On Synchronicity and Concurrency in Petri Nets

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Abstract. In the paper we extend the algebraic description of Petri nets based on rewriting logic by introducing a partial synchronous operation in order to distinguish between synchronous and concurrent occurrences of transitions. In such an extension one first needs to generate steps of transitions using a partial operation of synchronous composition and then to use these steps to generate process terms using partial operations of concurrent and sequential composition. Further, we define which steps are true synchronous. In terms of causal relationships, such an extension corresponds to the approach described in [6,7,9], where two kinds of causalities are defined, first saying (as usual) which transitions cannot occur earlier than others, while the second indicating which transitions cannot occur later than others. We illustrate this claim by proving a one-to-one correspondence between such extended algebraic semantics of elementary nets with inhibitor arcs and causal semantics of elementary nets with inhibitor arcs presented in [7].

1 Introduction

There are many extensions of Petri Nets that improve their modelling, suitability and/or their expressive power. These are almost all based on the same original model, augmented by capacities, context arcs, data structures or even object-oriented features. Conceptually, structures like capacities impose restrictions on the set of legal markings, whereas context arcs introduce new arrow types, which have to be accounted for in the occurrence rule. The definition of sequential semantics for these extensions of Petri Nets can be directly obtained by "iterating" the occurrence rule. However, we are even more interested in obtaining the non-sequential semantics[1]. For the classes of systems mentioned, this is obtained at the time being in an ad-hoc way. Naturally there arises the question if these approaches can be unified by defining some more general framework. In [5] there was recently presented one such unifying concept for extensions of Petri Nets based on a restriction of the occurrence rule. The approach extends and generalizes the idea of Winkowski (see [16]) that non-sequential semantics of elementary nets can be expressed in terms of concurrent rewriting. The principles of the approach are briefly described in the following paragraphs. For more details see [4,5].

[1] For comparative treatment of sequential and non-sequential semantics see [1].

A transition is understood to be an elementary rewrite term that allows replacing the marking $pre(t)$ by $post(t)$. A marking m is also considered to be a rewrite term that rewrites m by m itself. Assume that a suitable operation $+$ on the set of markings is given for each class of Petri Nets in interest such that for each transition $m \xrightarrow{t} m'$ there exist a marking x such that $x + pre(t) = m$ and $x + post(t) = m'$. Occurrence of t at m will be expressed by term $x \,\|\, t$. It may be the case that not all markings $x + pre(t)$ enable t. In that situations, x and t cannot be composed by $\|$. To describe such restriction, we introduce an abstract set of information I and the notion of independence of information elements. Each elementary term has an associated initial marking, final marking and an information set consisting of all information elements of elementary terms from which it is generated. A composition is allowed if and only if the associated information elements are independent. *The non-sequential behavior of a net is described by set of process terms, constructed from the elementary terms using operators of sequential and concurrent composition* ; *and* $\|$, *respectively.*

There are several works relating algebraic characterization and partial-order based description of non-sequential behavior of place/transitions Petri Nets, see [3,14]. These papers in common stem from the paper [10], which has inspired many to continue in this research direction. Thus, algebraic characteristics of non-sequential behavior based of sequential and concurrent composition of rewriting terms represents a suitable axiomatic semantics for the classes of nets which operational causal semantics can be based on partial order.

In a series of papers [6,7,9] authors illustrate that a simple partial-order is not enough expressive to characterize some kinds of causalities. They define more fine causal semantics, where two kinds of causalities are used, first saying (as usual by a partial-order based semantics) which transitions cannot occur earlier than others, while the second indicating which transitions have to occur later than others. Mathematically, this finer causal semantics is described using a relational structure with two relations, a partial order describing the "earlier than" causality and a relation representing the "not later than" causality. In [7,9] the principle is illustrated for a variant of nets with inhibitor arcs, where testing for absence of tokens precedes the execution of a transition (so called a-priori semantics). Thus, if a transition f tests a place for zero, which is in a post-set of another transition e (see Figure 1), this means that f cannot occur later than e and therefore they cannot occur concurrently or sequentially in order e, f - but still can occur synchronously or sequentially in order f, e. Moreover, there are cases where the pair of events e and f is executable neither concurrently nor sequentially, but still the a-priori semantics of transition firings allows them to fire synchronously at the same time. Such a situation is shown on figure 2.

Therefore, in this paper we extend our approach [5] to define an algebraic semantics which corresponds to the idea of finer causal semantics described in [6,7, 9]. Namely, we introduce a new partial operation of *synchronous composition* $\oplus$ which enable us to distinguish between synchronous and concurrent occurrences of transitions. In such an extension of our approach one first needs to generate synchronous steps from transitions using a partial operation of synchronous

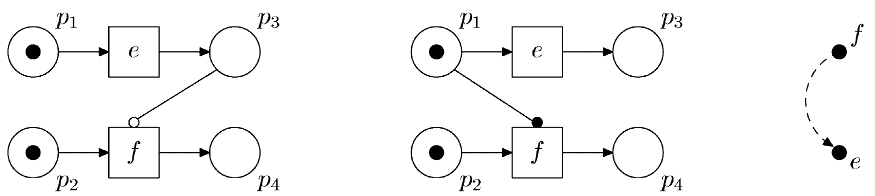

Fig. 1. A simple net with inhibitor arc (p_3, f) where partial ordered semantics does not describe veritably the behaviour of the net, a process of the net (where the inhibitor arc is modelled using an activator arc testing on presence of a token in place p_1, which is complementary to place p_3) and the associated relational structure. The "earlier than" partial order is empty, the "not later than" relation is represented by the dashed line.

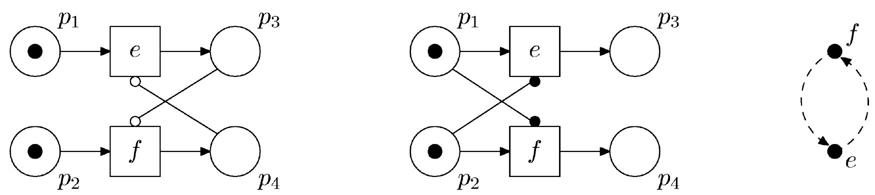

Fig. 2. A modification of the previous figure that shows a net with inhibitor arcs, a process of the net and and the associated relational structure. The "earlier than" partial order is again empty, the "not later than" relation is represented by the dashed lines.

composition and then to use these steps to generate process terms using partial operations of concurrent and sequential composition. Different process terms can still represent different partial or total sequentializations of the same run (the same process). Therefore, the process terms are further related modulo a set of axioms, which determine equivalence classes of process terms representing the same run. We will illustrate the approach on elementary nets with inhibitor arcs with a-priori semantics. As the main result, we prove a one-to-one correspondence between the new defined algebraic semantics of elementary nets with inhibitor arcs based on rewriting logic and causal semantics of elementary nets with inhibitor arcs presented in [7].

2 Mathematical Notation

We use the symbol id_A to denote the identity mapping on the set A. We write id to denote id_A whenever A is clear from the context.

We use partial algebra to define algebraic semantics of nets. A partial algebra is a set (called carrier) together with a couple of partial operations on this set (with possibly different arity). Given a partial algebra with carrier X, an equivalence $\sim$ on X satisfying the following conditions is a *congruence*: If op is an n-ary partial operation, $a_1 \sim b_1, \ldots, a_n \sim b_n$, $(a_1, \ldots, a_n) \in dom_{op}$ and $(b_1, \ldots, b_n) \in dom_{op}$, then $op(a_1, \ldots, a_n) \sim op(b_1, \ldots, b_n)$. If moreover $a_1 \sim b_1, \ldots, a_n \sim b_n$ and $(a_1, \ldots, a_n) \in dom_{op}$ imply $(b_1, \ldots, b_n) \in dom_{op}$

for each n-ary partial operation then the congruence $\sim$ is said to be *closed*. Thus, a congruence is an equivalence preserved by all operations of a partial algebra, while a closed congruence moreover preserves the domains of the operations. For a given partial algebra there always exists a unique greatest closed congruence. The intersection of two congruences is again a congruence. Given a binary relation on X, there always exists a unique least congruence containing this relation. In general, the same does not hold for closed congruences. Given a partial algebra $\mathcal{X}$ with carrier X and a congruence $\sim$ on $\mathcal{X}$, we write $[x]_\sim = \{y \in X \mid x \sim y\}$ and $X/_\sim = \bigcup_{x\in X}[x]_\sim$. A closed congruence $\sim$ defines the partial algebra $\mathcal{X}/_\sim$ with carrier $X/_\sim$, and with n-ary partial operation $op/_\sim$ defined for each n-ary partial operation $op : dom_{op} \to X$ of $\mathcal{X}$ as follows: $dom_{op/_\sim} = \{([a_1]_\sim, \dots, [a_n]_\sim) \mid (a_1, \dots, a_n) \in dom_{op}\}$ and, for each $(a_1, \dots, a_n) \in dom_{op}$, $op/_\sim([a_1]_\sim, \dots, [a_n]_\sim) = [op(a_1, \dots, a_n)]_\sim$. The partial algebra $\mathcal{X}/_\sim$ is called factor algebra of $\mathcal{X}$ with respect to the congruence $\sim$.

