On Unimodality of Independence Polynomials of some Well-Covered Trees^{*}

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Abstract

The stability number $\alpha(G)$ of the graph G is the size of a maximum stable set of G. If s_k denotes the number of stable sets of cardinality k in graph G, then $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is the *independence polynomial* of G (I. Gutman and F. Harary, 1983). In 1990, Y. O. Hamidoune proved that for any *claw-free graph* G(a graph having no induced subgraph isomorphic to $K_{1,3}$), I(G; x) is unimodal, i.e., there exists some $k \in \{0, 1, ..., \alpha(G)\}$ such that

 $s_0 \le s_1 \le \dots \le s_k \ge s_{k+1} \ge \dots \ge s_{\alpha(G)}.$

Y. Alavi, P. J. Malde, A. J. Schwenk and P. Erdös (1987) asked whether for trees (or perhaps forests) the independence polynomial is unimodal. J. I. Brown, K. Dilcher and R. J. Nowakowski (2000) conjectured that I(G; x) is unimodal for any *well-covered graph* G (a graph whose all maximal independent sets have the same size). V. E. Levit and E. Mandrescu (1999) demonstrated that every well-covered tree can be obtained as a join of a number of well-covered spiders, where a *spider* is a tree having at most one vertex of degree at least three.

In this paper we show that the independence polynomial of any well-covered spider is unimodal. In addition, we introduce some graph transformations respecting independence polynomials. They allow us to reduce several types of well-covered trees to claw-free graphs, and, consequently, to prove that their independence polynomials are unimodal.

key words: stable set, independence polynomial, unimodal sequence, wellcovered tree, claw-free graph.

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1 Introduction

Throughout this paper G = (V, E) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set V = V(G) and edge set E = E(G). If $X \subset V$, then G[X] is the subgraph of G spanned by X. By G - W we mean the subgraph G[V - W], if $W \subset V(G)$. We also denote by G - F the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we write shortly G - e, whenever $F = \{e\}$. The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{w : w \in V$ and $vw \in E\}$, and $N_G[v] = N_G(v) \cup \{v\}$; if there is no ambiguity on G, we use N(v)and N[v], respectively. If N(v) induces a complete graph in G, then v is a simplicial vertex of G. A simplicial vertex is *pendant* if its neighborhood contains only one vertex, and an edge is *pendant* if at least one of its endpoints is a pendant vertex. $K_n, P_n, C_n, K_{n_1, n_2, \dots, n_p}$ denote respectively, the complete graph on $n \ge 1$ vertices, the chordless path on $n \ge 1$ vertices, the chordless cycle on $n \ge 3$ vertices, and the complete p-partite graph on $n_1 + n_2 + \ldots + n_p$ vertices.

The disjoint union of the graphs G_1, G_2 is the graph $G = G_1 \amalg G_2$ having as a vertex set the disjoint union of $V(G_1), V(G_2)$, and as an edge set the disjoint union of $E(G_1), E(G_2)$. In particular, $\amalg G$ denotes the disjoint union of n > 1 copies of the graph G. If G_1, G_2 are disjoint graphs, then their Zykov sum, (Zykov, [18], [19]), is the graph $G_1 + G_2$ with

$$V(G_1 + G_2) = V(G_1) \cup V(G_2),$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}.$$

As usual, a *tree* is an acyclic connected graph. A tree having at most one vertex of degree ≥ 3 is called a *spider*, [8], or an *aster*, [5].

A stable set in G is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a maximum stable set of G, and the stability number of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G. Let s_k be the number of stable sets in G of cardinality $k, k \in \{1, ..., \alpha(G)\}$. The polynomial

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k, s_0 = 1,$$

is called the *independence polynomial* of G, (Gutman and Harary, [6]).

A number of general properties of the independence polynomial of a graph are presented in [6] and [2]. As important examples, we mention the following:

$$I(G_1 \amalg G_2; x) = I(G_1; x) \cdot I(G_2; x),$$

$$I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1$$

A finite sequence of non-negative real numbers $\{a_0, a_1, a_2, ..., a_n\}$ is said to be *unimodal* if there is some $k \in \{0, 1, ..., n\}$, called the *mode* of the sequence, such that

$$0 \le a_0 \le a_1 \le \dots \le a_k \ge a_{k+1} \ge \dots \ge a_n.$$

The mode is unique if $a_{k-1} < a_k > a_{k+1}$.

Unimodal sequences occur in many areas of mathematics, including algebra, combinatorics, and geometry (see Brenti, [3], and Stanley, [17]). In the context of our paper, for instance, if a_i denotes the number of ways to select a subset of *i* independent edges (a matching of size *i*) in a graph, then the sequence of these numbers is unimodal (Schwenk, [16]). As another example, if a_i denotes the number of dependent *i*-sets of a graph *G* (sets of size *i* that are not stable), then the sequence of $\{a_i\}_{i=0}^n$ is unimodal (Horrocks, [10]).

A polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ is called *unimodal* if the sequence of its coefficients is unimodal. For instance, the independence polynomial of K_n is unimodal, as $I(K_n; x) = 1 + nx$. However, the independence polynomial of the graph $G = K_{100} + \amalg 3K_6$ is not unimodal, since $I(G; x) = 1 + \mathbf{118}x + 108x^2 + \mathbf{206}x^3$ (for another examples, see Alavi et al [1]). Moreover, in [1] it is shown that for any permutation σ of $\{1, 2, ..., \alpha\}$ there exists a graph G, with $\alpha(G) = \alpha$, such that $s_{\sigma(1)} < s_{\sigma(2)} < ... < s_{\sigma(\alpha)}$, i.e., there are graphs for which $s_1, s_2, ..., s_\alpha$ is as "shuffled" as we like.

