Extrapolated Multistep Methods and Local-Global Step Size Control*

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Abstract. In recent papers [7], [8], [9], [11] the technique for a local and global errors estimation and the local-global step size control were presented to solve both ordinary differential equations and semi-explicit index 1 differential-algebraic systems by multistep methods with any reasonable accuracy obtained automatically. Now we extend those results to the concept of multistep extrapolation and demonstrate on numerical examples how such methods work in practice.

1 Introduction

In [7] we presented a new procedure to control a step size for linear multistep methods applied to semi-explicit index 1 differential-algebraic systems of the form

$$x'(t) = g\big(x(t), y(t), \alpha(t)\big), \tag{1a}$$

$$y(t) = f\big(x(t), y(t), \alpha(t)\big), \tag{1b}$$

$$x(t_0) = x^0,\ y(t_0) = y^0 \tag{1c}$$

where $t \in [t_0, t_0 + T]$, $x(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^n$, $\alpha(t) \in \mathbf{R}^l$ ($\alpha(t)$ is a known vector), $g : D \subset \mathbf{R}^{m+n+l} \to \mathbf{R}^m$, $f : D \subset \mathbf{R}^{m+n+l} \to \mathbf{R}^n$, and where initial conditions (1c) are consistent; i.e., $y^0 = f(x^0, y^0, \alpha(t_0))$. In contrast to the standard approach, the new error control mechanism was based on monitoring and controlling both the local error and the global error of multistep formulas. As a result, such methods with the local-global step size control solve differential-algebraic equations with any prescribed accuracy (up to round-off errors).

For implicit multistep methods, we gave the minimum number of both full and modified Newton iterations allowing the iterative approximations to be correctly used in the procedure of the local-global step size control. We also discussed a validity of fixed-point iterations for high accuracy solving index 1 differential-algebraic systems.

The only drawback of the local-global step size control was high execution time for some test problems. It was a consequence of recomputing the numerical solution as many times as the error control mechanism had required that. In

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order to decrease the execution time, we developed a new advanced version of the local-global step size control in [9] and [11]. We showed that this local-global step size control is not only effective, but it is also reliable in practice.

The conception of multistep extrapolation was presented for ordinary differential equations in [10]. Now we extend that approach in order to construct an extrapolation process in the class of linear multistep methods for index 1 differential-algebraic systems. Such extrapolation methods give a way to increase the order of an underlying multistep method without repeated integrations with smaller step sizes. Another advantage of multistep extrapolation methods is breaking the first Dahlquist barrier [1] which imposes a restrictive limitation on the order of stable multistep methods. For example, our procedure makes it possible to increase the accuracy of Backward Differentiation Formulas (BDFs) up to the 10-th order preserving their stability.

2 Local and Global Errors Estimation

In the rest of this paper we consider a differential-algebraic system (DAS) of the form

$$x'(t) = g\big(x(t), y(t)\big), \tag{2a}$$

$$y(t) = f\big(x(t), y(t)\big), \tag{2b}$$

where $t \in [0,T]$, $x(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^n$, $g : D \subset \mathbf{R}^{m+n} \to \mathbf{R}^m$, $f : D \subset \mathbf{R}^{m+n} \to \mathbf{R}^n$, and where the initial conditions $x(0) = x^0$, $y(0) = y^0$ are consistent; i.e., $y^0 = f(x^0, y^0)$. To solve problem (2) numerically, we introduce a nonuniform grid $w_\tau = \{t_{k+1} = t_k + \tau_k,\ k = 0, 1, \ldots, K-1,\ t_0 = 0,\ t_K = T\}$ on the interval $[0,T]$ and define τ as the diameter of the grid w_τ; i.e., $\tau = \max_{0 \le k \le K-1} \{\tau_k\}$. Having applied a stable linear l-step method to problem (2), we obtain the following discrete problem:

$$\sum_{i=0}^{l} a_i(k) x_{k+1-i} = \tau_k \sum_{i=0}^{l} b_i(k) g(x_{k+1-i}, y_{k+1-i}), \tag{3a}$$

$$y_{k+1} = f(x_{k+1}, y_{k+1}), \quad k = l-1, l, \ldots, K-1. \tag{3b}$$

Here, we have used the indirect approach to construct multistep methods for DASs and have taken into account that coefficients of the underlying multistep method may be variable [3], [4]. We consider the starting values x_i and y_i, $i = 0, 1, \ldots, l-1$, to be known.

