Quadratic correctness criterion for Non commutative Logic

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Abstract. The multiplicative fragment of Non commutative Logic (called MNL) has a proof nets theory [AR00] with a correctness criterion based on long trips for cut-free proof nets. Recently, R.Maieli has developed another criterion in the Danos-Regnier style [Mai00]. Both are in exponential time. We give a quadratic criterion in the Danos contractibility criterion style.

1 Introduction

Non commutative Logic (NL) is a unification of linear logic [Gir87] and cyclic linear logic [Gir89,Yet90,Abr91] (a classical conservative extension of the Lambek calculus [Lam58]). It includes all linear connectives: multiplicatives, additives, exponentials and constants. Recents results [AR00,Rue00,MR00] introduce proof nets, sequent calculus, phase semantics and all the importants theorems like cut elimination and sequentialisation. The central notion is the structure of order varieties. Let α be an order variety on a base set $X \cup \{x\}$, provided a point of view (the element x) α can be seen as a partial order on X. Order varieties can be presented in different ways by changing the point of view and are invariant under the change of presentation: one uses rootless planar trees called seaweeds. Thus this structure allows focusing on any formula to apply a rule.

Proof nets are graph representations of NL derivations. Then a proof net with conclusion A is obtained as an interpretation of a sequent calculus proof of A: we say that it can be sequentialized. But the corresponding cut-free derivation of the formula A is not unique in general. It introduces some irrelevant order on the sequent rules. For instance, a derivation Π ending with $\vdash A \parr B, C \parr D$ implies an order on the two rules introducing the principal connectives of $A \parr B$ and $C \parr D$, but the proof net corresponding to Π does not depend on such order.

A contracting proof structure is a hypergraph built in accordance with the syntax of proof nets and seaweeds. A proof structure is a particular contracting one. To know if a such structure is a proof net or not, we use a correctness criterion. The Maieli one is in the Danos-Regnier criterion style: at first it uses a switching condition and tests if we obtain an acyclic connected graph. Then for each ∇ link, we check the associated order varieties.

It is known that the proof nets of multiplicative linear logic have a linear time correctness criterion [Gue99]. The first step towards a linear algorithm is to have a contractibility criterion (the Danos one [Dan90]) which can be seen as a parsing algorithm. One can reformulate it in terms of a sort of unification. Then a direct implementation leads a quasi-linear algorithm, and sharp study give the exact complexity. Up to now, there was no polynomial criterion for MNL.

Here we present a set of shrinking rules for MNL proof structures characterising MNL proof nets as the only structures that contract to a seaweed. We show that this contractibility criterion is quadratic. This idea is extended by a presentation as a parsing algorithm. So this work may be a decisive step towards a linear MNL correctness criterion.

Notations. One writes $X \uplus Y$ for the disjoint union of the sets X and Y. Let ω and τ be orders respectively on the sets X and Y. Let x be in X. One writes $\omega[\tau/x]$ the order on $(X \backslash \{x\}) \cup Y$ defined by $\omega[\tau/x](y,z)$ iff $\omega(y,z)$ or $\tau(y,z)$ or $\omega(y,x)$ if $z \in Y$ or $\omega(x,z)$ if $y \in Y$. Let f and g be positive functions. One writes $g(n) = \Theta(f(n))$ to denote that $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$.

2 Order Varieties

2.1 Order Varieties and Orders

Definition 1 (order varieties). *Let X be a set. An* order variety *on X is a ternary relation α which is:*

$$\begin{cases} \textit{cyclic:} & \forall x,y,z \in X, \alpha(x,y,z) \Rightarrow \alpha(y,z,x), \\ \textit{anti-reflexive:} & \forall x,y \in X, \neg\alpha(x,x,y), \\ \textit{transitive:} & \forall x,y,z,t \in X, \alpha(x,y,z) \textit{ and } \alpha(z,t,x) \Rightarrow \alpha(y,z,t), \\ \textit{spreading:} & \forall x,y,z,t \in X, \alpha(x,y,z) \Rightarrow \alpha(t,y,z) \textit{ or } \alpha(x,t,z) \textit{ or } \alpha(x,y,t). \end{cases}$$

Definition 2 (series-parallel orders). *Let ω and τ be two partial orders on disjoint sets X and Y respectively. Their* serial sum *(resp.* parallel sum*) $\omega < \tau$ (resp. $\omega \parallel \tau$) is a partial order on $X \cup Y$ defined respectively by:*

$$\begin{array}{ll} (\omega < \tau)(x,y) & \textit{iff } x <_\omega y \textit{ or } x <_\tau y \textit{ or } (x \in X \textit{ and } y \in Y), \\ (\omega \parallel \tau)(x,y) & \textit{iff } x <_\omega y \textit{ or } x <_\tau y. \end{array}$$

Definition 3 (closure). *Let $\omega = (X,<)$ be a partial order on X and $z \in X$. Let $\overset{z}{<}$ denote the binary relation: $x \overset{z}{<} y$ iff $x < y$ and z is comparable neither with x nor y. The* closure *of ω is the ternary relation $\overline{\omega}$ on X defined by:*

$$\begin{array}{lllll} \overline{\omega}(x,y,z) & \textit{iff } x<y<z & \textit{or } y<z<x & \textit{or } z<x<y & \textit{or} \\ & \phantom{\textit{iff }} x \overset{z}{<} y & \textit{or } y \overset{x}{<} z & \textit{or } z \overset{y}{<} x. & \end{array}$$

Facts 1. i) If ω is a partial order on X then $\overline{\omega}$ is an order variety on X,

ii) The closure identifies serial and parallel sums of partial orders on disjoint sets.