Let $\mathcal{X}$ be a partial algebra with k operations $op_i^{\mathcal{X}}, i \in \{1, \dots, k\}$, and let $\mathcal{Y}$ be a partial algebra with k operations $op_i^{\mathcal{Y}}, i \in \{1, \dots, k\}$ such that the arity $n_i^{\mathcal{X}}$ of $op_i^{\mathcal{X}}$ equals the arity $n_i^{\mathcal{Y}}$ of $op_i^{\mathcal{Y}}$ for every $i \in \{1, \dots, k\}$. Denote by X the carrier of $\mathcal{X}$ and by Y the carrier of $\mathcal{Y}$. Then a function $f : X \to Y$ is called homomorphism if for every $i \in \{1, \dots, k\}$ and $x_1, \dots x_{n_i^{\mathcal{X}}} \in X$ we have: if $op_i^{\mathcal{X}}(x_1, \dots, x_{n_i^{\mathcal{X}}})$ is defined then $op_i^{\mathcal{Y}}(f(x_1), \dots, f(x_{n_i^{\mathcal{X}}}))$ is also defined and $f(op_i^{\mathcal{X}}(x_1, \dots, x_{n_i^{\mathcal{X}}})) = op_i^{\mathcal{Y}}(f(x_1), \dots, f(x_{n_i^{\mathcal{X}}}))$. A homomorphism $f : X \to Y$ is called closed if for every $i \in \{1, \dots, k\}$ and $x_1, \dots x_{n_i^{\mathcal{X}}} \in X$ we have: if $op_i^{\mathcal{Y}}(f(x_1), \dots, f(x_{n_i^{\mathcal{X}}}))$ is defined then $op_i^{\mathcal{X}}(x_1, \dots, x_{n_i^{\mathcal{X}}})$ is also defined. If f is a bijection, then it is called an isomorphism, and the partial algebras $\mathcal{X}$ and $\mathcal{Y}$ are called isomorphic. In the paper we distinguish between partial algebras up to isomorphism.

There is a strong connection between the concepts of homomorphism and congruence in partial algebras: If f is a surjective (closed) homomorphism from $\mathcal{X}$ to $\mathcal{Y}$, then the relation $\sim \subseteq X \times X$ defined by $a \sim b \iff f(a) = f(b)$ is a (closed) congruence and $\mathcal{Y}$ is isomorphic to $\mathcal{X}/_\sim$. Conversely, given a (closed) congruence $\sim$ of $\mathcal{X}$, the mapping $h : X \to X/_\sim$ given by $h(x) = [x]_\sim$ is a surjective (closed) homomorphism. This homomorphism is called the *natural homomorphism w.r.t.* $\sim$. For more details on partial algebras see e.g. [2].

3 General Approach

An algebraic Petri Net according to [10] is a graph with vertices representing markings and edges labelled by transitions. Moreover, there is an operation $+ : M \to M$, which is the marking addition. Thus M, the set of markings, and $+$, together with neutral element e (the empty marking) form a commutative monoid $\mathcal{M} = (M, +)$.

To obtain a process term semantics, in [5] we assign to each marking and transition an information element, used for determining concurrent composability of processes. Now we use this information element also for synchronous composability of processes. The set of information elements is then equipped with operation

$\dot{\|}$ (as in [5]) and in addition the operation $\dot{\oplus}$, denoting the information of concurrent and synchronous composition, respectively. Since concurrent realization of events admits their synchronous realization, the domain of concurrent composition is a subset of that of synchronous composition and $\dot{\|}$ is the restriction of $\dot{\oplus}$ to this domain. The partial algebra $\mathcal{I}$ of information elements is formally defined as follows:

Definition 1. *Let $\mathcal{I} = (I, dom_{\dot{\|}}, \dot{\|}, dom_{\dot{\oplus}}, \dot{\oplus})$, where I is a set (of information elements), $dom_{\dot{\|}} \subseteq dom_{\dot{\oplus}} \subseteq I \times I$, and $\dot{\|} : dom_{\dot{\|}} \to I$ and $\dot{\oplus} : dom_{\dot{\oplus}} \to I$ satisfying $\dot{\oplus}|_{dom_{\dot{\|}}} = \dot{\|}$ and:*

- *$\forall a, b \in I$: if $a \dot{\oplus} b$ is defined then $b \dot{\oplus} a$ is defined and $a \dot{\oplus} b = b \dot{\oplus} a$. Similarly, if $a \dot{\|} b$ is defined then $b \dot{\|} a$ is defined.*
- *$\forall a, b, c \in I$: if $(a \dot{\oplus} b) \dot{\oplus} c$ is defined then $a \dot{\oplus} (b \dot{\oplus} c)$ is defined and $(a \dot{\oplus} b) \dot{\oplus} c = a \dot{\oplus} (b \dot{\oplus} c)$. Similarly, if $(a \dot{\|} b) \dot{\|} c$ is defined then $a \dot{\|} (b \dot{\|} c)$ is defined.*

In comparison with [5], we first need to generate steps from transitions using a partial operation of synchronous composition and then to use these steps to generate process terms using partial operations of concurrent and sequential composition.

The following explanations are now exactly the same as in [5]: A process term $\alpha \colon m_1 \to m_2$ represents a process transforming marking m_1 to marking m_2. Process terms $\alpha \colon m_1 \to m_2$ and $\beta \colon m_3 \to m_4$ can be sequentially composed, provided $m_2 = m_3$, resulting in $\alpha; \beta : m_1 \to m_4$. This notation illustrates the occurrence of β after the occurrence of α. The set of information elements of the sequentially composed process term is the union of the sets of information elements of the single process terms. The process terms can also be composed concurrently to $\alpha \parallel \beta : m_1 + m_3 \to m_2 + m_4$, provided the set of information elements of α is independent from (concurrent composable with) the set of information elements of β. The set of information elements of $\alpha \parallel \beta$ contains the concurrent composition of each element of the set of information elements of α with each element of the set of information elements of β. Since process terms have associated sets of information elements, we lift the partial algebra $(I, \dot{\|}, dom_{\dot{\|}})$ to the partial algebra $(2^I, \{\dot{\|}\}, dom_{\{\dot{\|}\}})$, where

- $dom_{\{\dot{\|}\}} = \{(X, Y) \in 2^I \times 2^I \mid X \times Y \subseteq dom_{\dot{\|}}\}$.
- $X\{\dot{\|}\}Y = \{x \dot{\|} y \mid x \in X \wedge y \in Y\}$.

For sequential composition of process terms we need information about the start and the end of a process term, which are both single markings. For concurrent composition, we require that the associated sets of information elements are independent. Two sets of information elements A and B do not have to be distinguished, if for each set of information elements C either both A and B are independent from C or both A and B are not independent from C. Therefore, we can use any equivalence $\cong \in 2^I \times 2^I$ that is a congruence with respect to the operations $\{\dot{\|}\}$ (concurrent composition) and union $\cup$ (sequential composition) and satisfies $(A \cong B \wedge (A,C) \in dom_{\{\dot{\|}\}}) \Longrightarrow (B,C) \in dom_{\{\dot{\|}\}}$, i.e. which is a *closed congruence* of the partial algebra $\mathcal{X} = (2^I, \{\dot{\|}\}, dom_{\{\dot{\|}\}}, \cup)$. The equivalence classes of the greatest (and hence coarsest) closed congruence represent the minimal information assigned to process terms necessary for concurrent composition. This congruence is unique ([2]). Now we can define an algebraic $(\mathcal{M},\mathcal{I})$-net as given in [5].

Definition 2. *An algebraic $(\mathcal{M},\mathcal{I})$-net is a quadruple $\mathcal{A} = (M, T, pre\colon T \to M, post\colon T \to M)$ together with a mapping $inf : M \cup T \to I$ satisfying*

(a) $\forall x, y \in M : \quad (inf(x), inf(y)) \in dom_{\dot{\oplus}} \Longrightarrow inf(x+y) = inf(x) \dot{\oplus} inf(y)$.
(b) $\{inf(t)\} \cong \{inf(t), inf(pre(t)), inf(post(t))\}$.

Since $\dot{\|}$ is the restriction of $\dot{\oplus}$ to this domain, $\dot{\|}$ has the same property as $\dot{\oplus}$ in part (a). In the following definition we define steps of transitions, which represent their synchronous occurrences.

Definition 3. *Every step s has associated an initial marking $pre(s) \in M$, a final marking $post(s) \in M$, and an information element for concurrent and synchronous composition $inf(s) \in I$.*

The elementary step terms are transitions. If s, s' are step terms that satisfy $(inf(s), inf(s')) \in dom_{\dot{\oplus}}$, then their synchronous composition yields the step term $s \oplus s'$ with $pre(s \oplus s') = pre(s) + pre(s')$, $post(s \oplus s') = post(s) + post(s')$ and $inf(s \oplus s') = inf(s) \dot{\oplus} inf(s')$. The set of all step terms of $\mathcal{A}$ is denoted by $Step_{\mathcal{A}}$.