Further we assume that the underlying multistep formula is of order s and problem (2) satisfies the following conditions on the compact set $D_1 \subset \mathbf{R}^{m+n}$:
I. *The smoothness condition.* The mapping $G : D_1 \subset \mathbf{R}^{m+n} \to \mathbf{R}^{m+n}$ is sufficiently differentiable on the set D_1, where G is obtained by combining the mappings g and f; i.e., $G \stackrel{\text{def}}{=} (g^T, f^T)^T$.
II. *The nonsingularity condition.* The matrix $I_n - \partial_y f(x,y)$ is nonsingular for any $z \stackrel{\text{def}}{=} (x^T, y^T)^T \in D_1$, where I_n is the identity matrix of dimension n, and $\partial_y f(x,y)$ denotes the partial derivative of mapping f with respect to y.

III. *The inclusion condition*[1]. There exists a convex set D_0 such that $z^0 \stackrel{\text{def}}{=} ((x^0)^T, (y^0)^T)^T \in D_0 \subset D_1$.

If conditions I–III hold then problem (2) is of index 1 and it has a unique solution $z(t) \stackrel{\text{def}}{=} \big(x(t)^T, y(t)^T\big)^T \in D_0$. Moreover, the fixed-step size version of method (3) is convergent and is of order s (for example, see [2]–[6]). Moreover, the coefficients $a_i(k)$, $b_i(k)$, $i = 0, 1, \ldots, l-1$, and the step size ratios τ_{k+1}/τ_k, $k = l-1, l, \ldots, K-1$, have to be bounded, and $|a_0(k)| \geq a_0 > 0$ (a_0 is a constant) to insure convergence of variable-step size method (3). We further consider stable multistep methods satisfying these conditions and require additionally only grids w_τ with any bounded step size rations; i.e., τ_i/τ_k, $i, k = l-1, l, \ldots, K-1$.

It follows from the construction of multistep method (3) that its local truncation error at the point t_{k+1} is

$$\begin{aligned} L\big(t_{k+1}, z(t), \tau_k\big) &\stackrel{\text{def}}{=} \sum_{i=0}^{l} a_i(k) x(t_{k+1-i}) - \tau_k \sum_{i=0}^{l} b_i(k) g\big(z(t_{k+1-i})\big) \\ &= \sum_{i=0}^{l} a_i(k) x(t_{k+1-i}) - \tau_k \sum_{i=0}^{l} b_i(k) x'(t_{k+1-i}). \end{aligned} \tag{4}$$

We denote the approximate solution of DAS (2) from method (3) by $z_{k+1} \stackrel{\text{def}}{=} \big(x_{k+1}^T, y_{k+1}^T\big)^T$, $k = l-1, l, \ldots, K-1$, and the error of method (3) at the grid points by $\Delta z_{k+1-i} \stackrel{\text{def}}{=} z(t_{k+1-i}) - z_{k+1-i}$, $i = 1, 2, \ldots, l$. Now we subtract (3a) from (4) and (2b) from (3b) to derive the error equation[2]

$$\Delta z_{k+1} = Q_k^{-1}(z_{k+1}) \begin{pmatrix} c_k \\ 0 \end{pmatrix} + O\left(\sum_{i=0}^{l} \Delta z_{k+1-i}^2 \right), \quad k = l-1, l, \ldots, K-1, \tag{5}$$