A graph G is called *well-covered* if all its maximal stable sets have the same cardinality, (Plummer, [13]). In particular, a tree T is well-covered if and only if $T = K_1$ or it has a perfect matching consisting of pendant edges (Ravindra, [14]).

The roots of the independence polynomial of well-covered graphs are investigated by Brown et al in [4]. It is shown that for any well-covered graph G there is a well-covered graph H with $\alpha(G) = \alpha(H)$ such that G is an induced subgraph of H, where all the roots of I(H; x) are simple and real. As it is also mentioned in [4], a root of independence polynomial of a graph (not necessarily well-covered) of smallest modulus is real, and there are well-covered graphs whose independence polynomials have non-real roots. Moreover, it is easy to check that the complete *n*-partite graph $G = K_{\alpha,\alpha,\dots,\alpha}$ is well-covered, $\alpha(G) = \alpha$, and its independence polynomial, namely $I(G; x) = n(1 + x)^{\alpha} - (n - 1)$, has only one real root, whenever α is odd, and exactly two real roots, for any even α . In other words, the theorem of Newton (stating that if a polynomial with positive coefficients has only real roots, then its coefficients form a unimodal sequence) does not help in proving the following conjecture.

Conjecture 1.1 [4] The independence polynomial of any well-covered graph is unimodal.

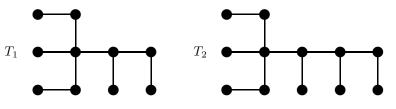


Figure 1: Two well-covered trees.

The claw-graph $K_{1,3}$ (see Figure 3) is a non-well-covered tree and $I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$ is unimodal, but has non-real roots. The trees T_1, T_2 in Figure 1 are well-covered, and their independence polynomials are respectively

$$I(T_1;x) = (1+x)^2 \cdot (1+2x) \cdot (1+6x+7x^2)$$

$$= 1 + 10x + 36x^{2} + 60x^{3} + 47x^{4} + 14x^{5},$$

$$I(T_{2}; x) = (1 + 6x + 10x^{2} + 5x^{3})^{2} - x^{2}(1 + x)^{3}(1 + 2x)$$

$$= 1 + 12x + 55x^{2} + 125x^{3} + 151x^{4} + 93x^{5} + 23x^{6},$$

which are both unimodal, while only for the first is true that all its roots are real. Hence, Newton's theorem is not useful in verifying the following conjecture, even for the particular case of well-covered trees.

Conjecture 1.2 [1] Independence polynomials of trees are unimodal.

A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. There are non-claw-free graphs whose independence polynomials are unimodal, e.g., the *n*-star $K_{1,n}$, $n \geq 3$. The following result of Hamidoune will be used in the sequel.

Theorem 1.3 [7] The independence polynomial of a claw-free graph is unimodal.

As a simple application of this statement, one can easily see that independence polynomials of paths and cycles are unimodal. In [2], Arocha shows that

$$I(P_n; x) = F_{n+1}(x), and I(C_n, x) = F_{n-1}(x) + 2xF_{n-2}(x),$$

where $F_n(x), n \ge 0$, are *Fibonacci polynomials*, i.e., the polynomials defined recursively by

$$F_0(x) = 1, F_1(x) = 1, F_n(x) = F_{n-1}(x) + xF_{n-2}(x).$$

Based on this recurrence, one can deduce that

$$F_n(x) = \binom{n}{0} + \binom{n-1}{1}x + \binom{n-2}{2}x^2 + \dots + \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}x^{\lfloor n/2 \rfloor},$$

(for example, see Riordan, [15], where this polynomial is discussed as a special kind of rook polynomials). It is amusing that the unimodality of the polynomial $F_n(x)$, which may be not so trivial to establish directly, follows now immediately from Theorem 1.3, since any P_n is claw-free. Let us notice that for $n \geq 5$, P_n is not well-covered.

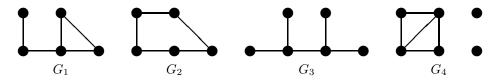


Figure 2: Two pairs of non-isomorphic graphs G_1, G_2 and G_3, G_4 satisfying $I(G_1; x) = I(G_2; x)$ and $I(G_3; x) = I(G_4; x)$.

Clearly, any two isomorphic graphs have the same independence polynomial. The converse is not generally true. For instance, while $I(G_1; x) = I(G_2; x) = 1 + 5x + 5x^2$, the well-covered graphs G_1 and G_2 are non-isomorphic (see Figure 2).

In addition, the graphs G_3, G_4 in Figure 2, have identical independence polynomials $I(G_3; x) = I(G_4; x) = 1 + 6x + 10x^2 + 6x^3 + x^4$, while G_3 is a tree, and G_4 is not connected and has cycles.

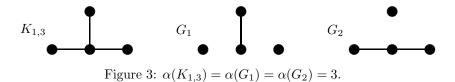
However, if I(G; x) = 1 + nx, $n \ge 1$, then G is isomorphic to K_n . Figure 3 gives us a source of some more examples of such uniqueness. Namely, the figure presents all the graphs of size four with the stability number equal to three. A simple check shows that their independence polynomials are different:

$$I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3,$$

$$I(G_1; x) = 1 + 4x + 5x^2 + 2x^3,$$

$$I(G_2; x) = 1 + 4x + 4x^2 + x^3.$$

In other words, if the independence polynomials of two graphs (one from Figure 3 and an arbitrary one) coincide, then these graphs are exactly the same up to isomorphism.