Finally, we are able to define inductively process terms for an algebraic $(\mathcal{M},\mathcal{I})$-net. In comparison with [5] steps are used to be elementary process terms instead of single transitions.

Definition 4. *Let $\mathcal{A}$ be an algebraic $(\mathcal{M},\mathcal{I})$-net. Every process term α has associated an initial marking $pre(\alpha) \in M$, a final marking $post(\alpha) \in M$, and an information for concurrent composition $Inf(\alpha) \in 2^I/_{\cong}$. In the following, for a process term α we write $\alpha : a \longrightarrow b$ to denote that $a \in M$ is the initial marking of α and $b \in M$ is the final marking of α.*

For each $a \in M$, $id_a : a \longrightarrow a$ is a process terms with associated information $Inf(id_a) = [\{inf(a)\}]_{\cong}$. For each $s \in Step_{\mathcal{A}}$, $s : pre(s) \longrightarrow post(s)$ is a process term with associated information $Inf(s) = [\{inf(s)\}]_{\cong}$.

If $\alpha : a_1 \longrightarrow a_2$ and $\beta : b_1 \longrightarrow b_2$ are process terms satisfying $(Inf(\alpha), Inf(\beta)) \in dom_{\{\dot{\|}\}}/_{\cong}$, their concurrent composition yields the process term

$$\alpha \,\|\, \beta\colon a_1 + b_1 \longrightarrow a_2 + b_2$$

with associated information $Inf(\alpha \,\|\, \beta) = Inf(\alpha)\ \{\dot{\|}\}/_{\cong}\ Inf(\beta)$.

If $\alpha : a_1 \longrightarrow a_2$ and $\beta : b_1 \longrightarrow b_2$ are process terms satisfying $a_2 = b_1$, their sequential composition (concatenation) yields the process term

$$\alpha; \beta \ : a_1 \longrightarrow b_2$$

with associated information $Inf(\alpha;\beta) = Inf(\alpha)\ \cup/_{\cong}\ Inf(\beta)$.

The partial algebra of all process terms with the partial operations concurrent composition and concatenation as defined above will be denoted by $\mathcal{P}(\mathcal{A})$.

Now, we define a congruence $\sim_t$ between process terms, saying when two terms are alternative descriptions of the same process.

Definition 5. *Let $\sim_t$ be defined on process terms as the smallest equivalence fulfilling the relations:*

1. $\alpha \parallel \beta \sim_t \beta \parallel \alpha$
2. $(\alpha \parallel \beta) \parallel \gamma \sim_t \alpha \parallel (\beta \parallel \gamma)$
3. $(\alpha; \beta); \gamma \sim_t \alpha; (\beta; \gamma)$
4. $\alpha = ((\alpha_1 \parallel \alpha_2); (\alpha_3 \parallel \alpha_4)) \sim_t$
   $\beta = ((\alpha_1; \alpha_3) \parallel (\alpha_2; \alpha_4))$
5. $\alpha \oplus \beta \sim_t \beta \oplus \alpha$
6. $(\alpha \oplus \beta) \oplus \gamma \sim_t \alpha \oplus (\beta \oplus \gamma)$
7. $(\alpha \oplus \beta) \sim_t (\alpha \parallel pre(\beta)); (post(\alpha) \parallel \beta)$
8. $(\alpha; post(\alpha)) \sim_t \alpha \sim_t (pre(\alpha); \alpha)$
9. $id_{(m+n)} \sim_t id_m \parallel id_n$
10. $(\alpha + id_\emptyset) \sim_t \alpha$

whenever the terms are defined (e.g. $\alpha \parallel \beta$ is defined iff $(A, B) \in dom_{\{\parallel\}}$, where $(A, B) = (Inf(\alpha), Inf(\beta))$. Axiom 4 holds whenever $Inf(\alpha) = Inf(\beta)$.

Henceforth, instead of writing id_m for any $m \in M$ we will usually only write just m. The only new axioms in comparison with [5] are axioms (5-7). Axioms (5) and (6) express commutativity and associativity of synchronous composition. The crucial axiom is the axiom (7), which enable to decompose (to sequentialize) synchronous steps. This axiom states that synchronous composition of two steps α and β and sequential composition of the step α (occurring concurrently with $pre(\beta)$) and the step β (occurring concurrently with $post(\alpha)$) are alternative decompositions of the same process whenever they are both defined. Surprisingly, as we prove later, this axiom is sufficient to identify all processes of nets with inhibitor arcs based on relational structures as defined in [7].

In the following the notion of a true synchronous step term is defined to be a set of transitions, which cannot be decomposed (cannot be sequentialized).

Definition 6. *Let s be a synchronous step term of an algebraic Petri net, such that for each pair of step terms s_1, s_2 satisfying $s_1 \oplus s_2 \sim_t s$ the term*

$$(s_1 \parallel pre(s_2)); (post(s_1) \parallel s_2)$$

is not a process term of the algebraic Petri net. Then s is called true synchronous step.

4 Elementary Nets with Inhibitor Arcs

In this section we will shortly describe the operationally defined a-priori step sequence and process semantics of elementary nets with inhibitor arcs, as defined in [7]. Although we restrict ourselves to elementary nets, the analogus results could be formulated for a-priori semantics of place/transition nets with inhibitor arcs defined in [9].

Definition 7. *A* net *is a triple $N = (P, T, F)$, where P and T are disjoint finite sets (of places and transitions, respectively) and $F \subseteq (P \times T) \cup (T \times P)$. An* elementary net with inhibitor arcs *is a quadruple $ENI = (P, T, F, Inh)$, where (P, T, F) is a net and $Inh \subseteq P \times T$ is an inhibitor relation satisfying $(F \cup F^{-1}) \cap Inh = \emptyset$.*

For a transition $t \in T$, ${}^{\bullet}t = \{p \in P \mid (p, t) \in F\}$ is the *pre-set* of t and $t^{\bullet} = \{p \in P \mid (t, p) \in F\}$ is the *post-set* of t, ${}^{-}t = \{p \in P \mid (p, t) \in Inh\}$ is the set of inhibiting places also called *negative context* of t. Elements of the inhibitor relation are graphically expressed by arcs ending with a circle (so called inhibitor arcs). Throughout the paper we assume that each transition has nonempty pre- and post-sets. A set $m \subseteq P$ is called *marking.* A transition t is enabled to occur in a marking m if no place from negative context ${}^{-}t$ belongs to m, every place from pre-set $\bullet t$ belongs to m, and no place from post-set $t\bullet$ belongs to m. The occurrence of t leads then to a new marking m', which is derived from m by removing the token from every place in $\bullet t$ and adding a token to every place in $t\bullet$. Thus, inhibiting places are tested on absence of tokens for the possible occurrence of a transition and this testing precedes the execution of the transition (so called a-priori semantics).

The following definitions introduce the basic notions of process semantics of elementary nets with inhibitor arcs introduced by [7].

Definition 8. *A* (labelled) occurrence net *is a labelled net $ON = (B, E, R, l)$ such that $(\forall b \in B)(|{}^{\bullet}b| \leq 1 \geq |b^{\bullet}|)$, the transitive closure F^{+} of the relation F is irreflexive (i.e. F^{+} is a partial order) and l is a labelling function for $B \cup E$. Elements of B are called conditions, elements of E events.*

Due to the fact that the absence of token in a place cannot be directly represented in an occurrence net, every inhibitor arc is replaced by an *activator* arc to a complement place. An activator arc (also called read arc, test arc, positive context arc) tests for the presence of a token in the place it is attached to. Moreover, the complement places remove possible contact situations, i.e. situations, when enabledness of a transition is violated by tokens in the post-set of the transition, i.e. by non-empty intersection of the actual marking and the post-set of the transition.

Definition 9. *Let $ENI = (P, T, F, Inh)$ be an elementary net with inhibitor arcs. Let P' be a set satisfying $|P'| = |P|$ and $P' \cap (P \cup T) = \emptyset$, let $c : P \to P'$ be a bijection. The* complementation $\overline{ENI} = (\overline{P}, T, \overline{F}, Act)$ *of ENI is defined by $\overline{P} = P \cup \{c(p) \mid p \in P\}$, $F = F \cup \{(t, c(p)) \mid (p, t) \in F \ \wedge \ (t, p) \notin F\} \cup \{(c(p), t) \mid (t, p) \in F \ \wedge \ (p, t) \notin F\}$ and $Act = \{(c(p), t) \mid (p, t) \in Inh\}$. If initial marking m_0 of ENI is given, its complementation $\overline{m_0}$ is given by $\overline{m_0} = m_0 \cup \{c(p) \mid p \in P \ \wedge \ p \notin m_0\}$.*

Observe that this construction of $\overline{N}$ from N is unique up to isomorphism. In proofs we will take advantage of the fact that we define sets P and P' as disjoint though this is certainly very clumsy in graphical formalism. In most cases, only few places need to be equipped with complement places (co-places) for the system to become contact-free.