where the $(m+n \times m+n)$-matrix

$$Q_k(z_{k+1}) \stackrel{\text{def}}{=} \begin{pmatrix} a_0(k) I_m - \tau_k b_0(k) \partial_x g\big(z_{k+1}\big) & -\tau_k b_0(k) \partial_y g\big(z_{k+1}\big) \\ -\partial_x f\big(z_{k+1}\big) & I_n - \partial_y f\big(z_{k+1}\big) \end{pmatrix},$$

the m-dimension vector

$$\begin{aligned} c_k \stackrel{\text{def}}{=} &\sum_{i=1}^{l} \Big(\tau_k b_i(k) \partial_x g(x_{k+1-i}, y_{k+1-i}) - a_i(k) I_m \Big) \Delta x_{k+1-i} \\ &+ \sum_{i=1}^{l} \tau_k b_i(k) \partial_y g(x_{k+1-i}, y_{k+1-i}) \Delta y_{k+1-i} + L\big(t_{k+1}, z(t), \tau_k\big), \end{aligned}$$

and 0 denotes the null vector of dimension n.

[1] The inclusion $D_0 \subset D_1$ implies that D_1 contains D_0 together with some neighborhood

[2] Here and further any raising of vectors to a power is component-wise

By definition, if we put $\Delta z_{k+1-i} = 0$, $i = 1, 2, \ldots, l$, in formula (5) and omit the higher order terms we obtain the local error of method (3) with accuracy $O(\tau^{2s+2})$; i.e.,

$$\Delta\tilde{z}_{k+1} \cong Q_k^{-1}(\tilde{z}_{k+1}) \begin{pmatrix} L\big(t_{k+1}, z(t), \tau_k\big) \\ 0 \end{pmatrix}, \quad k = l-1, l, \ldots, K-1, \tag{6}$$

where $\tilde{z}_{k+1} \stackrel{\text{def}}{=} \big((\tilde{x}_{k+1})^T, (\tilde{y}_{k+1})^T\big)^T$ is the solution of problem (3) provided that $z_{k-i} = z(t_{k-i})$, $i = 0, 1, \ldots, l-1$. Thus, we have derived formula (6) for a high accuracy computation of the local error of multistep method (3). The only problem is how to evaluate $L\big(t_{k+1}, z(t), \tau_k\big)$ at the grid nodes.

To do this, we expand the functions $x(t)$ and $x'(t)$ in (4) in Taylor series at the point t_{k+1} and use the definition of the order of variable-step size multistep formulas (see, for example, [3]) to estimate an approximate value of the local truncation error:

$$\begin{aligned} \tilde{L}\big(t_{k+1}, z(t), \tau_k\big) &\stackrel{\text{def}}{=} \frac{(-1)^{s+1}}{(s+1)!} x_{k+1}^{(s+1)} \\ &\times \sum_{i=1}^{l} \left(a_i(k) \sum_{j=0}^{i-1} \tau_{k-j} + (s+1)\tau_k b_i(k) \right) \left(\sum_{j=0}^{i-1} \tau_{k-j} \right)^s. \end{aligned} \tag{7}$$

Here, we have neglected higher order terms and have replaced the $(s+1)$-order derivative of the exact solution $x^{(s+1)}(t_{k+1})$ with the approximate value $x_{k+1}^{(s+1)}$ computed by an interpolation formula of sufficiently high degree. As a consequence, (6) and (7) yield the formula for estimating the local error of method (3) with accuracy $O(\tau^{s+2})$

$$\Delta\tilde{z}_{k+1} \cong Q_k^{-1}(\tilde{z}_{k+1}) \begin{pmatrix} \tilde{L}\big(t_{k+1}, z(t), \tau_k\big) \\ 0 \end{pmatrix}, \quad k = l-1, l, \ldots, K-1. \tag{8}$$

Finally, we must note that local error estimation (8) works correctly only if the solution z_k is improved in the course of numerical integration of DAS (2), for example, as follows:

$$\tilde{z}_k = z_k + \Delta\tilde{z}_k, \quad k = l, l+1, \ldots, K.$$

Actually, it is important to understand that in practice we can apply any appropriate way to get a numerical solution of problem (2) with the accuracy of $O(\tau^{s+2})$.

