Mechanical Properties of the Earth's Crust with Self-Similar Distribution of Faults

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Abstract. The mechanical behaviour of the Earth's curst with a self-similar structure of faults is modelled using a continuous sequence of continua each with its own size of the averaging volume element. The overall tensorial properties scale isotropically, i.e. according to a power law with common exponent. In fault systems modelled by sets of 2-D cracks, the scaling is isotropic for both isotropically oriented and mutually perpendicular cracks. In the extreme anisotropic case of one set of parallel cracks, all effective compliances but one become zero, the non-vanishing component being scale independent. For isotropically oriented or mutually orthogonal cracks the wave propagation is characterised by extreme scattering, since any wave path is almost certainly intersected by a crack. The waves are not intersected only in the case of one set of parallel cracks. However, in this case due to its extreme anisotropy only extremely long waves (longer than all faults) can propagate.

1 Introduction

The multiscale structure of the Earth's crust makes it discontinuous in the sense that usually the continuum modelling can be applied only at very large scales. This makes the modelling quite challenging since the existing methods of discontinuous mechanics (eg. molecular dynamics type methods such as the discrete element method) are still computationally prohibitive in multi-scale situations. The situation becomes considerably more tractable if there are reasons to believe that the region of the Earth's crust being modelled is characterised by self-similar structure since then the concepts of fractal modelling can be used (eg, [1–5]). Such a modelling is based on a non-traditional philosophy according to which detailed local modelling, which is still not possible, is sacrificed in favour of a general description of scaling laws.

The fractal modelling can be enhanced by the adaptation of traditional notions of continuum mechanics to intrinsically discontinuous fractal objects. This can be accomplished by introducing a continuous set of continua of different scales [6–8]. These continua provide a combined description of the fractal object, which in the spirit of fractal modelling concentrates on scaling laws of properties or state variables which are integral to each continuum. The following reasoning underpins the method.

P.M.A. Sloot et al. (Eds.): ICCS 2003, LNCS 2659, pp. 845–854, 2003.

Traditional continuum mechanics is based on the introduction of length *H* which is considered to be infinitesimal. In practical terms, this length must be much smaller than the problem scale, *L*. Since real materials are discontinuous, at least at some scale, the length *H* must be considerably greater than the characteristic dimension of the discontinuities, *l*. Thus continuum modelling is only applicable to the situations when $l<<H<<L$. It is the left part of this inequality which is impossible to satisfy for geomaterials usually possessing discontinuities with dimensions covering a wide range scales (eg, [1]). If the Earth's crust structure is fractal or self-similar, the continuum modelling can be achieved by the introduction of a continuous set of modelling continua, each continuum being characterised by its own length *H* which is treated as infinitesimal within the continuum.

Formally this can be done as follows [8]. Consider a volume of the Earth's crust with a fractal structure, chose a scale, *H*, and remove all discontinuities of size *H* and greater. Thus we obtain a material with truncated structure and simultaneously set up a scale at which the truncated material can be modelled as a continuum. Therefore, the actual continuum modelling can be conducted for an object that is not fractal but rather a continuum that models, in some averaging manner, material with a certain structure of discontinuities of sizes smaller than *H*. By varying *H* one can model self-similar fractal objects by a continuous set of continua each of them being characterised by its own yardstick, *H*, specifying the scale. Each continuum models the fractal material at the scale *H* in the sense that the volume elements of size *H* in both the original material and the continuum show similar response to uniform loading. The yardstick *H* determines the resolution: no features with dimensions smaller than *H* are viewable in the *H*-continuum, [6]. Thus the *H*-continuum replaces the original material with the one possessing modified microstructure in which only those microstructural elements present that have characteristic sizes less than *H*. This leads to a continuum fractal modelling (Fig. 1) in which a model is produced that possesses an additional dimension, the scale [9], on top of conventional dimensions (four in the general case - three spatial dimensions and time).

