The Reachability Problem in a Concave Region: A New Context

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Abstract. In this paper we present an algorithm that moves a chain confined in a T-shaped rectilinear region from an initial configuration to a final configuration where the end point of chain reaches a given point p. This work is an extension of the our previous results in concave region. In our algorithm links my cross over one another and none of end points of the link chain are fixed. It is shown that the algorithm takes a quadratic time and works when a certain condition is satisfied.

1 Introduction

The problem related to the movement of n-link chain in two dimensions is considered. The n-link chain consists of n links, which are line segments of arbitrary length. The links are joined end-to-end by freely rotating joints. One problem of interest in this area is the reachability problem. In this problem, we decide whether an n-link chain can be moved from one position in a bounded region to another position in order to reach a point in the region.

The movement of an n-link chain was first studied by Hopcroft *et al.* [1]. An algorithm running in O(n) time was then given for an n-link chain confined in a circular region of the plane. In [2] they showed that reachability decision problems are NP-hard if the arm constrained by arbitrary polygonal walls. The case when the arm is confined in a square has been discussed in [3].

All the results mentioned above are on the analysis of a given arm in a convex region. In [5] and[6] we extended the reachability problem for some special concave regions. We proved that under some conditions an n-linked chain that is located in one part of a T-shaped concave region could reach all points of the region. The present paper extends our previous works. We show that under some additional conditions we can improve our previous algorithm for a given n-linked chain without considering its placement in the region. The rest of the paper is organized as follows. Section 2 introduces the technical definitions and the problem statement. Section 3 presents our extended algorithm.

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2 Preliminaries and Previous Work

Assume $\Gamma[0,1,...n]$ is an n-link chain and l_i is the length of the i-th link L_i with endpoints A_{i-1} and A_i and $|\Gamma| = max_{1\leq i\leq n} l_i$. We say that an n-link chain Γ is *bounded by* b if $|\Gamma| < b$, i.e no link has length greater than or equal to b.

A simple motion is the one in which only a few joints are moved simultaneously (at most four angles), or the chain is translated or rotated as a rigid object.

We say that Γ is in *Rim Normal Form* (denoted RNF), if all its joints lie on ∂S.

We say that Γ is in *Ordered Normal Form* (denoted ONF), if Γ is in RNF and moving from A_0 toward A_n along Γ is always either clockwise or counterclockwise.

Consider two line segments γ_1 and γ_2 which intersect at q and $\angle\gamma_1\gamma_2$ is in $[\pi, 2\pi]$. Let ρ be the line segment which starts at q and divides the angle $\angle\gamma_1\gamma_2$ into two angles $\angle\gamma_1\rho$ and $\angle\rho\gamma_2$ in such a way that $\angle\gamma_1\rho$ is in $[\pi/2, \pi]$. Initial configuration of $\Gamma[1,2,3]$ is defined as follows: A_1 placed at point p on line segment γ_1, A_2 at q and A_3 at point r on line segment γ_2. By this assumption we can define our movement in a concave region as follows.

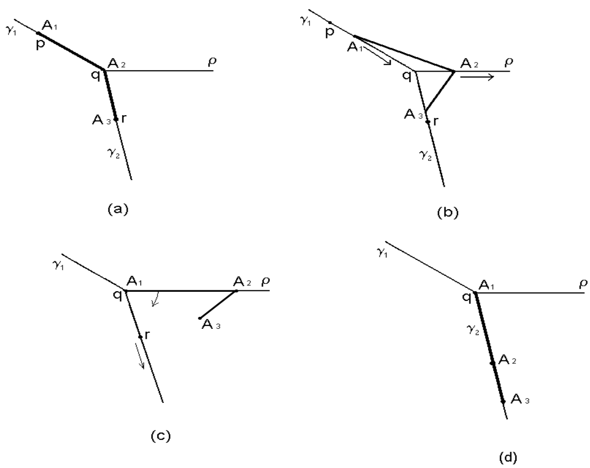

Fig. 1. *(a): Initial configuration of Γ, (b): middle-joint-up motion, (c): front-link-forward motion, (d): final configuration of Γ.*

Definition 1. The *mot(A_1, A_2, A_3, ρ)* movement changes the initial configuration of $\Gamma[1,2,3]$ to a final configuration by which Γ lies on γ_2. This is done by two consecutive motions (Fig 1):

– *Middle-joint-up*(A_1,A_2,A_3,ρ): moves A_2 along ρ away from q until A_1 reaches q. During the movement A_1 remains on γ_1.
– *Front-link-forward*(A_1,A_2,A_3,ρ): fixes A_1 at q then by turning counterclockwise A_2A_3 about A_2 and turning clockwise Γ about A_1 brings down A_3 on γ_2 (if not already there). To straighten Γ, it moves A_3 along γ_2 away from q.