Let us mention that the equality $I(G_1; x) = I(G_2; x)$ implies

$$|V(G_1)| = s_1 = |V(G_2)|$$
 and $|E(G_1)| = \frac{s_1^2 - s_1}{2} - s_2 = |E(G_2)|$.

Consequently, if G_1, G_2 are connected, $I(G_1; x) = I(G_2; x)$ and one of them is a tree, then the other must be a tree, as well.

In this paper we show that the independence polynomial of any well-covered spider is unimodal. In addition, we introduce some graph transformations respecting independence polynomials. They allow us to reduce several types of well-covered trees to claw-free graphs, and, consequently, to prove that their independence polynomials are unimodal.

2 Preliminary results

Let us notice that if the product of two polynomials is unimodal, this is not a guaranty for the unimodality of at least one of the factors. For instance, we have

$$I(K_{100} + \amalg 3K_6; x) \cdot I(K_{100} + \amalg 3K_6; x) = (1 + \mathbf{118}x + 108x^2 + \mathbf{206}x^3)^2$$

= 1 + 236x + 14140x² + 25900x³ + **60280**x⁴ + 44496x⁵ + 42436x⁶.

The converse is also true: the product of two unimodal polynomials is not necessarily unimodal. As an example, we see that:

$$I(K_{100} + \amalg 3K_7; x) \cdot I(K_{100} + \amalg 3K_7; x) = (1 + 121x + 147x^2 + 343x^3)^2 = 1 + 242x + 14935x^2 + 36260x^3 + 104615x^4 + 100842x^5 + 117649x^6.$$

However, if one of them is of degree one, we show that their product is still unimodal. **Lemma 2.1** If R_n is a unimodal polynomial, then $R_n \cdot R_1$ is unimodal for any polynomial R_1 .

Proof. Let $R_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a unimodal polynomial and $R_1(x) = b_0 + b_1x$. Suppose that $a_0 \le a_1 \le \ldots \le a_k \ge a_{k+1} \ge \ldots \ge a_n$ and $b_0 \le b_1$. Then, $R_n(x) \cdot R_1(x) = a_0b_0 + \sum_{i=1}^n (a_ib_0 + a_{i-1}b_1) \cdot x^i + a_nb_1 \cdot x^{n+1} = \sum_{i=0}^{n+1} c_i \cdot x^i$ and we show that $R_n \cdot R_1$ is unimodal, with the mode m, where

$$c_m = \max\{c_k, c_{k+1}\} = \max\{a_k b_0 + a_{k-1} b_1, a_{k+1} b_0 + a_k b_1\}.$$
 (1)

Firstly, $a_0b_0 \leq a_1b_0 + a_0b_1$ because $a_0b_0 \leq \max\{a_1b_0, a_0b_1\}$ and $0 \leq \min\{a_1b_0, a_0b_1\}$. Secondly, $a_{i-1} \leq a_i \leq a_{i+1}$ are true for any $i \in \{1, ..., k-1\}$, and these assure that $a_ib_0 + a_{i-1}b_1 \leq a_{i+1}b_0 + a_ib_1$. Further, $a_{i-1} \geq a_i \geq a_{i+1}$ are valid for any $i \in \{k+1, ..., n-1\}$, which imply that $a_ib_0 + a_{i-1}b_1 \geq a_{i+1}b_0 + a_ib_1$. Finally, $a_nb_0 + a_{n-1}b_1 \geq a_nb_1$, since $a_{n-1} \geq a_n$.

Similarly, we can show that $R_n \cdot R_1$ is unimodal, whenever $b_0 > b_1$.

The following proposition constitutes an useful tool in computing independence polynomials of graphs and also in finding recursive formulae for independence polynomials of various classes of graphs.

Proposition 2.2 [6], [9] Let G = (V, E) be a graph, $w \in V, uv \in E$ and $U \subset V$ be such that G[U] is a complete subgraph of G. Then the following equalities hold:

(i)
$$I(G; x) = I(G - w; x) + x \cdot I(G - N[w]; x);$$

(ii) $I(G; x) = I(G - U; x) + x \cdot \sum_{v \in U} I(G - N[v]; x);$
(iii) $I(G; x) = I(G - uv; x) - x^2 \cdot I(G - N(u) \cup N(v); x)$

The *edge-join* of two disjoint graphs G_1, G_2 is the graph $G_1 \ominus G_2$ obtained by adding an edge joining two vertices belonging to G_1, G_2 , respectively. If the two vertices are $v_i \in V(G_i), i = 1, 2$, then by $(G_1; v_1) \ominus (G_2; v_2)$ we mean the graph $G_1 \ominus G_2$.

Lemma 2.3 Let $G_i = (V_i, E_i), i = 1, 2$, be two well-covered graphs and $v_i \in V_i, i = 1, 2$, be simplicial vertices in G_1, G_2 , respectively, such that $N_{G_i}[v_i], i = 1, 2$, contains at least another simplicial vertex. Then the following assertions are true:

(i) $G = (G_1; v_1) \ominus (G_2; v_2)$ is well-covered and $\alpha(G) = \alpha(G_1) + \alpha(G_2);$ (ii) $I(G; x) = I(G_1; x) \cdot I(G_2; x) - x^2 \cdot I(G_1 - N_{G_1}[v_1]; x) \cdot I(G_2 - N_{G_2}[v_2]; x).$

Proof. (i) Let S_1, S_2 be maximum stable sets in G_1, G_2 , respectively. Since G_1, G_2 are well-covered, we may assume that $v_i \notin S_i, i = 1, 2$. Hence, $S_1 \cup S_2$ is stable in G and any maximum stable set A of G has $|A \cap V_1| \leq |S_1|, |A \cap V_2| \leq |S_2|$, and consequently we obtain:

$$|S_1| + |S_2| = |S_1 \cup S_2| \le |A| = |A \cap V_1| + |A \cap V_2| \le |S_1| + |S_2|,$$

i.e., $\alpha(G_1) + \alpha(G_2) = \alpha(G)$.