When a fractal (self-similar) object is modelled by a continuum of continua, the overall characteristics of each continuum (eg, effective moduli) become functions of *H*. Usually these characteristics have the same sign for all continua, so according to the general theorem (eg, [10]) they must be represented by power functions, $f(H)=f^*H^\alpha$, where f^* is a prefactor. This is essentially the consequence of the fact that the fractal (self-similar) objects have no characteristic length. It should also be noted that if a certain characteristic is bounded (for instance Poisson's ratio) the exponent in its scaling law must be zero. Therefore bounded integral quantities are scale-independent.

Many integral quantities associated with a continuum are tensorial (overall properties such as effective moduli or compliances, integral state variables such as average stress, etc.). The main feature of tensorial properties is that their components undergo linear transformations when the coordinate frame is rotated. Since the transformed components must also scale as power functions and because the power functions with different exponents are linearly independent (e.g. [11]), the components must be either zero or infinite or scale with the same exponent. In other words for any tensor

$$f_{ijk\cdots}(H) = f^*{}_{ijk\cdots} H^{\alpha_{ijk\cdots}}, \quad 0 < f^*{}_{ijk\cdots} < \infty, \quad \alpha_{ijk\cdots} = \alpha = const \qquad (1)$$

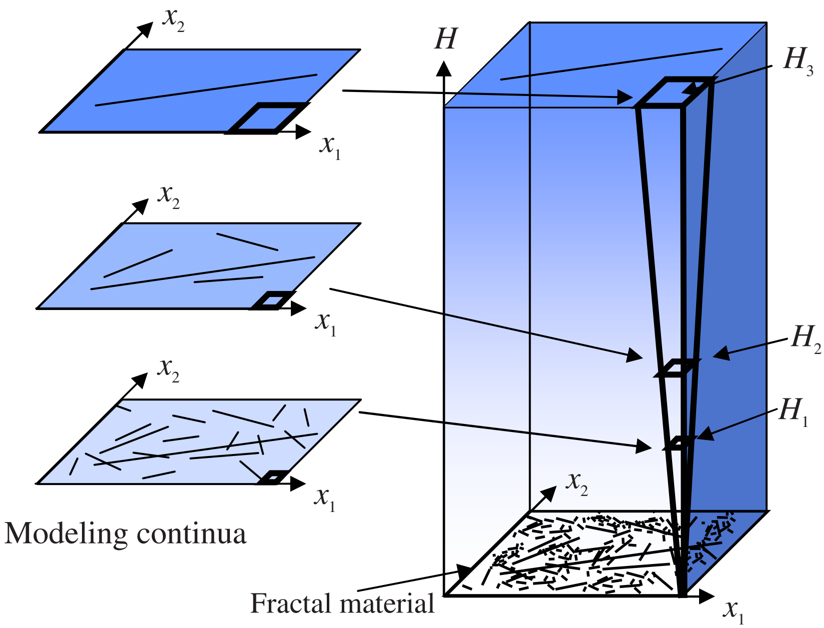

Fig. 1. Continuous fractal modeling. Only three continua out of a continuum of continua are shown

In particular, if the modelling continua are linearly elastic, the case considered hereafter, the tensors of general anisotropic moduli, C_{ijkl}, and compliances, A_{ijkl}, should scale as

$$C_{ijkl}(H) = c_{ijkl} H^{\alpha}, \quad A_{ijkl}(H) = a_{ijkl} H^{\beta}, \quad i, j, k, l = 1,2,3 \qquad (2)$$

Therefore, the scaling of elastic characteristics must be isotropic. The anisotropy is only captured by the prefactors, c_{ijkl} and a_{ijkl}. The particular values of α, β, c_{ijkl} and a_{ijkl} depend on the material structure.

The present paper uses this approach to consider effective properties and wave propagation in the Earth's crust with self-similar faulting in the case where the faults can be represented as 2-D cracks.