Definition 10. *A co-set of an occurrence net ON is a subset $S \subseteq B$ such that for no $a, b \in S$: $(a, b) \in R^+$. A slice is a maximal co-set. Let $Min(ON)$ denote the set of all minimal conditions of ON according to the partial order R^+. Similarly, let $Max(ON)$ denote the set of all maximal conditions of ON according to the partial order R^+.*

Definition 11. *Let $N = (P, T, F)$ be an elementary net and m_0 an initial marking. A* process of N *w.r.t. m_0 is a (labelled) occurrence net $ON = (B, E, R, l)$ such that these conditions are satisfied:*

1. *No isolated place of N is mapped by l to a co-place of N*[2].
2. *$l|_D$ is injective for every slice D of ON.*
3. *$l(Min(ON)) \cap P = m_0 \ \wedge \ l(Min(ON)) \subseteq \overline{m_0}$.*
4. *$\forall e \in E : l({}^\bullet e) = {}^\bullet l(e) \ \wedge \ l(e^\bullet) = l(e)^\bullet$, where ${}^\bullet l(e)$ and $l(e)^\bullet$ refer to complementation $\overline{N}$ of N.*

We use $on(N, m_0)$ to denote the set of all processes of N w.r.t. m_0 and $on(N) = \bigcup\limits_{m_0 \subseteq P} on(N, m_0)$ to denote the set of all processes of N.

Having defined the processes of “ordinary” elementary nets we may proceed to endow them with activator arcs. Let us first introduce the structure that is a generalization of the notion of partial order, suitable for the purpose of capturing both ”earlier than” and ”not later than” causality.

Definition 12. *A* relational structure *is a triple $\mathcal{S} = (X, \prec, \sqsubset)$. $\mathcal{S}$ is called a* stratified order structure *(so-structure) if the following conditions are satisfied:*

$$(C1)\, x \not\sqsubset x \qquad\qquad (C3)\, x \sqsubset y \sqsubset z \wedge x \neq z \Longrightarrow x \sqsubset z$$
$$(C2)\, x \prec y \Longrightarrow x \sqsubset y \qquad (C4)\, x \sqsubset y \prec z \vee x \prec y \sqsubset z \Longrightarrow x \prec z$$

Let $\mathcal{S} = (X, \prec, \sqsubset)$. The $\diamondsuit$-closure of $\mathcal{S}$ is the labelled relational structure

$$\mathcal{S}^{\diamondsuit} = (X, \prec_{S^\diamondsuit}, \sqsubset_{S^\diamondsuit}) = (X, (\prec \cup \sqsubset)^* \circ \prec \circ (\prec \cup \sqsubset)^*, (\prec \cup \sqsubset)^* \setminus id_X).$$

We say that a labelled relational structure $\mathcal{S}$ is $\diamondsuit$-acyclic if $\prec_{S^\diamondsuit}$ is irreflexive.

It is easy to see that $(X, \prec)$ is a partially ordered set. Notice also that $\mathcal{S}^{\diamondsuit}$ is a labelled *so*-structure if and only if $\prec_{S^\diamondsuit}$ is irreflexive. For these results see [7].

Definition 13. *A* labelled activator occurrence net *(ao-net) is a tuple $AON = (B, E, R, \, Act, l)$ such that: $ON = (B, E, R, l)$ is an occurrence net, $Act \subseteq B \times E$ are* activator arcs, *and the relational structure*

$$\mathcal{S}_{aux}(AON) = (E, \prec_{aux}, \sqsubset_{aux}) = (E, (R \circ R)|_{E \times E} \cup (R \circ Act), (Act^{-1} \circ R) \setminus id_E)$$

is $\diamondsuit$-acyclic.

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[2] Isolated places of ON represent ”unused” tokens of $\overline{m}_0$. We use only co-places which are necessary to get a contact-free system.

Definition 14. *An* activator process *of an elementary net with inhibitor arcs* $ENI = (P, T, F, Inh)$ *is an ao-net* $AON = (B, E, R, Act, l)$ *such that* $ON = (B, E, R, l) \in on(N)$ *(where* $N = (P, T, F)$*) and* $\forall b \in B \forall e \in E : (b, e) \in Act \iff (c^{-1}(l(b)), l(t)) \in Inh$*. The set of all activator processes of* ENI *is denoted by* $aon(ENI)$*.*

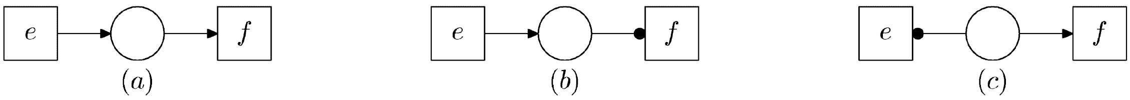

Fig. 3. Illustration of orders generation ([9]). Cases (a) and (b) generate $e \prec_{aux} f$, case (c) generates $e \sqsubset_{aux} f$.

In the diagrams, we draw activator arcs as arrows with black dots as heads and we write ${}^{+}t = \{b \mid (b, t) \in Act\}$.

Thus, $\prec_{aux}$ represent the causality "earlier than", while $\sqsubset_{aux}$ the causality "not later than". Because transitive closure of $\prec_{aux}$ is a partial order, there are no cycles formed by elements of $\prec_{aux}$. On the other hand, the transitive closure of $\sqsubset_{aux}$ is not necessary irreflexive, i.e. $\sqsubset_{aux}$ is not necessary acyclic. Then, cycles formed by elements of $\sqsubset_{aux}$ represent exactly the true synchronous step terms. Finally, the irreflexivity of $\prec_{S\diamond}$ expresses that there are no "combined" cycles, which obtain both elements of $\prec_{aux}$ and $\sqsubset_{aux}$.

5 Algebraic Representation of Elementary Nets with Inhibitor Arcs

In this section we represent elementary nets with inhibitor arcs with a-priori semantics as algebraic $(\mathcal{M}, \mathcal{I})$-nets. Let us consider the net $ENI = (P, T, F, Inh)$. We define $\mathcal{M} = (2^P, \cup)$. Further, let us define $pre(t) = {}^{\bullet}t$ and $post(t) = t^{\bullet}$ for each $t \in T$. To generate steps, we attach to transitions the information which consists of three disjoint components: the pre-set, the post-set, and the set of inhibiting places. So, we define (with $t \in T$ and $m \in 2^P$)

$$I = 2^P \times 2^P \times 2^P, \quad inf(t) = ({}^{\bullet}t, t^{\bullet}, {}^{-}t), \quad inf(m) = (m, m, \emptyset).$$

Synchronous composition of transitions t_1 and t_2 is possible only if places used by the transitions for token flow are disjoint, pre-set of t_1 is disjoint with the set of inhibiting places of t_2 and vice versa. In a-priori semantics, testing on absence of tokens precedes consuming/producing tokens. Therefore, we do not need to check whether the post-set places and the inhibiting places are disjoint in the case of the synchronous composition of transitions. Thus, we have $dom_{\dot\oplus} = \{((a, b, c), (d, e, f)) \in I \mid (a \cup b) \cap (d \cup e) = a \cap f = d \cap c = \emptyset\}$ and $(a, b, c) \dot\oplus (d, e, f) = (a \cup d, b \cup e, (c \cup f) \setminus (b \cup e))$.

For concurrent composition we want transitions to be independent and therefore we have to test whether both pre-sets and post-sets are disjoint with inhibiting places. Thus, we have $dom_{\dot{\|}} = \{((a,b,c),(d,e,f)) \in I \mid (a \cup b) \cap (d \cup e) = (a \cup b) \cap f = c \cap (d \cup e) = \emptyset\}$.

Finally, we need to find the greatest closed congruence $\cong$ of $(2^I, \{\dot{\|}\}, dom_{\{\dot{\|}\}}, \cup)$. We define a mapping *supp* which turns out to be the natural homomorphism of the congruence. The mapping yields the *support of the term*, i.e. the set of places that appear in the token flow, and the set of inhibiting places.

Definition 15. *Define two mappings* $s_1, s_2 : 2^I \to 2^P$ *by* $s_1(A) = \bigcup_{(a,b,c)\in A}(a \cup b)$, $s_2(A) = \bigcup_{(a,b,c)\in A} c$ *and* $supp : 2^I \to 2^P \times 2^P$ *by* $supp\,(A) = (s_1(A), s_2(A) \setminus s_1(A))$.