Let B be a stable set in G and $B_i = B \cap V_i$, i = 1, 2. Clearly, at most one of v_1, v_2 may belong to B. Since G_1, G_2 are well-covered, there exist S_1, S_2 maximum stable sets in G_1, G_2 , respectively, such that $B_1 \subseteq S_1, B_2 \subseteq S_2$.

Case 1. $v_1 \in B$ (similarly, if $v_2 \in B$), i.e., $v_1 \in B_1$. If $v_2 \notin S_2$, then $S_1 \cup S_2$ is a maximum stable set in G such that $B \subset S_1 \cup S_2$. Otherwise, let w be the other simplicial vertex belonging to $N_{G_2}[v_2]$. Then $S_3 = S_2 \cup \{w\} - \{v_2\}$ is a maximum stable set in G_2 , that includes B_2 , because $B_2 \subseteq S_2 - \{v_2\}$. Hence, $S_1 \cup S_3$ is a maximum stable set in G such that $B \subset S_1 \cup S_3$.

Case 2. $v_1, v_2 \notin B$. If $v_1, v_2 \in S_1 \cup S_2$, then as above, $S_1 \cup (S_2 \cup \{w\} - \{v_2\})$ is a maximum stable set in G that includes B. Otherwise, $S_1 \cup S_2$ is a maximum stable set in G such that $B \subset S_1 \cup S_2$.

Consequently, $G = (G_1; v_1) \oplus (G_2; v_2)$ is well-covered. (ii) Using Proposition 2.2(iii), we obtain that

$$I(G;x) = I(G - v_1v_2;x) - x^2 \cdot I(G - N_G(v_1) \cup N_G(v_2);x)$$

= $I(G_1;x) \cdot I(G_2;x) - x^2 \cdot I(G_1 - N_{G_1}(v_1);x) \cdot I(G_2 - N_{G_2}(v_2);x),$

which completes the proof. \blacksquare

By \triangle_n we denote the graph $\ominus nK_3$ defined as $\triangle_n = K_3 \ominus (n-1)K_3, n \ge 1$, (see Figure 4). \triangle_0 denotes the empty graph.

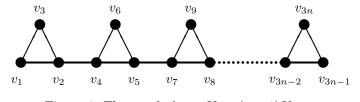


Figure 4: The graph $\triangle_n = K_3 \ominus (n-1)K_3$.

Proposition 2.4 The following assertions are true:

(i) for any $n \ge 1$, the graphs $\triangle_n, K_2 \ominus \triangle_n$ are well-covered; (ii) $I(\triangle_n; x)$ is unimodal for any $n \ge 1$, and

$$I(\triangle_n; x) = (1+3x) \cdot I(\triangle_{n-1}; x) - x^2 \cdot I(\triangle_{n-2}; x), n \ge 2,$$

where $I(\triangle_0; x) = 1, I(\triangle_1; x) = 1 + 3x;$ (iii) $I(K_2 \ominus \triangle_n; x)$ is unimodal for any $n \ge 1$, and

$$I(K_2 \ominus \triangle_n; x) = (1+2x) \cdot I(\triangle_n; x) - x^2 \cdot I(\triangle_{n-1}; x).$$

Proof. (i) We show, by induction on n, that \triangle_n is well-covered. Clearly, $\triangle_1 = K_3$ is well-covered. For $n \ge 2$ we have $\triangle_n = (\triangle_1; v_2) \ominus (\triangle_{n-1}; v_4)$, (see Figure 4). Hence, according to Lemma 2.3, Δ_n is well-covered, because v_2, v_3 and v_4, v_6 are simplicial vertices in $\triangle_1, \triangle_{n-1}$, respectively.

Therefore, \triangle_n is well-covered for any $n \ge 1$.

(ii) If $e = v_2 v_4$ and $n \ge 2$, then according to Proposition 2.2(iii), we obtain that

$$I(\triangle_{n}; x) = I(\triangle_{n} - e; x) - x^{2} \cdot I(\triangle_{n} - N(v_{2}) \cup N(v_{4}); x)$$

$$= I(K_{3}; x) \cdot I(\triangle_{n-1}; x) - x^{2} \cdot I(\triangle_{n-2}; x)$$

$$= (1 + 3x) \cdot I(\triangle_{n-1}; x) - x^{2} \cdot I(\triangle_{n-2}; x).$$

In addition, $I(\triangle_n; x)$ is unimodal by Theorem 1.3, because \triangle_n is claw-free.

(iii) Let us notice that both K_2 and \triangle_n are well-covered. The graph $K_2 \ominus \triangle_n = (K_2; u_2) \ominus (\triangle_n; v_1)$ is well-covered according to Lemma 2.3, and $I(K_2 \ominus \triangle_n; x)$ is unimodal for any $n \ge 1$, by Theorem 1.3, since $K_2 \ominus \triangle_n$ is claw-free (see Figure 5).

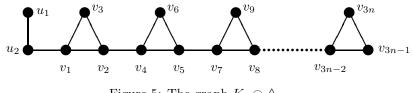


Figure 5: The graph $K_2 \ominus \triangle_n$.