2 Elastic Moduli of Earth's Crust with Self-Similar Distributions of Faults

Consider the case when the volume of Earth's crust with faults can be modeled in plane strain by a material containing a self-similar distribution of 2-D cracks. Suppose the crack size distribution is represented by a power distribution function

$$f(l) = \omega l^{-3} \tag{3}$$

Since the power functions are not integrable, we will choose the following normalization

$$\int_{l_{\min}}^{l_{\max}} l^2 f(l) dl = \Omega_t \tag{4}$$

Here, the dimensionless parameter ω is a concentration factor that ensures a specified total concentration, Ω_t. This distribution has an important property that distinguishes it from other self-similar distributions and makes it possible to determine the scaling laws for the effective characteristics [8]. In this distribution, the probability $P(n)$ that in a vicinity of a crack of size l, there are cracks of smaller sizes, from l/n to l, where $n>1$, is equal to $P(n)\sim\omega(n^2\text{-}1)$ which it does not depend upon the crack size, l. Since (3), (4) represent real distributions only asymptotically as $l_{max}/l_{min}\rightarrow\infty$, ie as $\omega\rightarrow 0$ (Ω_t=const), for any n the value of ω can be chosen sufficiently small to make this probability negligible for any crack size.

Therefore, in a vicinity of any crack only cracks of considerably smaller sizes can be found. Mechanically it means that the interaction between the cracks of similar sizes can be neglected; only the interaction between cracks of very different sizes is to be taken into account. Thus the differential self-consistent method, Salganik [12] can be used for calculating the effective characteristics.

According to the method, the compliance or modulus increments ΔA_{ij}, ΔC_{ij}, $i,j=1, \ldots, 6$, at each scale are determined by the contribution of non-interacting cracks of the corresponding scale considered in an effective continuum representing all cracks of smaller scales. The contribution is proportional to the concentration of this group of cracks, $\omega dH/H$. Therefore

$$A_{ij}(H + dH) = A_{ij}(H) + \omega S_{ij}(A_{11}, \ldots, A_{66})\, dH/H \tag{5}$$

$$C_{ij}(H + dH) = C_{ij}(H) + \omega \Lambda_{ij}(C_{11}, \ldots, C_{66})\, dH/H \tag{6}$$

where S_{ij}, Λ_{ij} are homogeneous functions of the first degree specific for the geometry and distribution of parameters of the cracks.

Substitution of (2) into (5) or (6) gives the scaling equations, which are equations to determine the exponent and prefactors:

$$\beta a_{ij} = \omega S_{ij}\left(a_{11}, a_{12}, \ldots a_{66}\right), \;\; \alpha c_{ij} = \omega \Lambda_{ij}\left(c_{11}, c_{12}, \ldots c_{66}\right) \qquad (7)$$

Isotropically Oriented Cracks

In the case of randomly oriented cracks, the material is isotropic. Then, the expressions for effective Young's modulus, E, and Poisson's ratio, ν, written for the case of non-interacting cracks are (e.g., [12]):

$$E = E_m\left[1 - \frac{\pi}{4}\Omega\right], \quad \nu = \nu_m\left[1 - \frac{\pi}{4}\Omega\right] \qquad (8)$$

where Ω is the crack concentration, E_m and ν_m are the Young's modulus and Poisson's ratio of the material. From here the corresponding components of tensor Λ_{ij} are Λ_E=-$E\pi/4$, Λ_ν=-$\nu\pi/4$. Then, from the second equation (7) one gets the following equations for the exponent, α, and the prefactors e and ν:

$$\alpha e = -\frac{\pi}{4}\omega e, \quad \alpha\nu = -\frac{\pi}{4}\omega\nu \qquad (9)$$

Since the Poisson's ratio is bounded, the second equation of (9) offers two solutions: either α=0 or ν=0. The first solution being substituted into the first equation of (8) leads to a trivial case of e=0. The second solution after substituting into the first equation of (8) gives the following scaling law

$$\nu = 0, \quad E = eH^{\alpha}, \quad \alpha = -\pi\omega/4 \qquad (10)$$

In this scaling law the prefactor e is undeterminable and should be found independently, for instance from measurements at a certain scale. Then the scaling law (10) will determine the modulus for other scales. This scaling exponent is not related to the fractal dimension being D=2 for the case of cracks, because the cracks are considered as ideal cuts with no internal volume (or area in 2-D).