Now we turn our attention to the global error estimation. Remember equation (5) obtained for the error of method (3). Assuming further the errors of the starting values Δz_i, $i = 0, 1, \ldots, l-1$, to be known and omitting the $O(\tau^{2s})$-order terms in (5), we have the following recursion relation for an evaluation of the real error of method (3) at the grid points:

$$\Delta z_{k+1} \cong Q_k^{-1}(z_{k+1}) \begin{pmatrix} c_k \\ 0 \end{pmatrix}, \quad k = l-1, l, \ldots, K-1. \tag{9}$$

It was shown above that having used the corrected values of approximate solution $\tilde{z}_i$, $i = k+1, k, \ldots, k+1-l$, we can find $L\big(t_{k+1}, z(t), \tau_k\big)$, which is necessary for computing the vector c_k in (9), with the accuracy of $O(\tau^{s+2})$ by formula (7). Thus, formulas (8) and (9) give a way to estimate the principal terms of the local and global errors of multistep method (3) for any grid w_τ with sufficiently small diameter τ provided that the order of the underlying multistep formula is $s \geq 3$. More precisely, we derive the following:

Theorem 1 *Let DAS* (2) *satisfy conditions I–III and the order of multistep method* (3) *be* $s \geq 3$. *Suppose that the starting values* z_i, $i = 0, 1, \ldots, l-1$, *are given with accuracy* $O(\tau^{s+1})$, *and the errors in the starting values* $\Delta z_i = z(t_i) - z_i$, $i = 0, 1, \ldots, l-1$, *are known with accuracy* $O(\tau^{s+2})$. *Then formulas* (8) *and* (9) *give the principal terms of the local and global errors of method* (3), *respectively, for any grid* w_τ *with sufficiently small diameter* τ *if the corrected approximate solution*

$$\tilde{z}_k = z_k + \Delta z_k, \quad k = l, l+1 \ldots, K, \tag{10}$$

is used for the computation of both the local truncation error $L\big(t_{k+1}, z(t), \tau_k\big)$ *and the local error* $\Delta\tilde{z}_k$ *of method* (3).

The proof of theorem 1 and full details concerning the local and global errors estimation can be found in [7]. The similar results for ordinary differential equations are presented in [8].

3 Extrapolated Multistep Methods

Obviously, if we increase the accuracy of computation of the local truncation error $L\big(t_{k+1}, z(t), \tau_k\big)$ formula (9) remains valid for a higher accuracy evaluation of the error of multistep method (3). To provide this, we replace (7) with the following formula giving an estimate of $L\big(t_{k+1}, z(t), \tau_k\big)$ with accuracy $O(\tau^{s+q+2})$:

$$\begin{aligned}\tilde{L}_q\big(t_{k+1}, z(t), \tau_k\big) \stackrel{\text{def}}{=} \sum_{r=s}^{s+q} \Bigg(& \frac{(-1)^{r+1}}{(r+1)!} x^{(r+1)}(t_{k+1}) \\ & \times \sum_{i=1}^{l} \Bigg(a_i(k) \sum_{j=0}^{i-1} \tau_{k-j} + (r+1)\tau_k b_i(k) \Bigg) \Bigg(\sum_{j=0}^{i-1} \tau_{k-j} \Bigg)^r \Bigg),\end{aligned} \tag{11}$$

$k = l-1, l, \ldots, K-1$, where q is a nonnegative integer.