In addition, applying Proposition 2.2(iii), we infer that

$$I(K_2 \ominus \triangle_n; x) = I(K_2 \ominus \triangle_n - u_2 v_1; x) - x^2 \cdot I(K_2 \ominus \triangle_n - N(u_2) \cup N(v_1); x)$$

$$= I(K_2; x) \cdot I(\triangle_n; x) - x^2 \cdot I(\triangle_{n-1}; x)$$

$$= (1+2x) \cdot I(\triangle_n; x) - x^2 \cdot I(\triangle_{n-1}; x),$$

that completes the proof. \blacksquare

Lemma 2.5 Let $G_i = (V_i, E_i), v_i \in V_i, i = 1, 2, and P_4 = (\{a, b, c, d\}, \{ab, bc, cd\}).$ Then the following assertions are true:

(i) $I(L_1; x) = I(L_2; x)$, where $L_1 = (P_4; b) \ominus (G_1; v)$, while L_2 has $V(L_2) = V(L_1)$, $E(L_2) = E(L_1) \cup \{ac\} - \{cd\}$. If G_1 is claw-free and v is simplicial in G_1 , then $I(L_1; x)$ is unimodal. (ii) I(G; x) = I(H; x), where $G = (G_3; c) \ominus (G_2; v_2)$ and $G_3 = (G_1; v_1) \ominus (P_4; b)$, while H has V(H) = V(G), $E(H) = E(G) \cup \{ac\} - \{cd\}$. If G_1, G_2 are claw-free and v_1, v_2 are simplicial in G_1, G_2 , respectively, then I(G; x) is unimodal.

Proof. (i) The graphs $L_1 = (P_4; b) \oplus (G_1; v)$ and $L_2 = (K_1 \amalg K_3; b) \oplus (G_1; v)$ are depicted in Figure 6.

Clearly, $I(P_4; x) = I(K_3 \amalg K_1; x) = 1 + 4x + 3x^2$. By Proposition 2.2(iii), we obtain:

$$I(L_1; x) = I(L_1 - vb; x) - x^2 \cdot I(L_1 - N(v) \cup N(b); x)$$

= $I(G_1; x) \cdot I(P_4; x) - x^2 \cdot I(G_1 - N_{G_1}[v]; x) \cdot I(\{d\}; x)$

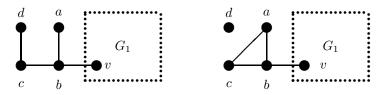


Figure 6: The graphs $L_1 = (P_4; b) \ominus (G_1; v)$ and $L_2 = (K_1 \amalg K_3; b) \ominus (G_1; v)$.

On the other hand, we get:

$$I(L_2; x) = I(L_2 - vb; x) - x^2 \cdot I(L_2 - N(v) \cup N(b); x)$$

= $I(G_1; x) \cdot I(K_3 \amalg K_1; x) - x^2 \cdot I(G_1 - N_{G_1}[v]; x) \cdot I(\{d\}; x).$

Consequently, the equality $I(L_1; x) = I(L_2; x)$ holds. If, in addition, v is simplicial in G_1 , and G_1 is claw-free, then L_2 is claw-free, too. Theorem 1.3 implies that $I(L_2; x)$ is unimodal, and, hence, $I(L_1; x)$ is unimodal, as well.

(ii) Figure 7 shows the graphs G and H.

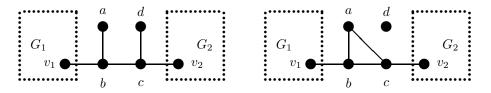


Figure 7: The graphs G and H from Lemma 2.5(ii).

According to Proposition 2.2(iii), we obtain:

$$I(G;x) = I(G - v_1b;x) - x^2 \cdot I(G - N(v_1) \cup N(b);x)$$

= $I(G_1;x) \cdot I(G - G_1;x) - x^2 \cdot I(G_1 - N_{G_1}[v_1];x) \cdot I(G_2;x) \cdot I(\{d\};x).$

On the other hand, using again Proposition 2.2(iii), we get:

$$I(H;x) = I(H - v_1b;x) - x^2 \cdot I(H - N(v_1) \cup N(b);x)$$

= $I(G_1;x) \cdot I(H - G_1;x) - x^2 \cdot I(G_1 - N_{G_1}[v_1];x) \cdot I(G_2;x) \cdot I(\{d\};x).$

Finally, let us observe that the equality $I(G-G_1; x) = I(H-G_1; x)$ holds according to part (i).

Now, if G_1, G_2 are claw-free and v_1, v_2 are simplicial in G_1, G_2 , respectively, then H is claw-free, and by Theorem 1.3, its independence polynomial is unimodal. Consequently, I(G; x) is also unimodal.

3 Independence polynomials of well-covered spiders

The well-covered spider $S_n, n \ge 2$ has one vertex of degree n + 1, n vertices of degree 2, and n + 1 vertices of degree 1 (see Figure 8).

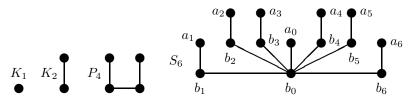


Figure 8: Well-covered spiders.

Theorem 3.1 The independence polynomial of any well-covered spider is unimodal, moreover,

$$I(S_n; x) = (1+x) \cdot \left\{ 1 + \sum_{k=1}^n \left[\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k \right\}, n \ge 2$$

and its mode is unique and equals $1 + (n-1) \mod 3 + 2(\lceil n/3 \rceil - 1)$.

Proof. Well-covered spiders comprise K_1, K_2, P_4 and $S_n, n \ge 2$. Clearly, the independence polynomials of K_1, K_2, P_4 are unimodal.