Plane with Two Mutually Orthogonal Sets of Cracks

Consider a plane with two mutually orthogonal sets of cracks, Fig. 2. It is assumed that the set of cracks perpendicular to the x_i axis is characterised by the distribution $\omega_i l^{-3}$ such that the total distribution has the concentration factor ω=ω_1 + ω_2.

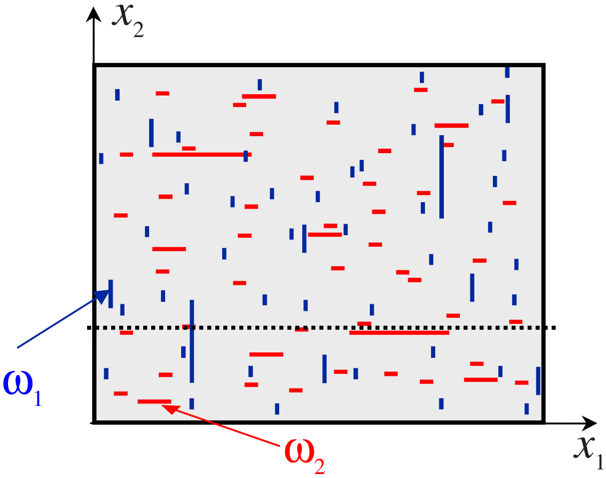

Fig. 2. Two mutually orthogonal sets of cracks with the concentration factors ω_1 and ω_2 respectively. Dotted line shows a possible wave path. It is almost certainly crosses a crack that is parallel to x_2 axis, however in the limiting case of a single set of parallel cracks (ω_1=0) the wave can propagate along the path shown without being interrupted

The analysis is based on Vavakin and Salganik's [13] solution for the effective compliances for an orthotropic plate with a set of non-interacting cracks aligned to one of the symmetry axes of the material. Generalizing their formula one can obtain the effective compliances for an orthotropic plate with two sets of non-interacting cracks of concentrations Ω_1 and Ω_2 normal to axes x_1 and x_2 (in a coordinate system x_1, x_2 aligned to the symmetry axes of the material):

$$\begin{cases} A_{11} = A_{11}{}^m + \dfrac{\pi}{4}\Omega_1 \sqrt{A_{11}{}^m\left(2A_{12}{}^m + A_{66}{}^m + 2\sqrt{A_{11}{}^m A_{22}{}^m}\right)} \\ A_{22} = A_{22}{}^m + \dfrac{\pi}{4}\Omega_2 \sqrt{A_{22}{}^m\left(2A_{12}{}^m + A_{66}{}^m + 2\sqrt{A_{11}{}^m A_{22}{}^m}\right)} \\ A_{66} = A_{66}{}^m + \dfrac{\pi}{4}\Omega_1 \sqrt{A_{22}{}^m\left(2A_{12}{}^m + A_{66}{}^m + 2\sqrt{A_{11}{}^m A_{22}{}^m}\right)} \\ \qquad\qquad + \dfrac{\pi}{4}\Omega_2 \sqrt{A_{11}{}^m\left(2A_{12}{}^m + A_{66}{}^m + 2\sqrt{A_{11}{}^m A_{22}{}^m}\right)} \\ A_{12} = A_{12}{}^m \end{cases} \tag{11}$$

Here $A_{11}{}^{m}$, $A_{22}{}^{m}$, $A_{12}{}^{m}$, $A_{66}{}^{m}$ are the compliances of the material, such that the Hook's law has a form: $\varepsilon_{11}=A_{11}{}^{m}\sigma_{11}+A_{12}{}^{m}\sigma_{22}$, $\varepsilon_{22}=A_{12}{}^{m}\sigma_{11}+A_{22}{}^{m}\sigma_{22}$, $\varepsilon_{12}=½A_{66}{}^{m}\sigma_{12}$.