Definition 4 (gluing). *Let ω and τ be two partial orders on disjoint sets X and Y respectively. The* gluing $\omega * \tau$ *of ω and τ is the following order variety on $X \cup Y$:*

$$\omega * \tau = \overline{\omega < \tau} = \overline{\omega \parallel \tau} = \overline{\tau < \omega}$$

Definition 5. *Let α be an order variety on a set X and $x \in X$. The order α_x* induced by α and x *is the partial order on $X \backslash \{x\}$ defined by:*

$$\alpha_x(y, z) \text{ iff } \alpha(x, y, z)$$

One writes x for the unique partial order on $\{x\}$.

Proposition 1. *Let α be an order variety on a set X, $x \in X$ and ω be a partial order on $X \backslash \{x\}$. Then*

$$\alpha_x * x = \alpha \quad \text{and} \quad (\omega * x)_x = \omega$$

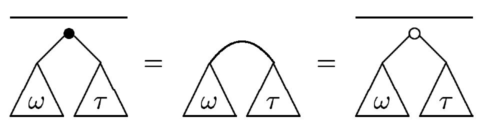
Fact 2. Let α be an order variety on a non-empty set. α is series-parallel iff there exists a series-parallel order ω such that $\alpha = \overline{\omega}$. In other words, series-parallel order varieties are exactly those can be represented by series-parallel orders.

Definition 6 (seaweed). *Let $\alpha = \overline{\omega}$ be a series-parallel order variety on X ($\sharp X \geq 2$) such that ω is written as a (non-unique) binary tree T with leaves labelled by elements of X, and root and nodes labelled by $\bullet$ (serial composition) or $\circ$ (parallel composition).*

A seaweed *S representing α is a rootless planar tree with leaves labelled by elements of X and ternary nodes labelled by $\bullet$ or $\circ$, defined by removing the root of T:*

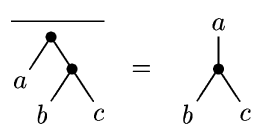
$$\alpha \quad = \quad \overline{\omega < \tau} \quad = \quad \omega * \tau \quad = \quad \overline{\omega \parallel \tau}$$

=



By convention orders are represented with top root and then seaweeds are oriented anti-clockwise:

$\overline{a < b < c}$ =

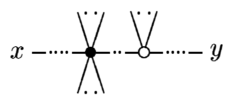


One extends the definition of seaweeds to the rootless planar trees on n-ary-nodes ($n \geq 3$).

Definition 7 (normal form). *Let α be a series-parallel order variety. Let the seaweeds representing α be considered modulo assocativity of $\circ$ and $\bullet$: there is not two nodes linked with a same label, and there is not binary or unary nodes. The equivalence class of such seaweeds modulo commutativity of $\circ$ has a unique representative which is said in* normal form.

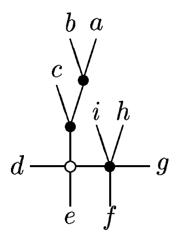
The uniqueness comes from the next proposition.

Remark 1. A seaweed is in normal form if it has n-ary nodes and verifies that all paths between two leaves are a sequence of alternate $\bullet$ and $\circ$ nodes. Afterwards for a seaweed (not specially in normal form) we denote such alternate paths between arbitrary leaves x and y by the following figure:



This notation does not presuppose that this alternate path starts by a $\bullet$-node and finishes by a $\circ$-node.

Example 1. Let α be the closure of $[((a < b) < c) \parallel d] < [e \parallel (f < (g < h) < i)]$. Then the path between d and g is



To be convenient we only use seaweeds in normal form. So $\circ$-nodes are commutative. When it is not ambiguous, we use an order variety instead of its representation.

2.2 Seesaw and Entropy

Definitions 8. *Let ω and τ be series-parallel orders on a same given set. The equivalence relation* seesaw *is defined by $\overline{\omega} = \overline{\tau}$. The relation* entropy $\trianglelefteq$ *is defined by $\omega \trianglelefteq \tau$ iff $\omega \subseteq \tau$ and $\overline{\omega} \subseteq \overline{\tau}$.*

Proposition 2. *In the case of series-parallel orders, seesaw (resp. entropy) turns out to be the least equivalence $\backsim$ (resp. the least reflexive transitive relation) given by:*

$$(\omega_1 \parallel \omega_2) \backsim (\omega_1 < \omega_2) \qquad\qquad \textit{(resp. } \omega[\omega_1 \parallel \omega_2] \trianglelefteq \omega[\omega_1 < \omega_2] \textit{)}$$

Facts 3. i) Entropy is a partial order, compatible with restriction and the serial and parallel sums of orders,

ii) entropy between orders corresponds to inclusion of order varieties: let α and β be order varieties on X, and $x \in X$, we have

$$\alpha \subseteq \beta \text{ iff } \alpha_x \trianglelefteq \beta_x.$$

This is independent from the choice of x,

iii) entropy is performed on seaweeds by changing some $\bullet$-nodes into $\circ$-nodes.

2.3 Wedge and Identification

Definitions 9 (wedge). *Let $(\omega_i)_{i\in I}$ be a non empty family of partial orders on a same set. The* wedge $\bigwedge_{i\in I}\omega_i$ *is the largest partial order (w.r.t. $\trianglelefteq$) such that*

$$(\bigwedge_{i\in I}\omega_i) \trianglelefteq \omega_i \textit{ for all } i \in I.$$

Let $(\alpha_i)_{i\in I}$ be a non empty family of order varieties on a set X. The wedge $\bigwedge_{i\in I}\alpha_i$ *is*

$$(\bigwedge_{i\in I}(\alpha_i)_x) * x$$

for an arbritrary $x \in X$.