It is possible to show that the mot(A_1,A_2,A_3,ρ) movement can be done in finite number of simple motions. This will be done by showing how each of middle-joint-up motion and front-link-forward motion can be done in finite number of simple motions (see [5] and[6]).

A T-shaped rectilinear S with boundary ∂S can be considered as the union of two rectangles S_1 and S_2, with boundaries ∂S_1 and ∂S_2, and with sides s_1, s_2 and s_3, s_4 respectively, attached together via the side ρ (Fig 2).

Suppose $p \in S_2$ and $\Gamma[0,1,...n]$ confined within S_1 and $|\Gamma| < \mathrm{Min}\{\frac{\sqrt{2}}{2}s_1, |\rho|\}$, then A_n can reach the point p using the following algorithm.

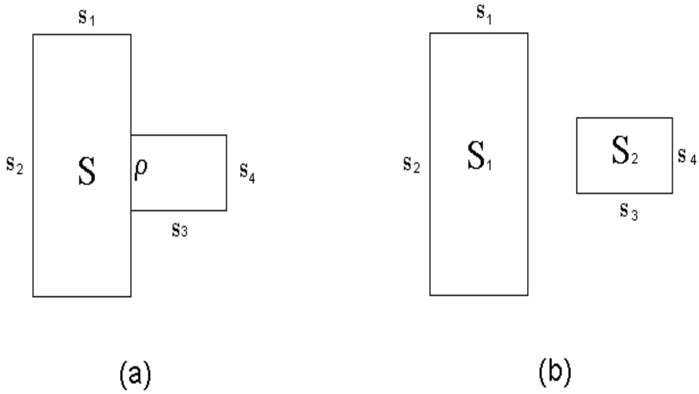

Fig. 2. *(a):A T-shaped rectilinear, (b):T-shaped rectilinear can be considered as the union of two rectangles with sides $s_1, s_2 and s_3$, s_4.*

Algorithm 1: Move all joints of Γ to ∂S_1, then move link $A_{n-1}A_n$ inside S_2 and bring joints A_{n-2}, A_{n-3},...inside S_2 consecutively, using mot(A_{i-1},A_i,A_{i+1},ρ) until p is reached by A_n.

Sketch of proof: We introduce an algorithm to bring A_n to the point p using O(n^2) simple motions, in the worst case. The algorithm consist of two cases:

$$d(p, v_1) \geq |A_{n-1}A_n| \text{ and } d(p, v_1) < |A_{n-1}A_n|.$$

In case 1, i.e $d(p,v_1) \geq |A_{n-1}A_n|$, A_n reaches p in three steps. In the first step we place all joints of Γ on ∂S_1 and reach v_1 by A_n. It takes O(n).

In the second step we move link $A_{n-1}A_n$ inside S_2, then define k_0 = min {k $|d(p,v_1) \geq \sum_{i=k+1}^{n} l_i\}$ and use mot(A_{i-1},A_i,A_{i+1},ρ), $k_0 < i \leq n-1$ to bring joints A_{n-2}, A_{n-3},...,A_{k_0} inside S_2 consecutively.

Finally p is reached during mot(A_{k_0-1},A_{k_0},A_{k_0+1},ρ). This step takes O(n^2) in the worst case.

In the third step, if k=0, we bring the whole Γ inside S_2, and then reach the point p (Fig 3-a).

In case 2, i.e $d(p,v_1) < |A_{n-1}A_n|$, p is close to v_1 and the technique of case 1 is not applicable. In this case we present three subcases (Fig 3-b, c and d). It has been shown that each case takes O(n) time to reach p. For more details see [5, 6].

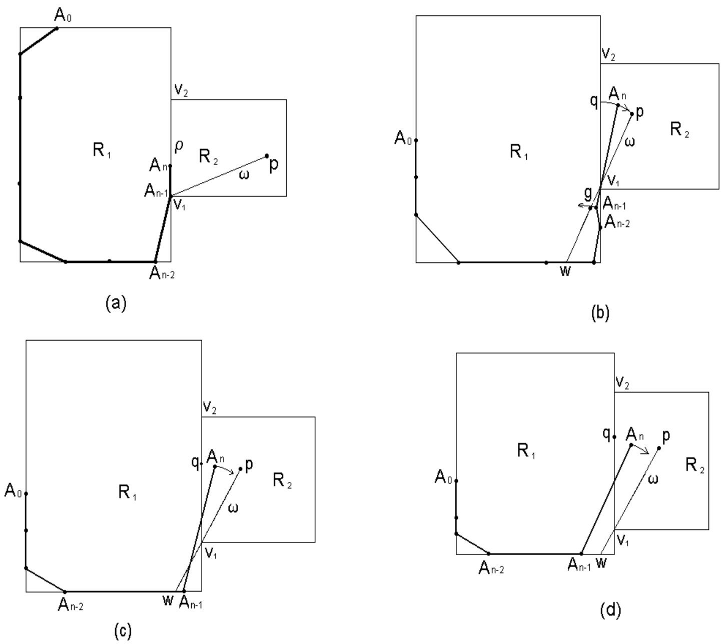

Fig. 3. *(a): $d(p,v_1) \geq |A_{n-1}A_n|$, (b): A_{n-1} belongs to the same edge as v_1, (c): A_n and A_{n-1} are in both sides of ω, (d): A_n and A_{n-1} are in the same side of ω.*