Let $\Delta_q\tilde{z}_{k+1} \stackrel{\text{def}}{=} \big((\Delta_q\tilde{x}_{k+1})^T, (\Delta_q\tilde{y}_{k+1})^T\big)^T$ denote the sum of the first q terms of local error expansion in the Taylor series at the point t_{k+1}. In addition, we assume that $\Delta_0\tilde{z}_{k+1} \equiv 0$. Then, for any integer $q \geq 0$ it is valid

$$\Delta\tilde{z}_{k+1} = \Delta_q\tilde{z}_{k+1} + O(\tau^{s+q+1}), \quad k = l-1, l, \ldots, K-1.$$

As above, the notation $x_{k+1}^{(r+1)}$ means x-components of the $(r+1)$-order derivative of the approximate solution of problem (2) at the point t_{k+1}. Further we suppose that

$$x^{(r+1)}(t_{k+1}) = x_{k+1}^{(r+1)} + O(\tau^{S-r+1}), \quad r = s, s+1, \ldots, S. \tag{12}$$

Then (5), (8) and (11) give the following:

Theorem 2 *Let DAS* (2) *satisfy conditions I–III. Suppose that the starting values* z_i, $i = 0, 1, \ldots, l-1$, *of stable* l*-step method* (3) *are given with accuracy* $O(\tau^{s+1})$, *and the errors in the starting values* $\Delta z_i = z(t_i) - z_i$, $i = 0, 1, \ldots, l-1$, *are known with accuracy* $O(\tau^{S+2})$, $s \le S \le 2s-3$. *Then, for any grid* w_τ *with sufficiently small diameter* τ *the iterative process*

$$\tilde{z}_{k+1}^{q-1} = z_{k+1} + \Delta_{q-1}\tilde{z}_{k+1}, \quad q = 1, 2, \ldots, S-s+1, \tag{13a}$$

$$\Delta_q \tilde{z}_{k+1} = Q_k^{-1}(\tilde{z}_{k+1}^{q-1}) \begin{pmatrix} \tilde{L}_{q-1}\big(t_{k+1}, z(t), \tau_k\big) \\ 0 \end{pmatrix} \tag{13b}$$

computes the local error of method (3) *with accuracy* $O(\tau^{S+2})$ *if we correct the numerical solution in the course of integration; i.e.,*

$$\tilde{z}_{k+1} = z_{k+1} + \Delta_{S-s+1}\tilde{z}_{k+1}, \quad k = l-1, l, \ldots, K-1. \tag{14}$$

In order to find the necessary derivatives of the solution of problem (2) for formula (11) and to provide condition (12) in iterative process (13), we suggest to differentiate a sufficiently high degree interpolation formula based on the improved values of numerical solution (13a) and (14).

Theorem 3 *Let DAS* (2) *satisfy conditions I–III. Suppose that the starting values* z_i, $i = 0, 1, \ldots, l-1$, *of stable* l*-step method* (3) *are given with accuracy* $O(\tau^{s+1})$, *and the errors in the starting values* $\Delta z_i = z(t_i) - z_i$, $i = 0, 1, \ldots, l-1$, *are known with accuracy* $O(\tau^{S+2})$, $3 \le s \le S \le 2s-3$. *Then, for any grid* w_τ *with sufficiently small diameter* τ *formula* (9) *gives the global error of method* (3) *with accuracy* $O(\tau^{S+1})$ *if the corrected numerical solution* (10) *has been used in iterative process* (13) *for an evaluation of the local error.*

Theorems 2 and 3 allow the error of any stable multistep method and the numerical solution of DAS (2) to be calculated with accuracy $O(\tau^{2s-2})$. The latter implies, for example, that the accuracy of BDFs can be increased up to order 10. Moreover, we conclude from the results mentioned above that the order of any stable l-step method does not exceed $2l+2$ for even l and $2l$ for the odd step number. Thus, we made weaker the first Dahlquist barrier (compare with [1], see also [3]).