Facts 4. i) Partial orders on a given set form a complete inf-semi-lattice for entropy and wedge,

ii) the wedge is not intersection in general,

iii) the wedge is not series-parallel in general, even if all ω_i are series-parallel,

iv) the wedge (partially) commutes with restriction:

$$\text{if } Y \subseteq |\omega_i| \text{ then } (\bigwedge_{i\in I}\omega_i) \upharpoonright Y \trianglelefteq (\bigwedge_{i\in I}\omega_i \upharpoonright Y),$$

v) the two notions of wedge are related by:

$$(\bigwedge_{i\in I}\alpha_i)_x = \bigwedge_{i\in I}(\alpha_i)_x \quad \text{and} \quad (\bigwedge_{i\in I}\omega_i) * x = \bigwedge_{i\in I}(\omega_i * x)$$

Definition 10 (identification). *Let α be an order variety on a set $X \uplus \{x\} \uplus \{y\}$, and let $z \notin X \cup \{x, y\}$. The* identification $\alpha[z/x, y]$ *of x and y into z in α is the order variety defined by:*

$$\alpha[z/x, y] = \alpha \upharpoonright_{X\cup\{x\}} [z/x] \ \wedge \ \alpha \upharpoonright_{X\cup\{y\}} [z/y]$$

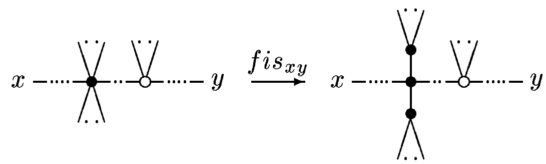
Lemma 1. *i) $\alpha[z/x, y]_z * (x \parallel y) \subseteq \alpha$,*

ii) Let α be an order variety on $X \uplus \{x\} \uplus \{y\}$ and ω be a partial order on X such that $\omega * (x \parallel y) \subseteq \alpha$. Then $\omega * (x \parallel y) \subseteq \alpha[z/x, y]_z * (x \parallel y)$, or equivalently $\omega \trianglelefteq \alpha[z/x, y]_z$.

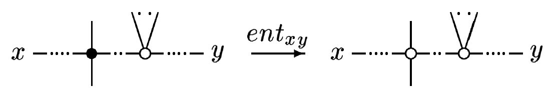
Proof. See the proof of lemma 3.35 in [Rue00]. □

Definition 11. *Let α be a series-parallel order variety represented by a seaweed S. We define the seaweed $S\langle z/x, y\rangle$ by the following sequence on the alternate path between x and y in S:*

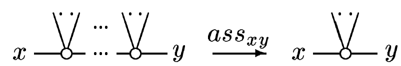
1. *fis_{xy}: transform every $\circ$-node belong the path between x and y. This is called "fission":*



2. *ent_{xy}: apply entropy belong the path between x and y:*



3. *ass_{xy}: apply associativity belong the path between x and y:*



4. *substitute z for $x \parallel y$.*

Lemma 2. *i) Identification in order varieties is monotonic (for the inclusion),*
ii) If v denotes a map such that $v(S)$ is the order variety corresponding to the seaweed S then, for S and T seaweeds,

$$v(S) \subseteq v(T) \implies v(S\langle z/x, y\rangle) \subseteq v(T\langle z/x, y\rangle)$$

Proof. Let α and β be order varieties on a set X such that $\alpha \subseteq \beta$. We have $\alpha[z/x, y] \subseteq \beta[z/x, y]$ i.e. identification is monotonic because the wedge is clearly monotonic. On the seaweeds, the only nodes which are different in the representation of α and β are the $\circ$-nodes in the representation of α which correspond to $\bullet$-nodes in the representation of β. If so,

- by definition, for all $x, y \in X$, $fis_{xy}(\alpha)$ and $fis_{xy}(\beta)$ represent always the same included order varieties,
- all differents nodes on the path between x and y in $ent_{xy}(fis_{xy}(\alpha))$ become $\circ$-nodes and stay $\circ$-nodes in $ent_{xy}(fis_{xy}(\beta))$,
- all others are unchanged.

Hence the order variety represented by $ent_{xy}(fis_{xy}(\alpha))$ is included in the one which is represented by $ent_{xy}(fis_{xy}(\beta))$ □

Proposition 3. *Let α be a series-parallel order variety on a set $X \uplus \{x\} \uplus \{y\}$, and let $z \notin X \cup \{x, y\}$. If the seaweed S represents α then the seaweed $S\langle z/x, y\rangle$ represents the identification $\alpha[z/x, y]$.*

Proof. Using the notations of lemma 2,

$\supseteq$) With the hypothesis, we have that $\alpha[z/x, y]_z * (x \parallel y) \subseteq \alpha$. Then by the previous lemma,

$$v((\alpha[z/x, y]_z * (x \parallel y))\langle z/x, y\rangle) \subseteq v(\alpha\langle z/x, y\rangle)$$

So by definition of $S\langle z/x, y\rangle$, we obtain that

$$\alpha[z/x, y]_z * z \subseteq v(\alpha\langle z/x, y\rangle)$$

For all $u \in |\alpha|$ $\alpha_u * u = \alpha$, thus

$$\alpha[z/x, y] \subseteq v(\alpha\langle z/x, y\rangle)$$

$\subseteq$) By definition, $fis_{xy}(\alpha)$ represents the same order variety as α and for all order variety β, $v(ent_{xy}(\beta)) \subseteq \beta$. Thus $v(ent_{xy}(fis_{xy}(\alpha))) \subseteq \alpha$. Then we again have that $v(S\langle z/x, y\rangle)_z * (x \parallel y) \subseteq \alpha$. Then by definition and as identification is monotonic we have