Using Proposition 2.2(i), we obtain the following formula for S_n :

$$I(S_n; x) = I(S_n - b_0; x) + x \cdot I(S_n - N[b_0]; x)$$

= $(1 + x) \cdot (1 + 2x)^n + x \cdot (1 + x)^n = (1 + x) \cdot R_n(x),$

where $R_n(x) = (1+2x)^n + x \cdot (1+x)^{n-1}$. By Lemma 2.1, to prove that $I(S_n; x)$ is unimodal, it is sufficient to show that $R_n(x)$ is unimodal. It is easy to see that

$$R_n(x) = 1 + \sum_{k=1}^n \left[\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k = 1 + \sum_{k=1}^n A(k) \cdot x^k,$$

$$A(k) = \binom{n}{k} \cdot 2^k + \binom{n-1}{k-1}, 1 \le k \le n, A(0) = 1.$$

To start with, we show that R_n is unimodal with the mode $k = n - 1 - \lfloor (n-2)/3 \rfloor$. Taking into account the proof of Lemma 2.1, namely, the equality 1, the mode of the polynomial

$$I(S_n; x) = (1+x) \cdot R_n(x) = 1 + \sum_{k=1}^n c_k \cdot x^k$$

is the index m, with $c_m = \max\{c_k, c_{k+1}\} = \max\{A(k) + A(k-1), A(k+1) + A(k)\}$. In other words, we will give evidence for

$$m = \begin{cases} k, & \text{if } A(k-1) > A(k+1), \\ k+1, & \text{if } A(k-1) \le A(k+1). \end{cases}$$
(2)

• Claim 1. If n = 3m + 1, then R_n is unimodal with the mode 2m + 1, $I(S_n; x)$ is also unimodal, and its unique mode equals 2m + 1.

We show that

$$\begin{array}{rcl} A(2m+i+1) & \geq & A(2m+i+2), 0 \leq i \leq m-1, \ and \\ A(2m+1-j) & \geq & A(2m-j), 0 \leq j \leq 2m. \end{array}$$

We have successively (for 2m + i + 1 = h):

$$\begin{aligned} A(h) - A(h+1) &= \left[\binom{3m+1}{h} \cdot 2^h + \binom{3m}{h-1} \right] - \left[\binom{3m+1}{h+1} \cdot 2^{h+1} + \binom{3m}{h} \right] \\ &= \frac{(3m+1)! \cdot 2^h \cdot \left[(h+1) - 2 \cdot (m-i) \right]}{(h+1)! \cdot (m-i)!} + \frac{(3m)! \cdot \left[h - (m-i) \right]}{h! \cdot (m-i)!} \\ &= \frac{(3m+1)! \cdot (3i+2) \cdot 2^h}{(h+1)! \cdot (m-i)!} + \frac{(3m)! \cdot (m+2i+1)}{h! \cdot (m-i)!} \ge 0. \end{aligned}$$

Further, we get (for 2m - j = h):

$$\begin{aligned} A(h+1) - A(h) &= \left[\binom{3m+1}{h+1} \cdot 2^{h+1} + \binom{3m}{h} \right] - \left[\binom{3m+1}{h} \cdot 2^{h} + \binom{3m}{h-1} \right] \\ &= \frac{(3m+1)! \cdot 2^{h} \cdot [2 \cdot (m+j+1) - (h+1)]}{(h+1)! \cdot (m+j+1)!} + \frac{(3m)! \cdot [(m+j+1) - h]}{h! \cdot (m+j+1)!} \end{aligned}$$

$$=\frac{(3m)!}{(h+1)!\cdot(m+j+1)!}\left[(3m+1)\cdot(3j+1)\cdot2^{h}-(m-2j-1)\cdot(h+1)\right]\geq0,$$

because $3m + 1 > m \ge m - 2j - 1$ and $2^h \ge h + 1$. To find the location of the mode of $I(S_n; x)$, we obtain

$$A(2m) - A(2m+2) =$$

$$= \binom{3m+1}{2m} \cdot 2^{2m} + \binom{3m}{2m-1} - \binom{3m+1}{2m+2} \cdot 2^{2m+2} + \binom{3m}{2m+1}$$
$$= \frac{(3m)! \cdot 2^{2m}}{(2m)! \cdot (m-1)!} \cdot \left[\frac{3m+1}{m \cdot (m+1)} \cdot \left(\frac{1}{2m} - \frac{1}{2m+1}\right] + \frac{(3m)!}{(2m-1)! \cdot (m-1)!} \cdot \left[\frac{1}{m \cdot (m+1)} - \frac{1}{2m \cdot (2m+1)}\right] > 0.$$

Consequently, in accordance with equality (2), the mode of $I(S_n; x)$ equals 2m + 1. Since A(2m) - A(2m+2) > 0, the mode is unique.

• Claim 2. If n = 3m, then R_n is unimodal with the mode 2m, $I(S_n; x)$ is also unimodal, and its unique mode equals 2m + 1.

We show that

$$\begin{array}{rcl} A(2m+i) & \geq & A(2m+i+1), 0 \leq i \leq m-1, \ and \\ A(2m-j) & \geq & A(2m-j-1), 0 \leq j \leq 2m-1. \end{array}$$

We have successively (for 2m + i = h):

$$A(h) - A(h+1) = \left[\binom{3m}{h} 2^{h} + \binom{3m-1}{h-1} \right] - \left[\binom{3m}{h+1} 2^{h+1} + \binom{3m-1}{h} \right]$$

$$=\frac{(3m)!\cdot 2^h\cdot [(h+1)-2\cdot (m-i)]}{(h+1)!\cdot (m-i)!}+\frac{(3m-1)!\cdot [h-(m-i)]}{h!\cdot (m-i)!}$$

$$=\frac{(3m)!\cdot(3i+1)\cdot 2^h}{(h+1)!\cdot(m-i)!}\cdot+\frac{(3m-1)!\cdot(m+2i)}{h!\cdot(m-i)!}\geq 0.$$