4 Numerical Experiments

Let us now consider some numerical examples confirming the theory presented in the previous sections. To do this, me take the DAS

$$x_1'(t) = 10t\,\exp\Big(5\big(y_2(t)-1\big)\Big)x_2(t), \quad x_2'(t) = -2t\,\ln\big(y_1(t)\big), \tag{15a}$$

$$y_1(t) = x_1(t)^{\frac{1}{5}}, \quad y_2(t) = \Big(x_2(t)^2 + y_2(t)^2\Big)/2 \tag{15b}$$

with the exact solution

$$x_1(t) = \exp\left(5\sin(t^2)\right), \quad x_2(t) = \cos(t^2), \tag{16a}$$

$$y_1(t) = \exp\left(\sin(t^2)\right), \quad y_2(t) = \sin(t^2) + 1. \tag{16b}$$

We will use the exact solution to determine and observe the actual order of extrapolated multistep methods. Here, we also do not take into account round-off errors and apply (16) to compute the starting values with zero errors.

Now we construct extrapolation process (9), (10), (13) on the base of BDFs of orders 4 and 6, for instance. To determine the real order of the multistep extrapolation technique, we integrate numerically problem (15) on the interval $[0.3, 1.4]$ with fixed step sizes τ. 5 different step sizes are chosen for every extrapolation number $q = 0, 1, 2$ in the first experiment and for $q = 0, 1, 2, 3, 4$ in the second one. It follows from (16) that all the conditions of theorems 2, 3 hold for (15) on the interval $[0.3, 1.4]$. Then multistep extrapolation is applicable in this case. The only problem is a correct implementation of the suggested extrapolated multistep methods because stable BDFs are implicit and algebraic constrains (15b) are nonlinear.

In practice, one can apply Newton-type iterations to treat arising algebraic systems, but any iterative method introduces an additional error that may influence the asymptotics of (9) and of (11), just making the extrapolation impossible. Thus, it is important to coordinate the iterative error with the accuracy of formulas (9), (11).

We take further the full Newton method. The sufficient quantity of iterations is obtained from the theoretical results presented in [7]. Of course, the modified Newton method can be also implemented correctly. Looking now at Tables 1 and 2, we see the total correspondence of the theory and the practice. The new extrapolation technique actually allows the accuracy of the fourth- and sixth-orders BDFs to be increased up to orders 6 and 10, respectively[3].

When implementing extrapolated multistep methods to solve ordinary differential equations, one has to fixed the number of simple (fixed-point) or Newton-type (full and modified) iterations per grid point following the theoretical results of paper [10].

[3] Note that the round-off errors influence the numerical solution when the order of the method is high and the step size is small

Table 1. Global errors of the extrapolated BDF of order 4

Extrapolation number q	fixed step size $\tau = 1.1/10$	$\tau = 1.1/20$	$\tau = 1.1/40$	$\tau = 1.1/80$	$\tau = 1.1/160$
0	$1.653 \cdot 10^{-01}$	$1.174 \cdot 10^{-02}$	$7.609 \cdot 10^{-04}$	$4.802 \cdot 10^{-05}$	$3.010 \cdot 10^{-06}$
1	$3.215 \cdot 10^{-02}$	$9.122 \cdot 10^{-04}$	$2.772 \cdot 10^{-05}$	$8.492 \cdot 10^{-07}$	$2.628 \cdot 10^{-08}$
2	$9.995 \cdot 10^{-03}$	$1.666 \cdot 10^{-04}$	$2.551 \cdot 10^{-06}$	$3.915 \cdot 10^{-08}$	$6.056 \cdot 10^{-10}$