3 Main Result

The main result is; Moving the chain in a T-shaped region from a given configuration to a final configuration in which A_n reaches p.

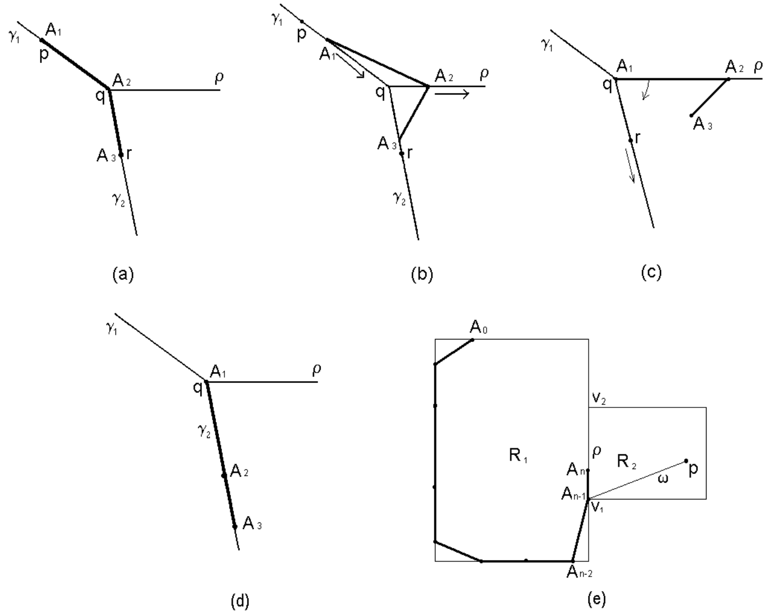

Fig. 4. *q is the farthest distance of A_i from $\partial S_1 \backslash \rho$*

Algorithm 2: Assume Γ is an n-link chain with $|\Gamma| < min\{s_1, s_2, s_3, s_4, \frac{|\rho|}{2}\}$ confined within T-shaped rectilinear S and p a point in S, then p is reachable by the end point of Γ.

We will present a few results that will be used in the explenation of the algorithm.

Assume $A_{i+1} \in S_2$, $\Psi = \partial S_1 \setminus \rho$,

$$q_i = min\{\text{Farthest distance of } A_i \text{ from } \psi, |A_i v_i|, |A_i v_2|\} \text{ (refer to Fig 4)},$$

and i is the lowest-indexed joint such that $A_i \in S_1 \backslash \rho$ and $A_{i+1} \in S_2$, and assume j is a index such that,

$$\Sigma_{m=j+1}^{i} l_m < q_i \leq \Sigma_{m=j}^{i} l_m,$$

(if j=i define $\Sigma_{m=j}^{i} l_m = \Sigma_{m=j+1}^{i} l_m = 0$).

Lemma 1. Assume $|\Gamma| < min\{s_1, s_2\}$, $A_0 \in S_1$, i is lowest-indexed joint such that $A_i \in S_1 \backslash \rho$ and $A_{i+1} \in S_2$ (at least i=0). For some j, as defined before, it is possible to bring $\Gamma[0,..j]$ to RNF or straighten $\Gamma[0,..i]$ with O(n) simple motion.

Proof: Bring the joints of $\Gamma[0,...j]$ one by one to ∂S_1 in decreasing order of their indices or straighten $\Gamma[0,..i]$ in two steps. In the first step A_j is brought into ∂S_1 or $\Gamma[0,..i]$ is straightened. In the second step $\Gamma[0,..j]$ is brought to ∂S_1.

Step 1: Let A_{i-1} and A_{i-2} move, turning A_{i-1} about A_i, keeping A_i and $\Gamma[0,...i-3]$ fixed (if i=2, let A_1 and A_0 move, if i=1, let A_0 move and if i=0 there is nothing to do) so that A_{i-1} gets close to ∂S_1. One of the following events occurs:

A_{i-1} hits ∂S_1 (Fig 5-a), A_{i-2} hits ∂S_1 (Fig 5-b), or the joint angle A_{i-2} straightens (Fig 5-c).