Using the method outlined above the scaling equations can be obtained:

$$\begin{cases} \beta a_{ii} = \dfrac{\pi}{4}\omega_i \sqrt{a_{ii}\left(a_{66}+2\sqrt{a_{11}a_{22}}\right)}, \quad i=1,2 \\ \beta a_{66} = \dfrac{\pi}{4}\omega_2 \sqrt{a_{11}\left(a_{66}+2\sqrt{a_{11}a_{22}}\right)} + \dfrac{\pi}{4}\omega_1 \sqrt{a_{22}\left(a_{66}+2\sqrt{a_{11}a_{22}}\right)} \end{cases} \tag{12}$$

The solution represents the scaling laws:

$$\begin{gathered} A_{ij} = a_{ij}H^{\beta}, \quad \beta = (\pi/2)\sqrt{\omega_1\omega_2} \\ a_{22} = a_{11}(\omega_2/\omega_1)^2, \quad a_{66} = 2a_{11}(\omega_2/\omega_1), \quad a_{12}=0 \end{gathered} \tag{13}$$

For the case of a single set of cracks, i.e. when the concentration of the other set vanishes, say $\omega_2 \to 0$, the exponent and all compliances except a_{11} vanish. The material becomes completely rigid in the direction x_2. This can easily be understood if one considers that in fractal materials with finite ω the total crack concentration is infinite implying the infinite total compliancy. What the scaling law determines is the way the moduli change in transition from one scale to another. Relative to such moduli the materials without cracks as well as the material in the directions not affected by cracks (the direction x_2 in this case) become infinitely rigid. In real situations we do not have true fractal materials as there are always lower and upper cut-offs. Then the complete rigidity simply means a very high modulus as compared to the values of other moduli.

3 On the Possibility of Wave Propagation

When a wave is sent through an Earth's crust with self-similar fault system its path may be intersected by the faults (as shown in Fig. 2). In order to determine the probability p of intersection we will use the renormalization technique. We introduce a scale, H and consider all faults with dimensions smaller than H. Take an arbitrary path of length $L \gg H$. The path length L should be large enough to make the probability p_H of faults intersecting the path independent of particular realizations of fault positions. Divide the path into $m \gg 1$ segments. Each segment can be intersected either by faults of dimension less than H/m – this would happen with probability $p_{H/m}$ – or by faults of dimensions between H/m and H. This last probability will be denoted by q. Probability p_H as any property should scale according to the power law. On the other hand, $0 \le p_H \le 1$. This implies that the exponent must be zero such that p_H =const. In particular, $p_{H/m}=p_H=p$. The path will not be intersected if all of its segments are not intersected. Therefore

$$1-p=(1-p)^m(1-q) \tag{14}$$

If $q>0$, i.e. faults of dimensions between H/m and H can intersect the path, the only solution of equation (14) is $p=1$. This for example corresponds to the cases of isotropically oriented faults or two sets of mutually orthogonal faults. When $q=0$, i.e. faults of dimensions between H/m and H cannot intersect the path, $p=0$. This is, for instance, the case of one set of cracks parallel to the x_1 axis ($\omega_1=0$in Fig. 2) with wave path being parallel to the cracks.

This result suggests that only in the case of special crack arrangements, like parallel cracks one can expect the wave transmission. In other cases there should be extreme scattering. (In principle, the fact that the probability of intersection is one does not mean that there are no paths without intersections, just there are too few of them. So, the question of wave propagation in these cases would require further study.)

We now analyze the wave propagation parallel to faults in the case of one set. In order to do this we will formally write the wave equations for the case of two mutually orthogonal sets of cracks (Fig. 2) and then set $\omega_1=0$.