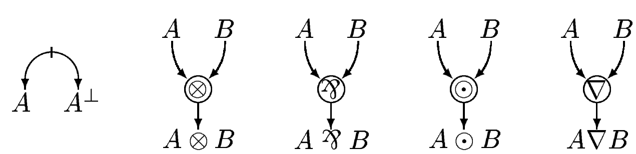
$$v(S\langle z/x, y\rangle)_z * z \subseteq \alpha[z/x, y] \qquad i.e. \qquad v(S\langle z/x, y\rangle) \subseteq \alpha[z/x, y]$$

□

3 MNL Proof Nets

We restrict us to the multiplicative fragment of NL i.e. to the formulae built from atoms $a, a^\perp, \dots$, the commutative conjonction and disjonction (resp. $\otimes$ and $⅋$) and the non commutative conjonction and disjonction (resp. $\odot$ and ∇).

Definitions 12 (links and proof structures). *A* link *is an object for which the* premises *(input edges) and the* conclusions *(output edges) are two disjoint sets of vertices:*



A proof structure *G over the vertices $V(G)$ is a set of links such that:*

- *every vertex in $V(G)$ is a conclusion of (only) one link,*
- *every vertex in $V(G)$ either is a* conclusion of *G (i.e. is not a premise of any link of G) or is a premise of (only) one link,*
- *the set γ of the conclusions of G (writen $G \vdash \gamma$) is not empty.*

3.1 Maieli Correctness Criterion

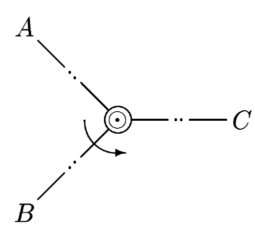
Definitions 13 (Switchings). *Let G a proof structure. A* switching *s for G is given by mutilating one premise-edge for each ∇-link and $⅋$-link. Any ∇-link (resp. $⅋$-link) admits a left/right mutilation wich is called the* left/right switch *of ∇ (resp. $⅋$). Any switching s for a proof structure G induces a graph on $V(G)$ which is called the* switched proof structure *$s(G)$.*

Fact 5. If a switched proof structure S induced by a proof structure $G \vdash \gamma$ is acyclic and connected then (viewing $\otimes$-nodes as $\circ$-nodes and $\odot$-nodes as $\bullet$-nodes, and effacing binary nodes implie that) S is a seaweed which represents a series-parallel order variety on γ.

Definition 14 (Suitable conclusion). *Let $G \vdash \gamma$ be a proof structure and s be a switching for G. Let a vertex of $s(G)$ labelled $A \nabla B$. A* conclusion suited to *$A \nabla B$ is a vertex $C \in \gamma$ such that there is no paths from $A \nabla B$ to C in $s(G)$ which is oriented in G.*

Definition 15 (M-correctness). *A proof structure G is* M-correct *iff for any switching s:*

1. *the switched proof structure $s(G)$ is acyclic and connected,*
2. *for any ∇-link labelled $A \nabla B$, for any suitable conclusion C, the intersection of the paths AB, AC and BC in the seaweed $s(G)$ is a $\odot$-node in G with the following anti-clockwise order:*



Theorem 1 ([Mai00]). *A proof structure G is M-correct iff G is sequentialisable.*

In the commutative fragment (multiplicative linear logic) the Maieli correctness criterion is exactly the Danos-Regnier's (the first step in the previous definition). The latter is well known to be in exponential time: if n is the number of $⅋$-links in a proof structure G then the Danos-Regnier correctness criterion checks 2^n graphs and cannot be inferred by the inspection of a fixed subset of the switches of G. So the Maieli correctness criterion is at least in exponential time.

3.2 The Size of a Proof Structure

If we call *size* of a proof structure G the number of registers $size(G)$ required for the memorisation of G on some ramdom access machine (RAM) then in any non redundant coding, $size(G)$ is linear in the number of vertices of G i.e. $size(G) = \Theta(|V(G)|)$. Moreover, since the number of links in G is linear in the number of vertices of G, $size(G) = \Theta(|G|)$ also. In the following, one shall analyse the worst case asymptotic complexity of correctness in terms of $size(G)$.

Remark 2. It is usual to describe a proof net with only one conclusion: built a tree of $\parr$-links of the conclusions. This description does not improved the worst case asymptotic complexity.

4 Sequent Calculus

Definition 16. *A* sequent $\vdash \alpha$ *consists of a series-parallel order variety* α *of formula occurences.*

Identity group

$$\frac{}{\vdash A * A^{\perp}}\ \text{(identity)} \qquad \frac{\vdash \omega * A \quad \vdash \omega' * A^{\perp}}{\vdash \omega * \omega'}\ \text{(cut)}$$

Structural group

$$\frac{\vdash \beta}{\vdash \alpha}\ \text{(entropy)},\ \alpha \subseteq \beta$$

Logic group

$$\frac{\vdash \omega * A \quad \vdash \omega' * B}{\vdash (\omega' < \omega) * A \odot B} \qquad \frac{\vdash \omega * (A < B)}{\vdash \omega * A \nabla B}$$

$$\frac{\vdash \omega * A \quad \vdash \omega' * B}{\vdash (\omega \parallel \omega') * A \otimes B} \qquad \frac{\vdash \omega * (A \parallel B)}{\vdash \omega * A \parr B}$$