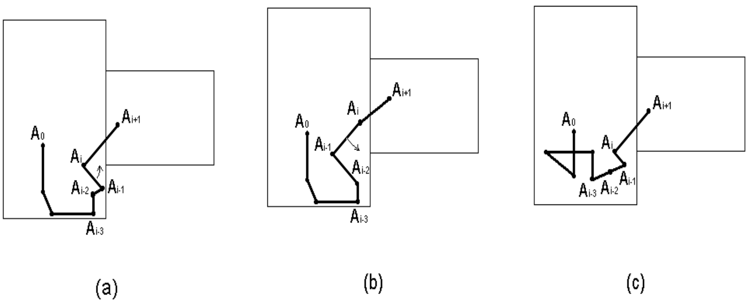

Fig. 5. *(a): A_{i-1} hits ∂S_1, (b): A_{i-2} hits ∂S_1, (c): angle A_{i-2} straightens.*

In cases A_{i-1} or A_{i-2} hits ∂S_1, move to the step 2, in case the joint angle A_{i-2} straightens begin again with L_{i-1} replaced by line segment $A_{i-1}A_{i-3}$ and A_{i-2} by A_{i-3}. Unless for some $k < i-3$, A_k hits ∂S_1, the angles will be successfully straighten, after which repeatedly apply the same procedure to successively lower-indexed joint angles. If all joint angles become straight, the chain $\Gamma[0,...i-1]$ is simply turned to the direction of the A_i-to-A_{i-1} until $\Gamma[0,..i]$ straightens or A_0 hits ∂S_1. This step takes O(n) simple motion at most.

Step 2: Consider for some $k <= j$, A_k lies on ∂S_1, first bring the joints of $\Gamma[0,...k]$ one by one to ∂S_1 in decreasing order of their index joints, then bring the joints between A_k and A_j (if $k \neq j$) to ∂S_1 (see [4]). This step also takes O(n) simple motion.

Lemma 2. Assume $|\Gamma| < min\{s_1, s_2, s_3, s_4, \frac{|\rho|}{2}\}$, $A_0 \in S_1$ and i is lowest-indexed joint such that $A_i \in S_1 \backslash \rho$ and $A_{i+1} \in S_2$ (at least i=0) and assume for some $j <= i$, $\Gamma[j,...i]$ is straight and $A_j \in \partial S_1$, if $j \neq 0$. It is possible to bring A_{i+1} to S_2.

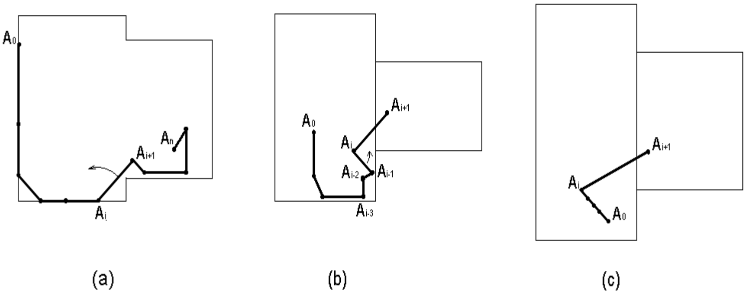

Fig. 6. *(a): turn A_{i+1} about A_i, (b): move A_{i+1} along λ toward the initial place of A_i, (c): A_i is simply rotated about A_0*

Proof: W.l.g. assume v_1 is farthest point of ρ from p. There are tree cases to consider as follows:

In case $j = i$, i.e. $A_i \in \partial S_1$, turn A_{i+1} about A_i until A_{i+1} exit S_2 and enter S_1. By assumption ($|\Gamma| < \frac{|\rho|}{2}$) during rotation A_{i+1} does not hit ∂S_2 (Fig 6-a).

In case $j < i$, i.e. $A_i \notin \partial S_1$, let λ be the straight line trough A_iA_{i+1}. MoveA_{i+1} along λ toward the initial place of A_i letting joint A_i move, keeping the joints between A_i and A_j straight, and fixing A_j. If A_{i+1} exits S_2, we are done. If A_i hits ∂S_1, do the same as the case j=i. (Fig 6-b).

In case j=0, $\Gamma[0,...i]$ is straight and $A_0 \notin \partial S_1$, A_i is simply rotated about A_0. All these cases takes a finite number of simple motion.

The algorithm consists of four consecutive steps as follows.

Step 1: Fix A_i and depending on distance of A_i from Ψ, for some j (as defined), bring $\Gamma[0,...j]$ to ∂S_1 or straighten $\Gamma[0,...i]$, using lemma 1.

Step 2: Move all chains of Γ to S_1 by repeatedly applying step.1 and lemma 2 successively.

Step 3: Move Γ to ONF in S_1 using Kantabutra Algorithm (see [4]).

Step 4: Move chain to a configuration in which A_n reaches p using algorithm 1. All these steps take O(n^2) simple motions together at most.

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