Further, we get (for 2m - j = h):

$$\begin{aligned} A(h) - A(h-1) &= \left[\binom{3m}{h} \cdot 2^h + \binom{3m-1}{h-1} \right] - \left[\binom{3m}{h-1} \cdot 2^{h-1} + \binom{3m-1}{h-2} \right] \\ &= \frac{(3m)! \cdot 2^{h-1} \cdot [2(m+j+1)-h]}{h! \cdot (m+j+1)!} + \frac{(3m-1)! \cdot [(m+j+1)-(h-1)]}{(h-1)! \cdot (m+j+1)!} \\ &= \frac{(3m-1)!}{h! \cdot (m+j+1)!} \cdot \left[3m \cdot \frac{3j+2}{2} \cdot 2^h - (m-2j-2) \cdot h \right] \ge 0, \end{aligned}$$

since $3m > m \ge m - 2j - 2$ and $2^h \ge h$. To determine the mode of $I(S_n; x)$, we obtain

$$A(2m-1) - A(2m+1) =$$

$$= \binom{3m}{2m-1} \cdot 2^{2m-1} + \binom{3m-1}{2m-2} - \binom{3m}{2m+1} \cdot 2^{2m+1} + \binom{3m-1}{2m}$$
$$= \frac{3}{2} \cdot \frac{(3m-1)!}{(2m+1)! \cdot (m+1)!} \cdot \left[(m-1) \cdot (2m+1) - m \cdot 2^{2m} \right] < 0.$$

Consequently, in accordance with equality (2), the mode of $I(S_n; x)$ equals 2m + 1. Since A(2m-1) - A(2m+1) < 0, the mode is unique.

• Claim 3. If n = 3m - 1, then R_n is unimodal with the mode 2m - 1, $I(S_n; x)$ is also unimodal, and its unique mode equals 2m.

We show that

$$\begin{array}{rcl} A(2m+i-1) & \geq & A(2m+i), 0 \leq i \leq m-1, \ and \\ A(2m-j-1) & \geq & A(2m-j-2), 0 \leq j \leq 2m-2. \end{array}$$

We have successively (for 2m + i = h):

$$A(h-1) - A(h) = \left[\binom{3m-1}{h-1} \cdot 2^{h-1} + \binom{3m-2}{h-2} \right] - \left[\binom{3m-1}{h} \cdot 2^h + \binom{3m-2}{h-1} \right]$$

$$=\frac{(3m-1)!\cdot 2^{h-1}\cdot [h-2\cdot (m-i)]}{h!\cdot (m-i)!}+\frac{(3m-2)!\cdot [(h-1)-(m-i)]}{(h-1)!\cdot (m-i)!}$$

$$=\frac{(3m-1)!\cdot 3i\cdot 2^{h-1}}{h!\cdot (m-i)!}+\frac{(3m-2)!\cdot (m+2i-1)}{(h-1)!\cdot (m-i)!}\geq 0.$$

Further, we get (for 2m - j - 1 = h):

$$A(h) - A(h-1) = \left[\binom{3m-1}{h} \cdot 2^h + \binom{3m-2}{h-1} \right] - \left[\binom{3m-1}{h-1} \cdot 2^{h-1} + \binom{3m-2}{h-2} \right]$$

$$=\frac{(3m-1)!\cdot 2^{h-1}\cdot [2\cdot (3m-h)-h]}{h!\cdot (3m-h)!}+\frac{(3m-2)!\cdot [(3m-h)-(h-1)]}{(h-1)!\cdot (3m-h)!}$$

$$=\frac{(3m-2)!}{h!\cdot(m+j+1)!}\cdot\left[(3m-1)\cdot\frac{3j+3}{2}\cdot 2^{h}-h\cdot(m-2j-3)\right]\geq 0,$$

because $3m-1 > m \ge m-2j-1$ and $2^h \ge h$.

Finally, we obtain that

$$A(2m-2) - A(2m) =$$

$$= \binom{3m}{2m-2} \cdot 2^{2m-2} + \binom{3m-1}{2m-3} - \binom{3m-1}{2m} \cdot 2^{2m} + \binom{3m-2}{2m-1}$$

$$=\frac{(3m-1)!}{(2m-1)!\cdot(m+1)!}\cdot\left[m-2-3\cdot2^{2m-2}\right]<0.$$

Consequently, in accordance with equality (2), the mode of $I(S_n; x)$ equals 2m. Since A(2m-1) - A(2m+1) < 0, the mode is unique.

4 Transformations of some well-covered trees

If both v_1 and v_2 are vertices of degree at least two in G_1, G_2 , respectively, then $(G_1; v_1) \ominus (G_2, v_2)$ is an *internal edge-join* of G_1, G_2 . Notice that the edge-join of two trees is a tree, and also that two trees can be internal edge-joined provided each one is of order at least three. An alternative characterization of well-covered trees is the following:

Theorem 4.1 [11] A tree T is well-covered if and only if either T is a well-covered spider, or T is the internal edge-join of a number of well-covered spiders.

As examples, $W_n, n \ge 4$, and $G_{m,n}, m \ge 2, n \ge 3$, (see Figures 9 and 12) are internal edge-join of a number of well-covered spiders, and consequently, they are wellcovered trees. The aim of this section is to show that the independence polynomials of W_n and of $G_{m,n}$ are unimodal. The idea is to construct, for these trees, some claw-free graphs having the same independence polynomial, and then to use Theorem 1.3. We leave open the question whether the procedure we use is helpful to define a claw-free graph with the same independence polynomial as a general well-covered tree.