Table 2. Global errors of the extrapolated BDF of order 6

Extrapolation number q	fixed step size $\tau = 1.1/10$	$\tau = 1.1/20$	$\tau = 1.1/40$	$\tau = 1.1/80$	$\tau = 1.1/160$
0	$3.018 \cdot 10^{-02}$	$5.544 \cdot 10^{-04}$	$8.608 \cdot 10^{-06}$	$1.319 \cdot 10^{-07}$	$2.035 \cdot 10^{-09}$
1	$5.968 \cdot 10^{-03}$	$4.091 \cdot 10^{-05}$	$3.510 \cdot 10^{-07}$	$2.707 \cdot 10^{-09}$	$2.102 \cdot 10^{-11}$
2	$3.103 \cdot 10^{-03}$	$1.277 \cdot 10^{-05}$	$4.596 \cdot 10^{-08}$	$1.715 \cdot 10^{-10}$	$6.450 \cdot 10^{-13}$
3	$4.427 \cdot 10^{-04}$	$2.328 \cdot 10^{-06}$	$5.070 \cdot 10^{-09}$	$9.871 \cdot 10^{-12}$	$3.144 \cdot 10^{-14}$
4	$8.178 \cdot 10^{-04}$	$9.364 \cdot 10^{-07}$	$7.930 \cdot 10^{-10}$	$7.376 \cdot 10^{-13}$	$1.231 \cdot 10^{-14}$

5 Local-Global Step Size Control

With a practical standpoint, the most important and difficult question is the error arising in real numerical computations. To treat automatically this problem for multistep methods applied to both differential equations and index 1 differential-algebraic ones, we presented the local-global step size control in [7], [8]. If we now take ϵ_l and ϵ_g as limits for the local and global errors, respectively, and choose the maximum step size τ, then that algorithm can be given as follows:

Step 1. $k := l - 1$, $M := 0$; {we set $\tau < 1$ and $t_{l-1} + \tau_{l-1} \leq T$}
Step 2. While $t_k < T$ do
go to *Step 3*,
else go to *Step 13*;
Step 3. $t_{k+1} := t_k + \tau_k$, compute $\tilde{z}_{k+1}$, $\Delta\tilde{z}_{k+1}$;
Step 4. $\tau_k^* := \tau_k \left(\epsilon_l / \|\Delta\tilde{z}_{k+1}\|\right)^{1/(s+1)}$;
Step 5. If $\|\Delta\tilde{z}_{k+1}\| > \epsilon_l$,
then $\tau_k := \tau_k^*$, go to *Step 3*;
Step 6. Compute z_{k+1}, Δz_{k+1};
Step 7. $\tau_k^{**} := \tau_k \left((\epsilon_g - \|\Delta\tilde{z}_{k+1}\|) / \|\Delta z_{k+1}\| \right)^{1/s}$;
Step 8. If $\|\Delta z_{k+1}\| \leq \epsilon_g$,
then go to *Step 12*;
Step 9. $\tau_k := \tau_k^{**}$, $M := M + 1$;
Step 10. If $M < 2$,
then go to *Step 3*;
Step 11. $\tau := \tau \left(\epsilon_g / \|\Delta z_{k+1}\|\right)^{1/s}$, go to *Step 1*;

Step 12. $\tau_{k+1} := \min\{\tau, \tau_k^*, \tau_k^{**}, T - t_{k+1}\}$,
$k := k + 1$, $M := 0$, go to *Step 2*;

Step 13. Stop.

Here, we have assumed that the starting values z_i, $i = 0, 1, \ldots, l-1$, are known with sufficiently high accuracy and, hence, their errors Δz_i, $i = 0, 1, \ldots, l-1$, are expected to be zero. We refer to [7], [8] for the complete description of the procedure of local and global errors control mechanism and for the numerical examples showing its advantage over the classical (local) step size selection.

Another profit of the new extrapolation technique is an ability to use the high accuracy estimates of the local and global errors of the numerical solution obtained by multistep methods to control a step size. This idea was implemented in [9] and [11] to construct the advanced local-global step size control based on a computation of the first two principle terms of the local and global errors expansions. There, we proved that the advanced version of step size control is more efficient and reliable then the standard one. Moreover, taking into account the results presented above we can easily extend the advanced step size control version to more terms from the asymptotic expansions of the local and global errors of multistep methods, just making it better for practical usage.

6 Conclusion

In this paper we have introduced the conception of extrapolation technique in the class of multistep methods for index 1 DAS. We have given the necessary theory and shown how such methods work in practice. This technique is not only useful for increasing the accuracy of numerical integration, but also to construct stable multistep methods of higher order.

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