Lemma 1. *Denote* $J = \{(x,y) \in 2^P \times 2^P \mid x \cap y = \emptyset\}$. *Let* $\circ$ *be the binary operation on* J *defined by* $(w,p) \circ (w',p') = (w \cup w', (p \cup p') \setminus (w \cup w'))$, $dom_{\overline{\|}} = \{((a,b),(c,d)) \in J \times J \mid a \cap c = b \cap c = a \cap d = \emptyset\}$ *and* $\overline{\|} = \circ|_{dom_{\overline{\|}}}$. *Then the mapping supp* : $(2^I, \{\dot{\|}\}, dom_{\{\dot{\|}\}}, \cup) \to (J, \overline{\|}, dom_{\overline{\|}}, \circ)$ *is a surjective closed homomorphism.*

Lemma 2. *The closed congruence* $\cong \subseteq 2^I \times 2^I$ *defined by* $A \cong B \iff$ *supp*$(A) =$ *supp*(B) *is the greatest closed congruence on* $\mathcal{X} = (2^I, \{\dot{\|}\}, dom_{\{\dot{\|}\}}, \cup)$.

The proofs of the previous two lemmas are similar to those in [5] for a-posteriori semantics of elementary nets with inhibitor arcs. Easy computation, using $({}^\bullet t \cup t^\bullet) \cap {}^- t = \emptyset$ proves condition (b) of Definition 2, i.e. $supp(\{inf(t)\}) = supp(\{inf(t), inf(pre(t)), inf(post(t))\})$. The reader may observe that $\overline{\|}$ and J correspond to $\|$ and I in [5]; thus analogous results hold.

The partial algebra $(J, \overline{\|}, dom_{\overline{\|}}, \circ)$ is isomorphic with the greatest closed congruence on $\mathcal{X}$. Therefore, by construction of process terms (using concurrent and sequential composition) it is enough to save just the set of flow places and the set of inhibiting places which are not in the flow of the process as the information for deciding whether the processes are independent (concurrent composable). Now we are able to represent an elementary net with inhibitor arcs as an algebraic $(\mathcal{M}, \mathcal{I})$-net.

Theorem 1. *Let* $ENI = (P, T, F, Inh)$ *be an elementary net with inhibitor arcs, together with* $\mathcal{M}, \mathcal{I}, pre, post, inf$ *defined throughout this section. Then the quadruple* $\mathcal{A}_{ENI} = (2^P, T, pre, post)$ *together with the mapping inf is an algebraic* $(\mathcal{M}, \mathcal{I})$*-net.*

Definition 16. *The algebraic* $(\mathcal{M}, \mathcal{I})$*-net from the previous theorem is called the corresponding* $(\mathcal{M}, \mathcal{I})$*-net to the elementary net with inhibitor arcs* ENI.

Example 1. In the net from Figure 1 the expressions $(f \,\|\, \{p_1\}); (e \,\|\, \{p_4\})$, $e \oplus f$ and $f \oplus e$ are defined process terms, but $f \,\|\, \{p_3\}$ is not defined, because inhibiting place p_3 of f is also pre and post place of the elementary process term $\{p_3\}$

which means that the information elements of terms f and $\{p_3\}$ cannot be composed concurrently. By this the expression $(e \,\|\{p_2\}); (f \,\|\{p_3\})$ is not a defined process term, i.e. the sequence ef cannot be executed in the net from Figure 1. Similarly, in the net from Figure 2 neither expression $(e \,\|\{p_2\}); (f \,\|\{p_3\})$ nor expression $(f \,\|\{p_1\}); (e \,\|\{p_4\})$ are defined process terms, i.e. neither sequence ef nor sequence fe can occur in the net. The only expressions containing both e and f, which are defined process terms are $e \oplus f$ and $f \oplus e$, i.e. the only possibility to occur both e and f is to do it synchronously.

6 Activator Processes versus Process Terms

In this section we establish the main result on the relationship between the newly defined process terms and the activator processes of elementary nets with inhibitor arcs with a-priori semantics as introduced in [7]. We have the following pattern of the proof: First, we define a mapping τ that associates an activator occurrence net with each process term. Second, we prove that τ is surjective, i.e. every activator process can be represented by a process term. Then we show that $\alpha \sim_t \beta \Rightarrow \tau(\alpha) = \tau(\beta)$, i.e. equivalent process terms are mapped by τ on the same activator process. Finally, we prove that $\tau(\alpha) = \tau(\beta) \Rightarrow \alpha \sim_t \beta$. Hence, at the end of this section we have formulated the main theorem of the paper, which states that activator processes correspond bijectively to $\sim_t$-equivalence classes of process terms.

Because the process terms corresponding to markings are determined in a process term α by the value $pre(\alpha)$, in the following we will often omit them in process terms. In the sequel we consider an elementary net with inhibitor arcs $ENI = (P, T, F, Inh)$, its corresponding $(\mathcal{M}, \mathcal{I})$-net $\mathcal{A}_{ENI}$, and two actuator processes $K_1 = (B_1, E_1, R_1, Act_1, l_1)$ and $K_2 = (B_2, E_2, R_2, Act_2, l_2)$ of ENI. We will often state that $K_1 = K_2$ even if this is not exactly true and only the *graphical representations* of the nets are the same, i.e. there exists a bijective mapping that preserves labelling, flow and read arcs. If needed, we use $\approx$ to denote this isomorphism. Every time we use l without indices we mean the labelling function constructed as "union" of l_1 and l_2. Exactly, $l|_{B_1 \cup E_1} = l_1$ and $l|_{B_2 \cup E_2} = l_2$. From the definitions we can verify that it is always possible to construct such a function. We will commonly use the shorthand "places match". We precisely mean: Let $b_1 \in B_1, b_2 \in B_2$. We say that b_1 and b_2 match if $l_1(b_1) = l_2(b_2)$. In definitions of the compositional net operations we usually remove one of the matching places and attach the arcs adjacent with the removed place to the other. We then say these places were *glued*.

Definition 17 (Notation). *Let $K = (B, E, R, Act, l)$ be an ao-net. We define the set of* isolated places *by* $^{\emptyset}K = \{b \in B \mid \forall e \in E : b \notin {}^{\bullet}e \cup e^{\bullet} \cup {}^{+}e\}$*, the set of* write places *(flow places) by* $^{\diamond}K = \bigcup_{e \in E}({}^{\bullet}e \cup e^{\bullet}) \cup {}^{\emptyset}K$ *and the set of* purely read places *by* $^{+}K = (\bigcup_{e \in E} {}^{+}e) \setminus {}^{\diamond}K$.

In the following definition we define the mapping τ associating activator processes to process terms representing markings and single transitions.

Definition 18.
Let $m \in M$ *and* $\alpha = m : m \to m$ *be the related process term of* $\mathcal{A}_{ENI}$*. Define* $\tau(\alpha) = K_\alpha = (m, \emptyset, \emptyset, \emptyset, id_m)$*. Let* $t \in T$ *and* $\alpha = t : pre(t) \to post(t)$*. Define* $\tau(\alpha) = K_\alpha = ({}^\bullet t \cup t^\bullet \cup {}^+t, \{t\}, ({}^\bullet t \times \{t\}) \cup (\{t\} \times t^\bullet), {}^+t \times \{t\}, id_{{}^\bullet t \cup t^\bullet \cup {}^+t \cup \{t\}})$*, where* ${}^\bullet t$*,* $t^\bullet$ *and* ${}^+t$ *are with respect to* $\overline{ENI}$*.*

Observe, that K_m and K_t are activator processes. For a (step or process) term α we say $\tau(\alpha)$ to be the corresponding activator process. Beginning with processes representing markings and single transitions we define inductively the corresponding activator processes for synchronous composed step terms α_1 and α_2, and concurrent and sequential composed process terms α_1 and α_2, using activator processes K_1 and K_2 corresponding to the terms α_1 and α_2. In definitions we always assume that $B_1 \cap B_2 = E_1 \cap E_2 = \emptyset$. We can always achieve this by appropriate renaming. We also define $Int_{\|} = \{p \in P \mid (\exists b_1 \in {}^+K_1)(\exists b_2 \in {}^+K_2)(l_1(b_1) = l_2(b_2) = p)\}$ as the set of places that are used in both processes as purely read places.

To obtain the activator process corresponding to synchronous composed step terms we define the *synchronous interface* $Int^i_\oplus$ as the set of places that are used as read places in the process K_i and as write places in the other process. We delete places of $Int^i_\oplus$ and $Int_{\|}$ from K_i (places that are only used as read in one process remain), put the processes side-by-side, we glue read places of processes with their write counterparts in the other process. Then we add the "purely read" places in one copy and restore the *Act* relation.

