. Then the embedding $f(z_i) = w_i$ $(1 \leq i \leq r)$ maps T into a non-crossing copy of T in $G(100)$, and we are done.

The case when $G(001)$ contains a complete subgraph of size r can be treated similarly.

Assume now that G' has a complete subgraph with r vertices, $w_1 \prec w_2 \prec \ldots \prec w_r$. It is easy to see that if two edges, w_iw_j $(i < j)$ and w_kw_l $(k < l)$, cross each other, then we have $i < k < j < l$ or $k < i < l < j$. In other words, if two edges of this subgraph cross each other, the corresponding edges also cross in a drawing on the same vertex set, weakly isomorphic to the *convex* drawing C_r. Clearly, C_r contains a non-crossing copy of every tree with r vertices, so the same is true for G'. □

In view of the last claim, it remains to prove that at least one of $G(100)$, $G(001)$, and G' has a complete subgraph of size $r = \lceil m^{1/3} \rceil$. Suppose, in order to obtain a contradiction, that this is not the case.

If some element $u \in U = \{u_1, u_2, \ldots, u_m\}$ had at least $r-1$ larger neighbors in $G(001)$ with respect to the ordering $\prec$, then, by Claim 3.2 (ii), these neighbors together with u would induce a complete subgraph in $G(001)$, a contradiction.

Now we recursively construct a sequence $w_1 \prec w_2 \prec \ldots$ consisting of at least $m^{2/3}$ elements of U, which form an independent set in $G(001)$ (i.e., they induce a complete subgraph in $G(100) \cup G'$).

Let $W_0 := \emptyset$ and $U_0 := \{u_1, u_2, \ldots, u_m\}$. Suppose that, for some $p < m^{2/3}$, we have already found two subsets $W_p = \{w_1, w_2, \ldots, w_p\}$ and $U_p \subset \{u_1, u_2, \ldots, u_m\}$, such that

1. W_p is an independent set in $G(001)$,
2. every element of W_p precedes every element of U_p,
3. there is no edge between W_p and U_p,
4. $|U_p| \geq m - p(r-1)$.

If $U_p \neq \emptyset$, let w_{p+1} be the smallest element of U_p with respect to the ordering $\prec$, and set $W_{p+1} := W_p \cup \{w_{p+1}\}$. Let U_{p+1} denote the set obtained from U_p by the deletion of w_{p+1} and its larger neighbors. Clearly, we have $|U_{p+1}| \geq |U_p| - r + 1$, so that this procedure can be repeated at least $\lceil m^{2/3} \rceil$ times.

Define the *rank* of any element $w \in W := \{w_1, w_2, \ldots\}$, as the number of vertices of the longest monotone path (with respect to $\prec$) which ends at w in the subgraph of $G(100)$ induced by W. There is no element whose rank is at least $m^{1/3}$, otherwise, by Claim 3.2 (i), the vertices of the corresponding path would induce a complete subgraph of size at least r in $G(100)$, contradicting our assumptions.

Therefore, we can suppose that at least $m^{1/3}$ elements of W have the same rank. According to the definitions, these elements form an independent set in $G(100)$ as well as in $G(001)$. Thus, they induce a complete subgraph in G', again a contradiction. This proves Theorem 1.

4 Concluding Remarks

I. It follows from the proof of Theorem 1 that Theorem 2 can be slightly strengthened.

Theorem 4.1. *Every complete topological graph with n vertices has a complete topological subgraph with $m \geq c \log^{1/6} n$ vertices, weakly isomorphic to a twisted complete graph T_m, or a complete topological subgraph with $p \geq c \log\log n$ vertices, weakly isomorphic to a convex complete graph C_p.*

II. The following statement is a direct corollary of the first result in [PSS96].

Theorem 4.2. *Every complete topological graph of n vertices contains at least $c \log n / \log\log n$ pairwise crossing edges.*

III. Both C_m and T_m, the convex and the twisted topological graphs with m vertices, respectively, determine precisely $\binom{m}{4}$ edge crossings. Therefore, the following theorem of Harborth, Mengersen, and Schelp [HMS95] is an immediate consequence of Theorem 2.

Corrollary 4.3 *For any positive integer m, there exists a smallest number $n(m)$ such that every complete topological graph with at least $n(m)$ vertices has a complete subgraph with m vertices and with $\binom{m}{4}$ crossings between its edges.*

In fact, for large values of m, Theorem 2 implies a better bound on the function $n(m)$ than the proof given in [HMS95].

IV. Let F denote the graph obtained from a complete graph of 5 vertices by subdividing each of its edges with an extra vertex. Given a complete topological graph K_n of n vertices, define an abstract graph G. Let the vertex set of G consist of $\lfloor n/2 \rfloor$ edges of K_n, no two of which share an endpoint. Let two vertices, $e, e' \in E(K_n)$, be joined by an edge of G if and only if e and e' cross each other. It is easy to see that G does not contain F as an induced subgraph (see e.g. [EET76]).

It follows from a theorem of Erdős and Hajnal [EH89] that, if a graph with m vertices does not contain some fixed induced subgraph F, then it must have either an empty or a complete subgraph with at least $e^{c\sqrt{\log m}}$ vertices, where $c > 0$ is a constant depending on F. Putting these two facts together, we obtain

Corollary 4.4. *Any topological complete graph with n vertices has at least $e^{c\sqrt{\log n}}$ edges that are either pairwise disjoint or pairwise crossing.*

This suggests that the bounds in Theorems 1 and 4.2 are far from being optimal. We conjecture that both estimates can be replaced by n^δ, for some $\delta > 0$. As was pointed out in the Introduction, this holds for geometric graphs.