We can have a sequent calculus without an explicit rule for entropy: only the $\parr$-rule need this rule. So we can substitute the entropy rule and the $\parr$-rule by the following one given in [AR00]:

$$\frac{\vdash \alpha[A, B]}{\vdash \alpha[A \parr B / A, B]}\ (\parr\star\text{-rule})$$

where $\alpha[A \parr B/A, B]$ is the identification of definition 10. Indeed in the multiplicative fragment the two versions are equivalent: by lemma 1, we have

- $\alpha[A \parr B/A, B]_{A \parr B} * (A \parallel B) \subseteq \alpha$, so entropy and $\parr$-rule can mimic the $\parr\star$-rule,
- $\omega * (A \parallel B) \subseteq \alpha$ implies $\omega * (A \parallel B) \subseteq \alpha[A \parr B/A, B]_{A \parr B} * (A \parallel B)$, so $\parr\star$-rule is an optimized version of $\parr$-rule where entropy has been minimized.

See [Rue00] for a detailled explaination and consequences of removing the entropy rule in the full NL.

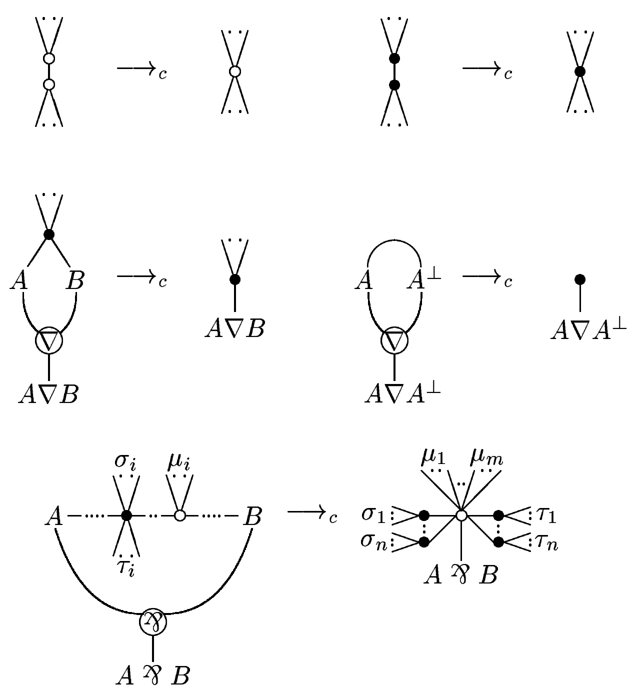
5 Contractibility Criterion

Definition 17 (Contracting proof structure). *A* contracting proof structure *G over the vertices $V(G)$ is a set of links and seaweeds such that:*

- *every vertex in $V(G)$ either is a conclusion of (only) one link or is an extremity of (only) one seaweed,*
- *every vertex in $V(G)$ either is a* conclusion of *G (i.e. is not a premise of any link of G) or is a premise of (only) one link or is an extremity of (only) one seaweed,*
- *the set γ of the conclusions of G (writen $G \vdash \gamma$) is not empty.*

We consider the following system of rewriting rules called *contraction rules* which is applied from contracting proof sub-structures to seaweeds:

- no rules for axiom-link, $\otimes$-link, $\odot$-link: an axiom-link is already a seaweed, a $\otimes$-link is viewed as a $\circ$-node and a $\odot$-link as $\bullet$-node,
- associativity rules, sequential rules and par rule:



The par contraction rule corresponds to the transformation of a seaweed S and a $\parr$-link in $S\langle A \parr B/A, B\rangle$. We have $\mid n - m \mid \leq 1$ due to the alternate path between A and B.

Note that proof structures are particular contracting proof structures.

Definition 18 (Contractibility criterion). *A contracting proof structure G is c-correct if $\rightarrow_c^*$ reduces G to a seaweed.*

Theorem 2 (Confluence). *The system of contraction rules is confluent.*

Proof. There is no problems to do interactions with local rules like ∇-rule and associativity rules. The cases $\parr$-rule vs $\parr$-rule and $\parr$-rule vs ∇-rule are treated in the appendix. The $\parr$-rule vs associativity rules are exactly the same as vs ∇-rule. $\square$

Theorem 3 (Sequentialisation). *A proof structure G is c-correct iff G is sequentialisable.*

The proof can be deduced from the sequentialisation theorem from next section by using proposition 3.

Corollary 1 (Correctness). *A proof structure G is c-correct iff G is M-correct.*

This correctness criterion acts on an initial contracting proof structure G with $size(G)$ links and nodes of seaweeds (recall that axiom-links are seaweeds). Let $n = \Theta(size(G))$ be the sum of weighted number of links and the number of nodes. The analysis of each step of reduction shows that the number of links always decreases and that:

- the associativity decreases the number of nodes of the seaweed,
- the ∇-rule decreases the number of links without changing the number of nodes. In the degenerated case, to assign a weight of 2 to ∇-links allows to decrease n.
- the $\parr$-rule acts on an alternate path. Let r and s be respectively the number of $\bullet$-nodes and $\circ$-nodes on this path. The contraction rule reduces the $r + s$ nodes to $2r + 1$ nodes with $\mid r - s \mid \leq 1$ due to the alternate. Then in the worst case, the difference is of 2. So to assign a weight of 3 to $\parr$-links allows to decrease n.

So in the worst case (when G is c-correct), the number of steps of reduction in this criterion is linear in $size(G)$. Each step of reduction is a choice of a rule and the application of this rule. This decreases n down to 0.

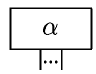
Expect in the case of $\parr$-rule, the complexity of choosing a rule is linear in $size(G)$. In order to enable the choice of $\parr$-rule to have the same complexity mark each seaweed with an integer.