Definition 19. *Define the* synchronous interfaces*:*

$$Int^1_\oplus = \{p \in P \mid (\exists b_1 \in {}^+K_1)(\exists b_2 \in {}^\diamond K_2)(l_1(b_1) = l_2(b_2) = p)\}$$
$$Int^2_\oplus = \{p \in P \mid (\exists b_2 \in {}^+K_2)(\exists b_1 \in {}^\diamond K_1)(l_1(b_1) = l_2(b_2) = p)\}$$
$$B'_i = (B_i \setminus (l_i^{-1}(Int^i_\oplus) \cap {}^+K_i)) \setminus l_i^{-1}(Int_{\|})$$
$$Act'_i = Act_i \cap (B'_i \times E_i)$$

Define $\tau(\alpha_1 \oplus \alpha_2) = K_{\alpha_1 \oplus \alpha_2} = (B, E, R, Act, l) = (B, E_1 \cup E_2, R_1 \cup R_2, Act, l)$*, where* $B = B'_1 \cup B'_2 \cup l_1^{-1}(Int_{\|})$ *and*

$$\begin{aligned} Act = {} & Act'_1 \cup Act'_2 \cup \\ & \{(b_1, e_2) \mid b_1 \in B'_1 \wedge (\exists b_2 \in l_2^{-1}(Int^2_\oplus))((b_2, e_2) \in Act_2 \wedge l_1(b_1) = l_2(b_2))\} \cup \\ & \{(b_2, e_1) \mid b_2 \in B'_2 \wedge (\exists b_1 \in l_1^{-1}(Int^1_\oplus))((b_1, e_1) \in Act_1 \wedge l_1(b_1) = l_2(b_2))\} \cup \\ & \{(b, e_i) \mid b \in l^{-1}(Int_{\|}) \wedge e_i \in E_i \wedge (\exists b_i \in B_i)(l_i(b_i) = l(b) \wedge (b_i, e_i) \in Act_i), \\ & \text{for } i = 1, 2\}. \end{aligned}$$

Lemma 3. *If* $\alpha_1 \oplus \alpha_2$ *is a defined step term, then* $K_{\alpha_1 \oplus \alpha_2}$ *as defined in the previous definition is an activator process.*

Proof. (Sketch) Observe that the activator processes corresponding to step terms consist of three "layers". The first consists of pre-places, the second of transitions and the third of post-places. We show the following:

(i) (B, E, R, l) is a labelled occurrence net (that means R^+ is irreflexive). This follows immediately from $R = R_1 \cup R_2$.
(ii) (B, E, R, l) is a process (see definition 11). Only the injectivity of the labelling on all slices is not obvious. It follows from the facts, that every slice

D of the composed net is of the form $D = D_1 \cup D_2$ with slices D_1 of K_1 and D_2 of K_2, and that from the precondition ($\alpha_1 \oplus \alpha_2$ is a defined step term) the labelling images of the flow places of K_1 and K_2 are disjoint (see [5] for a detailed proof of a similar statement).

(iii) $K_{\alpha_1 \oplus \alpha_2}$ is an ao-net (that means $\mathcal{S}_{aux}(K_{\alpha_1 \oplus \alpha_2})$ is $\diamondsuit$-acyclic). This follows from the observation, that activator arcs are only connected with places from the pre-places layer.

(iv) $K_{\alpha_1 \oplus \alpha_2}$ is an activator process (that means the labelling respects the inhibiting relation of ENI). That is obvious.

To obtain the activator process corresponding to concurrent composed process terms, we remove the matching read places from K_2, put the two processes together and restore activator arcs incident with erased places, using same-labelled places of K_1.

Definition 20. *Set* $B_2' = B_2 \setminus l_2^{-1}(Int_{\|})$, $Act_2' = Act_2 \cap (B_2' \times E_2)$ *and define* $\tau(\alpha_1 \,\|\, \alpha_2) = K_{\alpha_1 \,\|\, \alpha_2} = (B, E, R, Act, l) = (B_1 \cup B_2', E_1 \cup E_2, R_1 \cup R_2, Act, l_1 \cup l_2)$, *where* $Act = Act_1 \cup Act_2' \cup \{(b_1, e_2) \in B_1 \times E_2 \mid (\exists b_2 \in {}^+K_2)(l_1(b_1) = l_2(b_2) \;\wedge\; (b_2, e_2) \in Act_2)\}$.

Lemma 4. *If* $\alpha_1 \,\|\, \alpha_2$ *is a defined process term, then* $K_{\alpha_1 \,\|\, \alpha_2}$ *as defined in the previous definition is an activator process.*

Proof. (Sketch) The structure of the proof is the same as in the proof of the previous lemma. The statements (i), (ii) and (iv) can be proven analogously. Let $\mathcal{S}_{aux}(K_{\alpha_1 \,\|\, \alpha_2}) = (E, \prec_{aux}, \sqsubset_{aux})$, $\mathcal{S}_{aux}(K_1) = (E, \prec_{aux}^1, \sqsubset_{aux}^1)$ and $\mathcal{S}_{aux}(K_2) = (E, \prec_{aux}^2, \sqsubset_{aux}^2)$. Because no read place of the one process is matched and glued with a write place of the other process, we have $\prec_{aux} = \prec_{aux}^1 \cup \prec_{aux}^2$ and $\sqsubset_{aux} = \sqsubset_{aux}^1 \cup \sqsubset_{aux}^2$. Statement (iii) follows.

To obtain the activator process corresponding to sequential composed process terms we remove those minimal places of K_2 that match a maximal place in K_1 (the sequential interface $Int_;$), and attach the arcs originally attached to the minimal elements to these maximal elements.

Definition 21. *We define the* sequential interface *of the processes* K_1, K_2 *by* $Int_;(K_1, K_2) := \{p \in P \mid (\exists b_1 \in Max(K_1))(\exists b_2 \in Min(K_2))(l(b_1) = l(b_2) = p)\}$.

$$
\begin{aligned}
\textit{Set} \qquad B_2' &= B_2 \setminus \{b_2 \in Min(K_2) \mid l(b_2) \in Int_;(K_1, K_2)\} \\
R_2' &= R_2 \cap ((E_2 \times B_2') \cup (B_2' \times E_2)) \\
Act_2' &= Act_2 \cap (B_2' \times E_2) \;\; \textit{and} \\
\textit{define} \quad \tau(\alpha_1; \alpha_2) = K_{\alpha_1;\alpha_2} &= (B, E, R, Act, l) = (B_1 \cup B_2', E_1 \cup E_2, R, Act, l), \\
\textit{where} \qquad R &= R_1 \cup R_2' \cup \{(b_1, e_2) \mid b_1 \in Max(K_1) \wedge \\
&\qquad (\exists b_2 \in Min(K_2) : l_1(b_1) = l_2(b_2) \;\wedge\; (b_2, e_2) \in R_2)\} \\
Act &= Act_1 \cup Act_2' \cup \{(b_1, e_2) \mid b_1 \in Max(K_1) \wedge \\
&\qquad (\exists b_2 \in B_2 : l_1(b_1) = l_2(b_2) \;\wedge\; b_2, e_2) \in Act_2)\}.
\end{aligned}
$$

Lemma 5. *If* $\alpha_1; \alpha_2$ *is a defined process term, then* $K_{\alpha_1;\alpha_2}$ *as defined in the previous definition is an activator process.*

Proof. (Sketch) The structure of the proof is the same as in the proof of the previous lemma. The statements (i), (ii) and (iv) follow from construction. Let $\mathcal{S}_{aux}(K_{\alpha_1;\alpha_2}) = (E, \prec_{aux}, \sqsubset_{aux})$. From construction we have $\forall e_1 \in E_1,\ e_2 \in E_2 :$ $e_2 \not\prec_{aux} e_1 \wedge e_2 \not\sqsubset_{aux} e_1$. Therefore (iii) is satisfied.

Remark 1. Let $\alpha_1;\alpha_2$ be a defined process term and let $\mathcal{S}^{\diamondsuit}_{aux} = (E, \prec, \sqsubset)$ be the so-structure associated with $\tau(\alpha_1;\alpha_2)$. Let E_1, E_2 denote the set of events of $\tau(\alpha_1), \tau(\alpha_2)$, respectively. Then $\forall e_1 \in E_1,\ e_2 \in E_2 :\ e_2 \not\prec e_1 \wedge e_2 \not\sqsubset e_1$.

Naturally, the relationships between "earlier than" and "not later than" causalities on one side and definition domains of concurrent and synchronous composition play a crucial rôle in the proof of correspondence between the process term semantics and the activator process semantics.

If (copies of) transitions (in an activator process) are not ordered by "earlier than" causality, then they may be executed synchronously.