Using (13) the Hook's law can be expressed in the following form

$$\varepsilon_{11}=a_{11}\sigma_{11},\quad \varepsilon_{22}=a_{11}\left(\frac{\omega_1}{\omega_2}\right)^2 H^{\beta}\sigma_{22},\quad \varepsilon_{12}=\frac{1}{2}a_{11}\left(\frac{\omega_1}{\omega_2}\right)H^{\beta}\sigma_{12} \tag{15}$$

Expressing σ_{ij} from (15) through strains and then through displacements and substituting the result into 2-D equations of motion $\partial^2\sigma_{ij}/\partial x_j^2=\rho\partial^2 u_i/\partial t^2$, where $i=1,2$, ρ is the density (scale independent in the case of cracks), one gets

$$\begin{aligned} &\frac{\partial^2 u_1}{\partial x_1^{\;2}}+\frac{1}{2}\frac{\omega_1}{\omega_2}\left(\frac{\partial^2 u_1}{\partial x_2^{\;2}}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)=\rho a_{11}\ddot{u}_1 H^{\beta}\\ &\frac{1}{2}\frac{\omega_1}{\omega_2}\left(\frac{\partial^2 u_2}{\partial x_1^{\;2}}+\frac{\partial^2 u_1}{\partial x_1\partial x_2}\right)+\left(\frac{\omega_1}{\omega_2}\right)^2\frac{\partial^2 u_2}{\partial x_2^{\;2}}=\rho a_{11}\ddot{u}_2 H^{\beta} \end{aligned} \tag{16}$$

After the limiting transition $\omega_1\to 0$ one has $\beta=0$ and system (16) reduces to a single equation

$$\frac{\partial^2 u_1}{\partial x_1^{\;2}}=\rho a_{11}\ddot{u}_1 \tag{17}$$

Solution of wave equation (17) will be sought in the form of the longitudinal wave

$$u_1(x_1,x_2,t)=\Phi(x_2)\exp\bigl(ik(x_1-\nu t)\bigr) \tag{18}$$

where k is the wave number, ν is the frequency. Function Φ is determined after substitution of (18) into (17). It assumes the form

$$\Phi(x_2)=A\sinh(\nu/v\; x_2)+B\cosh(\nu/v\; x_2),\ \text{where}\ v=(\rho a_{11})^{-1/2} \tag{19}$$

To determine constants A and B consider a case when the wave is between two large faults (much longer than the wave length). Let these faults be at a distance $2h$ apart. Then, recalling that the faults are modeled as cracks at which surfaces $\sigma_{12}=0$ one obtains $A=0$ and $\nu=0$. Therefore the waves that can propagate have vanishing frequency and, if the velocity ν/k is finite, the infinite length. This is the consequence of the extreme anisotropy associated with the fault distribution of this type. Recalling that the fractal medium is only an approximation to the Earth's crust one can conclude that even in the case of parallel faults only very long (very low frequency) longitudinal waves can propagate and they propagate with very low velocity.

4 Conclusions

The continuum fractal modelling of mechanical behaviour of the Earth's crust with self-similar structure is based on representing the object as a continuum of continua of different scales. It concentrates on scaling of overall mechanical properties and integral state variables, which is described by power laws. The tensorial quantities scale by power laws with exponents common for all components of the tensors.

Thus, effective elastic characteristics of the Earth's crust with self-similar faulting structure always scale isotropically. Even in the extreme case of one set of parallel faults when all compliances but one vanish, since the non-vanishing component is scale independent, the exponents are formally zero, so the scaling is still isotropic.

Any wave path will be intersected by a fault with probability one if the faults are not all parallel. Only in the case of a single set of parallel faults the waves are not intersected, but in this case due to the extreme anisotropy of the medium, only extremely long waves can propagate.

Acknowledgment. The author is grateful to the Australian Research Council for the financial support (Large grant A00104937).

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