Applying a reduction rule is linear in $size(G)$ in the worst case: the associativity rules and the ∇-rules are in constant time, the $\parr$-rule is linear in the length of the path. Indeed this latter rule consists of an $S\langle z/x, y\rangle$ operation of some A and B into $A \parr B$: this requires a linear time for fis_{AB} as well as for ent_{AB} and for ass_{AB}.

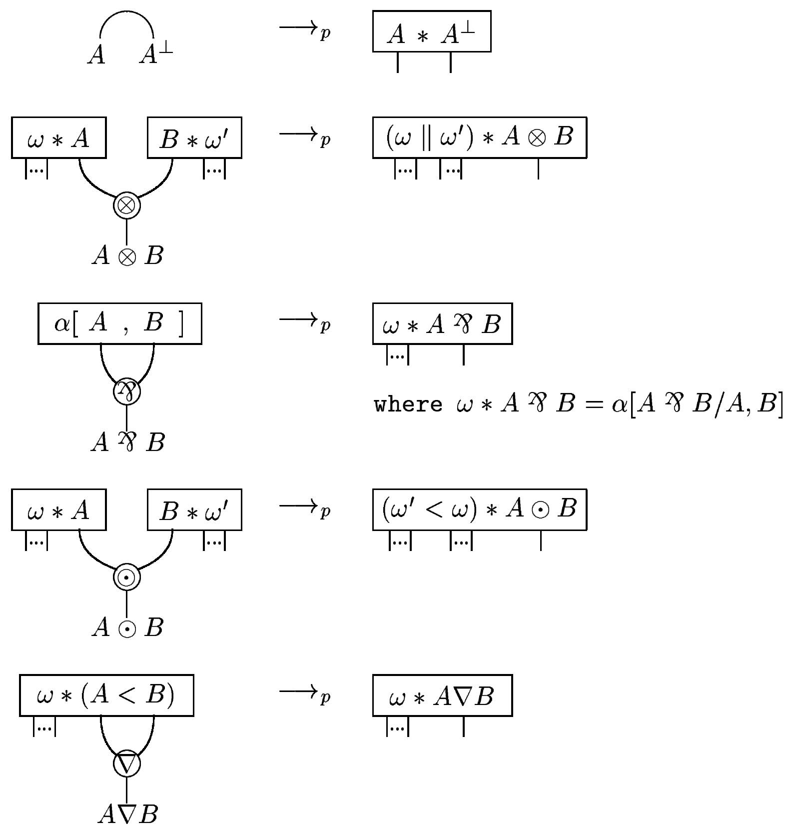
Therefore this correctness criterion is in quadratic time.

6 Parsing

In the previous section, we are dealing with contracting proof structure i.e. with seaweeds. Here is the same quadratic time parsing algorithm that checks the correctness of a proof structure but the objects are directly order varieties. From the sequent calculus one can find a non determinist algorithm for the sequentialisation of proof structures. We present here a determinist reformulation. In order to show this, we introduce the parsing box which contains an order variety: let α be an order variety on a set X,



is called the *parsing box* α. This a kind of link without premises which has one conclusion for each element of X. We use the following set of parsing rules $\rightarrow_p$:



By the properties of $\rightarrow_c$ and proposition 3, we obtain the confluence of $\rightarrow_p$.

Lemma 3. *If Π is a proof in cut-free MNL of $\vdash \alpha$ then we can naturally associate with Π a proof net Π^- which reduces to the parsing box $\beta \supseteq \alpha$.*

Proof. The proof net Π^- is defined by induction on Π as follow:

Case 1: Π is an axiom $\vdash A * A^\perp$; one must define Π^- as the axiom link: it is reduced to the parsing box $A * A^\perp$.

Case 2: Π is obtained by a $\otimes$-rule from λ_1 and λ_2 which are respectively the proofs of $\vdash \omega * A$ et $\vdash B * \omega'$; By induction hypothesis, λ_1^- and λ_2^- are respectively reduced to the parsing boxes $\beta * A \supseteq \omega * A$ and $B * \beta' \supseteq B * \omega'$. Then we define Π^- as the tensor on A and B of λ_1 and λ_2: it is reduced to the parsing box $(\beta \parallel \beta') * A \otimes B \supseteq (\omega \parallel \omega') * A \otimes B$.

Case 3: Π is obtained by a $⅋$-rule from λ which is a proof of $\vdash \alpha[A, B]$; By induction hypothesis, λ is reduced to the parsing box $\beta[A, B] \supseteq \alpha[A, B]$. Then we define Π^- as the par on A and B of λ: it is reduced to the parsing box $\beta[A ⅋ B/A, B] \supseteq \alpha[A ⅋ B/A, B]$ by lemma 2.

Case 4: Π is obtain by an entropy rule from λ which is a proof of $\vdash \beta$ with $\beta \supseteq \alpha$. Then we define Π^- as λ.