Lemma 6. *Let $K = (B, E, R, Act, l)$ be an activator process and let $e_1, e_2 \in E$. Let $\mathcal{S}^{\diamondsuit}_{aux} = (E, \prec, \sqsubset)$ be the associated so-structure. If $e_1 \not\prec e_2$ and $e_2 \not\prec e_1$ then $l(e_1) \oplus l(e_2)$ is a defined process term.*

Proof. (Sketch) Assume the term $l(e_1) \oplus l(e_2)$ is not defined (although $e_1 \not\prec e_2$ and $e_2 \not\prec e_1$), i.e. the associated information elements of e_1 and e_2 are not composable by $\dot{\oplus}$. This means, one of the following cases must be fulfilled:

(i) ${}^\bullet l(e_1) \cap {}^\bullet l(e_2) \neq \emptyset$. Observe that ${}^\bullet e_1 \cap {}^\bullet e_2 = \emptyset$ and $e_1^\bullet \cap e_2^\bullet = \emptyset$ by the definition of process nets. So there are places $b_1, b_2 \in B$ with $b_1 \in {}^\bullet e_1$, $b_2 \in {}^\bullet e_2$, $l(b_1) = l(b_2)$ and $b_1 \neq b_2$. Because the labelling is injective on slices, b_1 and b_2 are in different slices, and therefore are ordered by the transitive closure R^+ of the flow relation. Since conditions are unbranched, e_1 and b_2 or e_2 and b_1 must be ordered by R^+. Since $R^+ \subseteq \prec = (\prec_{aux} \cup \sqsubset_{aux})^* \circ \prec_{aux} \circ (\prec_{aux} \cup \sqsubset_{aux})^*$, this contradicts $e_1 \not\prec e_2 \wedge e_2 \not\prec e_1$. The proof for $l(e_1)^\bullet \cap l(e_2)^\bullet \neq \emptyset$ is analogous.

(ii) ${}^\bullet l(e_1) \cap l(e_2)^\bullet \neq \emptyset$. If ${}^\bullet e_1 \cap e_2^\bullet \neq \emptyset$, we have directly a contradiction of $e_1 \not\prec e_2 \wedge e_2 \not\prec e_1$. If not, the proof is similar to (i). ${}^\bullet l(e_1) \cap {}^+ l(e_2) \neq \emptyset$ and ${}^+ l(e_1) \cap l(e_2)^\bullet \neq \emptyset$ are proven in the same way.

Moreover, if (copies of) transitions (in an activator process) are neither ordered by "earlier than" causality nor by "not later than" causality, then they may be executed concurrently.

Lemma 7. *Let $K = (B, E, R, Act, l)$ be an activator process and let $e_1, e_2 \in E$. Let $\mathcal{S}^{\diamondsuit}_{aux} = (E, \prec, \sqsubset)$ be the associated so-structure. If $e_1 \not\prec e_2$, $e_2 \not\prec e_1$, $e_1 \not\sqsubset e_2$ and $e_2 \not\sqsubset e_1$, then $l(e_1) \parallel l(e_2)$ is a defined process term.*

Proof. (Sketch) Assuming the term $l(e_1) \parallel l(e_2)$ is not defined one can prove the lemma in a same way as the previous one. The only new inequality which has to be fulfilled is ${}^+ l(e_1) \cap {}^\bullet l(e_2) \neq \emptyset$. Because $R^+ \subseteq \sqsubset = (\prec_{aux} \cup \sqsubset_{aux})^* \setminus id_E$, this can be proven analogously as (i) in the previous lemma.

Theorem 2 (τ is surjective). *For every activator process K of ENI there is a process term α, such that $\tau(\alpha) = K$.*

Proof. With lemma 6 we can prove the statement analogous to the corresponding theorem in [5] (replacing $\|$ by $\oplus$) using sequential composition of maximal synchronous step terms. The searched term α is of the form $\alpha = \alpha_0; \ldots; \alpha_n$, where α_k is a maximal synchronous step term.

If two process terms are alternative decompositions of the same process, then they should naturally have the same corresponding activator process. This is indeed so. The proof consists of verifying the claim for each of the ten axioms in definition 5 and we omit it. The interested reader may find the proof of the corresponding theorem in [5], proving the statement for the axioms (1)-(4) and (8)-(10). For the axioms (5) and (6) the statement is obvious. For axiom (7) the statement is proven in a similar way as for axiom (4).

Theorem 3. *Let α,β be process terms of $\mathcal{A}_N$. Then $\alpha \sim_t \beta \Longrightarrow \tau(\alpha) = \tau(\beta)$.*

The proof of the converse, the theorem similar to theorem 7 found in [5] (the crucial theorem of that paper) does not work anymore, since there can be transitions in a step term, which cannot be sequentialized, because they are true synchronous. So every process term can only be sequentialized down to true synchronous step terms. We want to identify subsets of events of an activator process, which correspond to true synchronous step terms via the mapping τ. Clearly a set of events, that can be cyclicly ordered by the "not later than" causality, cannot occur sequentially and therefore is a candidate for such a subset. We will need exactly such sets which are maximal w.r.t. the $\subseteq$-relation.

Definition 22. *We say that a process term α_{seq} is* maximally sequentialized *if and only if it is of the form $(a_1 \| s_1); \ldots; (a_k \| s_k)$, where s_i is a true synchronous step term and $a_i \in M$ for all $i \in \{1, \ldots, k\}$.*

Lemma 8. *Let α be a process term. Then there exists a term α_{seq} such that $\alpha \sim_t \alpha_{seq}$ and α_{seq} is a maximally sequentialized process term.*

Proof. (Sketch) Inductively, replace $\alpha \| \beta$ with $\alpha; \beta$. Replace $\alpha \oplus \beta$ with either $\alpha; \beta$ or $\beta; \alpha$, whichever is defined. If none is defined, then it can be proved that $\alpha \oplus \beta$ is a part of a true synchronous step term. The algorithm define functions denoted by $\cdot_{seq}$.

Definition 23. *Let $K = (B, E, R, Act, l)$ be an activator process, $\prec_{aux}$ be the associated "earlier than" causality on E and $\sqsubset_{aux}$ be the associated "not later than" causality on E. A set $\eta \subseteq E$ is called a* cyclic event*, if it either contains exactly one element or the following two conditions are fulfilled:*

(i) The events from E are pairwise unordered w.r.t. $\prec_{aux}$.
(ii) There is a sequence $e_1 e_2 \ldots e_k$ of events from η, such that $e_i \sqsubset_{aux} e_{i+1}$ and $e_1 = e_k$ $(i \in \{1, \ldots, k-1\})$[3]. In other words there is cycle w.r.t. $\sqsubset_{aux}$ through all events in η.

An synchronous event *of K is a cyclic event, that is maximal w.r.t. the $\subseteq$-relation.*

[3] Of course it is allowed, that some events of η appear more than once in the sequence

Clearly, synchronous events of an activator process are disjoint. Moreover, we can extend the lemma 6 as follows:

Lemma 9. *Let $K = (B, E, R, Act, l)$ be an activator process and let η be its synchronous event. Then $s = \bigoplus_{e\in\eta} l(e)$ is a defined process term.*

Applying this lemma, the property of $\mathcal{S}^{\diamondsuit}_{aux}$, and the definition of $\dot{\|}$ we can also extend the lemma 7.

Lemma 10. *Let $K = (B, E, R, Act, l)$ be an activator process, and η_1, η_2 its synchronous events. Let $\mathcal{S}^{\diamondsuit}_{aux} = (E, \prec, \sqsubset)$ be the associated so-structure. , If $e_1 \not\prec e_2$, $e_2 \not\prec e_1$, $e_1 \not\sqsubset e_2$ and $e_2 \not\sqsubset e_1$ for some $e_1 \in \eta_1, e_2 \in \eta_2$, then $\bigoplus_{e\in\eta_1} l(e) \,\|\, \bigoplus_{e\in\eta_2} l(e)$ is a defined process term.*

We can characterize cyclic events in the following way.

Lemma 11. *Let $K = (B, E, R, Act, l)$ be an activator process. A set $\eta \subseteq E$ with at least two elements is a cyclic event, if and only if*

(i) The events from η are pairwise unordered w.r.t. $\prec_{aux}$.
(ii) For every nonempty subset $\varphi \subset \eta$ we have: There are events $e_1, e_2 \in \varphi$, $f_1, f_2 \in \eta \setminus \varphi$, such that $e_1 \sqsubset_{aux} f_1$ and $f_2 \sqsubset_{aux} e_2$.

In this statement condition (ii) can be equivalently replaced by condition

(ii)' For every nonempty subset $\varphi \subset \eta$ we have: There are conditions $b \in \bigcup_{e\in\varphi} {}^\bullet e$, $b' \in \bigcup_{e\in\eta\setminus\varphi} {}^\bullet e$ and events $f \in \varphi$, $f' \in \eta \setminus \varphi$, such that $b \in {}^+f'$ and $b' \in {}^+f$.

Proof. The equivalence between conditions (ii) and (ii)' follows directly from the definition of $\sqsubset_{aux}$.
if: Assume η is not a cyclic event, that means there are two events $e, f \in \eta$ with $e \not\sqsubset^*_{aux} f$, although condition (ii) is fulfilled. Set $\varphi = \{e' \in \eta \mid e \sqsubset^*_{aux} e'\}$. Obviously $e' \not\sqsubset^*_{aux} f$ for all $e' \in \varphi$. This contradicts (ii).
only if: $e_i \sqsubset_{aux} e_{i+1}$ implies by the definition of $\sqsubset_{aux}$, that there is a condition $b_{i+1} \in {}^\bullet b_{i+1}$ with $b_{i+1} \in {}^+e_i$. Assume there is a subset $\varphi \subset \eta$, which does not fulfil condition (ii), although η is a cyclic event. Without loss of generality assume that there is no condition $b \in \bigcup_{e\in\varphi} {}^\bullet e$, such that there exists an event $f' \in \eta \setminus \varphi$ with $b \in {}^+f'$. In other words $(\bigcup_{e\in\varphi} {}^\bullet e) \cap (\bigcup_{e\in\eta\setminus\varphi} {}^+e) = \emptyset$, i.e. $\forall e \in \eta, \forall f \in \eta \setminus \varphi : f \not\sqsubset_{aux} e$. It follows, that the transitive closure of $\sqsubset_{aux}$ restricted to η cannot be symmetric, which is a contradiction to the fact, that η is a cyclic event.