V. In the case of *geometric* graphs, one can introduce several partial orderings on the set of edges (cf. [PT94], [PA95]). This allows us to apply Dilworth's Theorem in place of Ramsey's Theorem, to find much larger homogeneous substructures.

References

[AH66] S. Avital and H. Hanani: Graphs (Hebrew), *Gilyonot Lematematika* **3** (1966), 2–8.

[AR88] D. Archdeacon and B. R. Richter: On the parity of crossing numbers, *J. Graph Theory* **12** (1988), 307–310.

[C81] Ch. J. Colbourn: On drawings of complete graphs, *J. Combin. Inform. System Sci.* **6** (1981), 169–172.

[CN00] G. Cairns and Y. Nikolayevsky: Bounds for generalized thrackles, *Discrete Comput. Geom.* **23** (2000), 191–206.

[EET76] G. Ehrlich, S. Even, and R. E. Tarjan: Intersection graphs of curves in the plane, *J. Combinatorial Theory, Ser. B* **21** (1976), 8–20.

[EG73] P. Erdős and R. K. Guy: Crossing number problems, *Amer. Math. Monthly* **80** (1973), 52–58.

[EH89] P. Erdős and A. Hajnal: Ramsey-type theorems, *Discrete Appl. Math.* **25** (1989), 37–52.

[ES35] P. Erdős and G. Szekeres: A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470.

[ES60] P. Erdős and G. Szekeres: On some extremum problems in elementary geometry, *Ann. Universitatis Scientiarum Budapestinensis, Eötvös, Sectio Mathematica* **III–IV** (1960–61), 53–62.

[GJ85] M. R. Garey and D. S. Johnson: Crossing number is NP-complete, *SIAM J. Algebraic Discrete Methods* **4** (1983), 312–316.

[GMPP91] P. Gritzmann, B. Mohar, J. Pach, and R. Pollack: Embedding a planar triangulation with vertices at specified points, *Amer. Math. Monthly* **98** (1991), 165-166.

[GH90] H.-D. Gronau and H. Harborth: Numbers of nonisomorphic drawings for small graphs, in: *Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989), Congr. Numer.* **71** (1990), 105–114.

[G69] R. K. Guy: The decline and fall of Zarankiewicz's theorem, in: *Proof Techniques in Graph Theory*, Academic Press, New York, 1969, 63–69.

[HM74] H. Harborth and I. Mengersen: Edges without crossings in drawings of complete graphs, *J. Combinatorial Theory, Ser. B* **17** (1974), 299–311.

[HM90] H. Harborth and I. Mengersen: Edges with at most one crossing in drawings of the complete graph, in: *Topics in Combinatorics and Graph Theory (Oberwolfach, 1990)*, Physica, Heidelberg, 1990, 757–763.

[HM92] H. Harborth and I. Mengersen: Drawings of the complete graph with maximum number of crossings, in: *Proceedings of the Twenty-third Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1992), Congr. Numer.* **88** (1992), 225–228.

[HMS95] H. Harborth, I. Mengersen, and R. H. Schelp: The drawing Ramsey number $\mathrm{Dr}(K_n)$, *Australasian J. Combinatorics* **11** (1995), 151–156.

[HT94] H. Harborth and Ch. Thürmann: Minimum number of edges with at most s crossings in drawings of the complete graph, in: *Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), Congr. Numer.* **102** (1994), 83–90.

[K79] Y. S. Kupitz: *Extremal Problems in Combinatorial Geometry, Lecture Notes Series* **53**, Aarhus Universitet, Matematisk Institut, Aarhus, 1979.

[LPS97] L. Lovász, J. Pach, and M. Szegedy: On Conway's thrackle conjecture, *Discrete and Computational Geometry* **18** (1997), 369–376.

[M78] I. Mengersen: Die Maximalzahl von kreuzungsfreien Kanten in Darstellungen von vollständigen n-geteilten Graphen (German), *Math. Nachr.* **85** (1978), 131–139.

[P99] J. Pach: Geometric graph theory, in: *Surveys in Combinatorics, 1999 (Canterbury), London Math. Soc. Lecture Note Ser.* **267**, Cambridge Univ. Press, Cambridge, 1999, 167–200.

[PA95] J. Pach and P. K. Agarwal: *Combinatorial Geometry*, Wiley-Interscience, New York, 1995.

[PSS96] J. Pach, F. Shahrokhi, and M. Szegedy: Applications of crossing numbers, *Algorithmica* **16** (1996), 111–117.

[PT94] J. Pach and J. Törőcsik: Some geometric applications of Dilworth's theorem, *Discrete and Computational Geometry* **12** (1994), 1–7.

[PT98] J. Pach and G. Tóth: Which crossing number is it, anyway? *Proceedings of the 39th Annual Symposium on Foundations of Computer Science,* 1998, 617–626.

[R64] G. Ringel: Extremal problems in the theory of graphs, in: *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, Publ. House Czechoslovak Acad. Sci., Prague, 1964, 85–90.

[TV98] G. Tóth and P. Valtr: Note on the Erdős-Szekeres theorem, *Discrete Comput. Geom.* **19** (1998), 457–459.

[T77] P. Turán: A note of welcome, *J. Graph Theory* **1** (1977), 7–9.

[W71] D. R. Woodall: Thrackles and deadlock, in: *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, Academic Press, London, 1971, 335–347.

[WB78] A. T. White and L. W. Beineke: Topological graph theory, in: *Selected Topics in Graph Theory (L. W. Beineke and R. J. Wilson., eds.),* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1983, 15–49.