Case 5: Π is obtained by a $\odot$-rule or a ∇-rule; one can build Π^- like respectively in cases 2 and 3 if we recall that $\beta \subseteq \alpha[\omega < \omega']$ implies $\beta[\omega < \omega']$. $\square$

Lemma 4. *If a proof net λ is reduced to the parsing box α then we can find a proof Π in sequent calculus of $\vdash \alpha$ such that $\Pi^- = \lambda$.*

Proof. By induction on the length of the reduction:

i) one step of reduction: λ is an axiom link which is reduced in the parsing box $A * A^\perp$. The claim is proved by taking as Π the axiom $\vdash A * A^\perp$.

ii) several steps of reduction: the system of parsing rules is confluent, so the last rule applied to λ is one of the followings:

- Tensor parsing rule: we have a proof net λ reduced in a parsing box $\beta = (\omega \parallel \omega') * A \otimes B$. So by the last step, there are the proof nets λ_1 and λ_2 reduced respectively in the parsing boxes $\omega * A$ and $B * \omega'$. By induction hypothesis, there is the proofs Π_1 and Π_2 in sequent calculus resp. of $\vdash \omega * A$ and $\vdash B * \omega'$ such that $\Pi_1^- = \lambda_1$ and $\Pi_2^- = \lambda_2$.
  So by taking as Π the tensor of $\vdash \omega * A$ and $\vdash B * \omega'$ we obtain a proof of $\vdash \beta$ such that $\Pi^- = \lambda$.
- Par parsing rule: we have a proof net λ reduced in a parsing box $\beta = \alpha[A ⅋ B/A, B]$. So by the last step, there is a proof net λ_1 reduced in a parsing box $\alpha[A, B]$. By induction hypothesis, there is a proof Π_1 in sequent calculus of $\vdash \alpha[A, B]$ such that $\Pi_1^- = \lambda_1$. One can take as Π the $⅋\star$-rule of $\alpha[A, B]$ then $\Pi^- = \lambda$.
- The others parsing rules can be treated as in previous cases. $\square$

Theorem 4 (Sequentialisation). *Let us say that the proof structure G is p-correct when $\rightarrow_p$ reduces G to a parsing box. Then, G is p-correct iff G is sequentialisable.*

Proof. Deduce from lemma 3 and 4. $\square$

Corollary 2. *A proof structure G is p-correct iff G is M-correct.*

7 Conclusion

These criteria are like the others one from the cut management point of view. Given a sequent calculus proof of NL with cuts P, there is an associate proof net with cuts. The standard cut elimination gives a cut-free proof net which can be sequentialised in a cut-free sequent calculus proof. Then this proof can be obtained from P by cut elimination. The question is to know what happens during the intermediate steps of cut elimination: is there a correctness criterion? i.e. is there a sequentialisation theorem extended to proof nets with cuts? In the commutative part of NL, the sequentialisation of proof nets with cuts can be solved by seeing a cut like a tensor for correctness. Detailed explanations can be found in [Laf95]. But these cannot be done here[1]. So how to deal with a contractibility correctness criterion for proof nets with cuts?

The obtained correctness criterion is quadratic but there is a linear alternative in the case of linear logic [Gue99]. This result comes from a reformulation of Danos contractibility criterion which is essentially based on unification. This gives a quasi-linear time algorithm. Guerrini's approach does not trivially generalize to this case. One can derive a trivial unification algorithm for NL from the parsing one but without improving the complexity. In fact, the needed information to make the unification is exactly that which is contained in the structure of order varieties. This new Danos contractibility style criterion for NL is a first step to obtain a linear correctness criterion.

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A Appendix: Confluence Proof

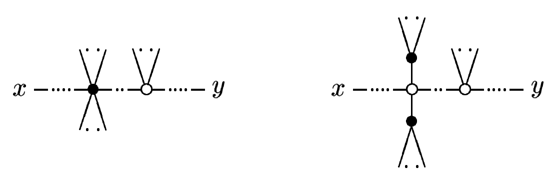
A.1 $\parr$-parsing rule v.s. $\parr$-parsing rule

Let α be a order variety on a set X. We want to prove that for all distincts $A, B, C, D \in X$ we have $\alpha\langle A \parr B/A, B\rangle\langle C \parr D/C, D\rangle = \alpha\langle C \parr D/C, D\rangle\langle A \parr B/A, B\rangle$. In fact the $\alpha\langle x \parr y/x, y\rangle$ operation can be decomposed in steps on sub-seaweeds: fis_{xy}, ent_{xy} and ass_{xy} are well defined for all nodes x, y in α. Note that the nodes x and y are not transformed in this processes: they are not in the open path between x and y (denoted $path(x, y)$).

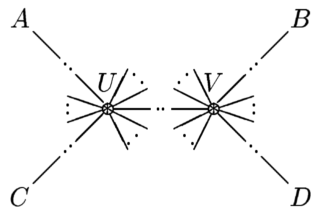
We are only interrested in fis_{xy} and ent_{xy}. So for all x, y, z, t nodes in α we have the followings equations:

$$\begin{aligned} ent_{xy}(fis_{xy}(S_{xy})) &= T_{xy} \\ ent_{xy}(fis_{xy}(T_{zt})) &= T_{zt} \\ ent_{xy}(fis_{xy}(S_{zt})) &= S_{zt} \text{ if } path(x, y) \cap path(z, t) = \emptyset \end{aligned}$$

where we denote respectively by S_{xy} and T_{xy} the following forms of sub-seaweed of α which belong the path between x and y:



Let A, B, C, D be in X and U, V be two nodes in α such that $path(A, B) \cap path(C, D) = path(U, V)$. Then α is represented by:



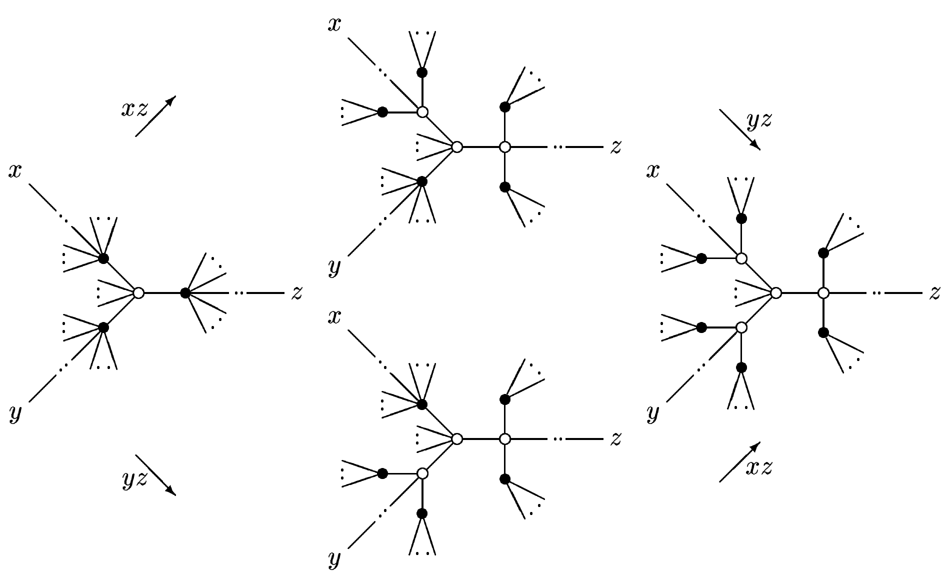
where U and V are undefined node. We have the following confluents diagrams:

$$
\begin{tikzcd}
S_{AU} \arrow[r, "CD"] \arrow[d, "AB"] & S_{AU} \arrow[d, "AB"] \\
T_{AU} \arrow[r, "CD"] & T_{AU}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
S_{UV} \arrow[r, "CD"] \arrow[d, "AB"] & T_{UV} \arrow[d, "AB"] \\
T_{UV} \arrow[r, "CD"] & T_{UV}
\end{tikzcd}
$$

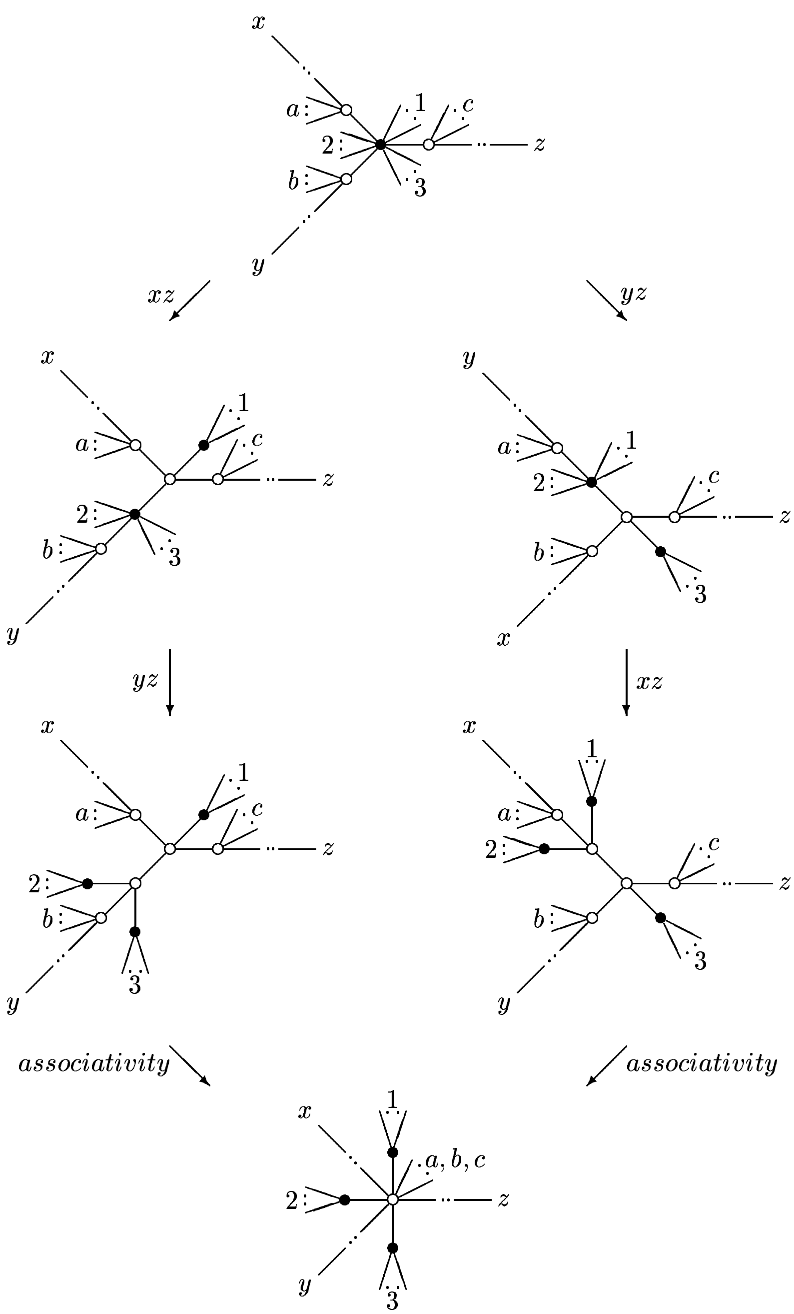
where $\xrightarrow{xy}$ stands for $\xrightarrow{ent_{xy}(fis_{xy}(\cdot))}$. And we have the same from S_{VB} (resp. S_{CU} and S_{VD}) to T_{VB} (resp. T_{CU} and T_{VD}).

So what happens to U and V? They can be treated in the same way: it depends only on the nature of the node.

- It is a parallel node:



- It is a serial node:



A.2 ⅋-parsing rule v.s. ∇-parsing rule