Lemma 12. *Let s be a true synchronous step term and $\tau(s) = (B, E, R, Act, l)$ be the corresponding activator process. Then E is a synchronous event of $\tau(s)$.*

Proof. Assume E is not a synchronous event although s is a true synchronous step term. Then there exists $\varphi \subseteq E$ such that there are either no activator arcs from $\bigcup_{e\in\varphi} {}^\bullet e$ to $E \setminus \varphi$ or no activator arcs from $\bigcup_{e\in E\setminus\varphi} {}^\bullet e$ to φ. Without loss of generality assume that there are no activator arcs from $\bigcup_{e\in\varphi} {}^\bullet e$ to $E \setminus \varphi$. From lemma 6 and commutativity and associativity of $\oplus$, terms

$\bigoplus_{e\in\varphi} l(e)$ and $\bigoplus_{f\in E\setminus\varphi} l(f)$ are defined terms. From definition of $dom_{\|}$ the composition $\bigoplus_{e\in\varphi} l(e) \,\|\, post(\bigoplus_{f\in E\setminus\varphi} l(f))$ is defined and therefore also the term $(\bigoplus_{f\in E\setminus\varphi} l(f) \,\|\, pre(\bigoplus_{e\in\varphi} l(e)); (\bigoplus_{e\in\varphi} l(e) \,\|\, post(\bigoplus_{f\in E\setminus\varphi} l(f))$ is defined. This contradicts the fact that s is a true synchronous step term (i.e. that s cannot be sequentialized).

Lemma 13. *Let $\alpha = s_1; \dots ; s_k$ be a maximally sequentialized process term. Let $SE = \{\eta_1, \dots , \eta_n\}$ be the set of all synchronous events of the activator process $\tau(\alpha)$. Then $n = k$ and there exist a permutation v such that: $\alpha = \bigoplus_{e\in\eta_{v_1}} l(e); \dots ; \bigoplus_{e\in\eta_{v_n}} l(e)$.*

Proof. By induction: From previous lemma the statement is valid for $k = 1$. If it is valid for $i < k$, then from the proof of lemma 5 adding $\tau(s_{i+1})$ to $\tau(s_1; \dots , s_i)$ we cannot extend neither any existing synchronous event of $\tau(s_1; \dots , s_i)$ by an event from $\tau(s_{i+1})$ nor the added synchronous event of $\tau(s_{i+1})$ by an event from $\tau(s_1; \dots , s_i)$. Since the set of events E_{i+1} of $\tau(s_{i+1})$ forms itself a synchronous event satisfying $s_{i+1} = \bigoplus_{e\in E_{i+1}} l(e)$, the statement is valid for $i + 1$.

Corollary 1. *Let $\alpha = s_1; \dots ; s_k$ and $\beta = r_1; \dots ; r_n$ be maximally sequentialized process terms. If $\tau(\alpha) = \tau(\beta)$ then $n = k$ and there exist a permutation v such that $\alpha = r_{v_1}; \dots ; r_{v_n}$.*

Now we can prove the last part needed for the correspondence between process term semantics and activator process semantics.

Theorem 4. *Let α,β be process terms. Then $\tau(\alpha) = \tau(\beta) \Longrightarrow \alpha \sim_t \beta$.*

Proof. Lemma 8 provides us with maximally sequentialized process terms α_{seq} and β_{seq} such that $\alpha_{seq} \sim_t \alpha$ and $\beta_{seq} \sim_t \beta$. By theorem 3 we have $\tau(\alpha_{seq}) = \tau(\alpha) = \tau(\beta) = \tau(\beta_{seq})$. Thus, it suffices to show that $\alpha_{seq} \sim_t \beta_{seq}$. Denote $\alpha_{seq} = s_1; \dots ; s_k$ and $\beta_{seq} = r_1; \dots ; r_n$. By corollary 1 we have $n = k$ and there exists a permutation v such that $\alpha_{seq} = r_{v(1)}; \dots ; r_{v(n)}$. If $\alpha_{seq} \neq \beta_{seq}$, consider that i is the first index satisfying $v(i) \neq i$ (obviously $v(i) > i$). The idea is to "bubble-sort" step term $r_{v(i)}$ from the position $v(i)$ in β_{seq} backwards to the position i, which it has in α_{seq}, and repeat this procedure until there is no such i. Thus, it suffices to prove that $r_1; \dots ; r_{v(i)-1}; r_{v(i)}; \dots ; r_n \sim_t r_1; \dots ; r_{v(i)}; r_{v(i)-1}; \dots ; r_n$, i.e. that we can exchange $r_{v(i)-1}$ and $r_{v(i)}$ in β_{seq}. A sufficient condition for this is that $r_{v(i)-1} \,\|\, r_{v(i)}$ is a defined process term. Since i was the first index with the property $v(i) \neq i$, the position of $r_{v(i)-1}$ in α_{seq} is at least $i + 1$, i.e. $\alpha_{seq} = r_{v(1)}; \dots ; r_{v(i-1)}; r_{v(i)}; \dots ; r_{v(i)-1}; \dots r_{v(n)}$. Thus, from β_{seq} and Remark 1 we hav e$\forall e \in E_{v(i)-1}, f \in E_{v(i)} : f \not\prec e \wedge f \not\sqsubset e$, where $E_{v(i)-1}, E_{v(i)}$ denote the set of events of $\tau(r_{v(i)-1}), \tau(r_{v(i)})$, respectively. On the other hand, from α_{seq} and Remark 1 we have $\forall e \in E_{v(i)-1}, f \in E_{v(i)} : e \not\prec f \wedge e \not\sqsubset f$. From lemma 10 follows that $r_{v(i)-1} \,\|\, r_{v(i)}$ is a defined process term, what finishes the proof.

Now we are prepared to state the main result of the paper, which is an immediate consequence of theorems proved in this section.

Theorem 5. *Let ENI be an elementary net with inhibitor arcs. Let $\mathcal{A}_{ENI}$ be the corresponding algebraic $(\mathcal{M},\mathcal{I})$-net. Then there is a one-to-one correspondence between $\sim_t$-equivalence classes of $\mathcal{A}_N$ and (isomorphism classes) of activator processes of ENI.*

7 Conclusion

In this paper we have presented an abstract axiomatic semantics for Petri nets, which enables to distinguish between synchronous and concurrent occurrence of steps. Within this framework, we have also defined notion of true synchronicity. The approach is based on rewriting logic with restricted definition domains of operations. We claim that our approach offers a dual description of processes based on "earlier than" and "not later than" causalities defined in the seminal papers [6,7]. We illustrate our claim proving a one-to one correspondence between our process semantics and those described in [7] for elementary nets with inhibitor arcs.

There are several works following the ideas of [7] in literature, e.g. [8] for nets with priorities and [9] for place/transition nets with inhibitor arcs. Many other papers also discuss weak causality of nets with inhibitor/read arcs [11,13, 12], or nets with read arcs [15]. However, they exclude the case where transitions are cyclically ordered by inhibitor arcs (or read arcs). In ([13], [12]) the weak causality is rather understood as an asymmetric conflict. In [15] the nets with read arcs are investigated. Duration is supposed and the causalities "e necessarily ends before f starts" and "e necessarily starts before f starts" are used. Thus, in the example from Figure 1, for e and f both to occur, f has to start before b. According [15], in the situation from Figure 2, e and f cannot both occur, because intuitively they block each other (to occur both, first the test on the *presence* of tokens has to be done (i.e. one event starts earlier) and after that the token is consumed (the second event starts), and after that occurrence is finished (tokens in post-sets are produced). A similar intuition (if a duration is assumed, then consuming a token take a time and during this time absence of tokens is post-sets can be tested) allow occurrence of both events in Figure 2.

An advantage of the algebraic approach we have presented is the fact, that it offers an abstract framework, where by "tuning" the underlying algebra of information elements and the definition domain of synchronous and concurrent composition, one can define non sequential semantics of different variants and dialects of nets in a unifying way. Moreover, true synchronous steps play a crucial role in Petri nets enriched by signals arcs, which are extensively used in modelling and control of engineering systems. Presently, we are working on non-sequential semantics for this class of nets. Another area of our present research consists in developing a suitable general mechanism which will allow to produce causal relations directly from the process terms.

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