A centipede is a tree denoted by $W_n = (A, B, E), n \ge 1$, (see Figure 9), where $A \cup B$ is its vertex set, $A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_n\}, A \cap B = \emptyset$, and the edge set $E = \{a_i b_i : 1 \le i \le n\} \cup \{b_i b_{i+1} : 1 \le i \le n-1\}.$

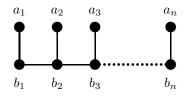


Figure 9: The centipede W_n .

The following result was proved in [12], but for the sake of self-consistency of this paper and to illustrate the idea of a hidden correspondence between well-covered trees and claw-free graphs, we give here its proof.

Theorem 4.2 [12] For any $n \ge 1$ the following assertions hold:

(i) $I(W_{2n}; x) = (1+x)^n Q_n(x) = I(\triangle_n \amalg nK_1; x)$, where $Q_n(x) = I(\triangle_n; x)$; $I(W_{2n+1}; x) = (1+x)^n Q_{n+1}(x) = I((K_2 \ominus \triangle_n) \amalg nK_1; x)$, where $Q_{n+1}(x) = I(K_2 \ominus \triangle_n; x)$; (ii) $I(W_n; x)$ is unimodal and

$$I(W_n; x) = (1+x) \cdot (I(W_{n-1}; x) + x \cdot I(W_{n-2}; x)), n \ge 2,$$

where $I(W_0; x) = 1, I(W_1; x) = 1 + 2x$.

Proof. (i) Evidently, the polynomials $I(W_n; x), 1 \le n \le 3$, are unimodal, since

$$I(W_1; x) = 1 + 2x, I(W_2; x) = 1 + 4x + 3x^2, I(W_3; x) = 1 + 6x + 10x^2 + 5x^3.$$

Applying $\lfloor n/2 \rfloor$ times Lemma 2.5(ii), we obtain that for $n = 2m \ge 4$,

$$I(W_n; x) = I(\triangle_m \amalg mK_1; x) = I(\triangle_m; x) \cdot (1+x)^m,$$

while for $n = 2m + 1 \ge 5$,

$$I(W_n; x) = I(K_2 \ominus \triangle_m \amalg mK_1; x) = I(K_2 \ominus \triangle_m; x) \cdot (1+x)^m.$$

(iii) According to Proposition 2.4 and Lemma 2.1, it follows that $I(W_n; x)$ is unimodal. Further, taking $U = \{a_n, b_n\}$ and applying Proposition 2.2(ii), we obtain:

$$\begin{split} I(W_n; x) &= I(W_n - U; x) + x \cdot (I(W_n - N[a_n]; x) + I(W_n - N[b_n]; x)) \\ &= I(W_{n-1}; x) + x \cdot I(W_{n-1}; x) + x \cdot (1 + x) \cdot I(W_{n-2}; x)) \\ &= (1 + x) \cdot (I(W_{n-1}; x) + x \cdot I(W_{n-2}; x)), \end{split}$$

which completes the proof. \blacksquare

It is worth mentioning that the problem of finding the mode of the centipede is still unsolved.

Conjecture 4.3 [12] The mode of $I(W_n; x)$ is k = n - f(n) and f(n) is given by

$$\begin{aligned} f(n) &= 1 + \lfloor n/5 \rfloor, 2 \leq n \leq 6, \\ f(n) &= f(2 + (n-2) \mod 5) + 2 \lfloor (n-2)/5 \rfloor, n \geq 7. \end{aligned}$$

Proposition 4.4 The following assertions are true:

- (i) $I(G_{2,4};x)$ is unimodal, moreover $I(G_{2,4};x) = I(3K_1 \amalg K_2 \amalg (K_4 \ominus K_3);x)$ (see Figure 12);
- (ii) I(G; x) = I(L; x), where $G = (G_{2,4}; v_4) \ominus (H; w)$ and $L = 3K_1 \amalg K_2 \amalg (K_4 \ominus K_3) \ominus H$ (see Figure 10); if w is simplicial in H, and H is claw-free, then I(G; x) is unimodal;
- (iii) I(G; x) = I(L; x), where $G = (H_1; w_1) \ominus (v; G_{2,4}; u) \ominus (H_2; w_2)$ and $L = 3K_1 \amalg K_2 \amalg ((H_1; w_1) \ominus (v; K_3) \ominus (K_4; u) \ominus (w_2; H_2)$ (see Figure 11); if w_1, w_2 are simplicial in H_1, H_2 , respectively, and H_1, H_2 are claw-free, then I(G; x) is unimodal.

Proof. (i) Using Proposition 2.2(iii) and the fact that $I(W_4; x) = 1 + 8x + 21x^2 + 22x^3 + 8x^4 = (1+x)^2(1+2x)(1+4x)$, we get that

$$I(G_{2,4};x) = I(G_{2,4} - b_2v_2;x) - x^2 \cdot I(G_{2,4} - N(b_2) \cup N(v_2);x)$$

= 1 + 12x + 55x² + 125x³ + **150**x⁴ + 91x⁵ + 22x⁶,

which is clearly unimodal. On the other hand, it is easy to check that

$$I(G_{2,4};x) = (1+x)^3(1+2x)(1+7x+11x^2) = I(3K_1 \amalg K_2 \amalg (K_4 \ominus K_3);x).$$

(ii) According to Proposition 2.2(iii), we infer that

$$I(G;x) = I(G - vw; x) - x^{2} \cdot I(G - N(v) \cup N(w); x)$$

= $I(G_{2,4}; x) \cdot I(H; x) - x^{2} \cdot (1 + x) \cdot I(H - N_{H}[w]; x) \cdot I(W_{4}; x)$
= $I(G_{2,4}; x) \cdot I(H; x) - - x^{2} \cdot (1 + x)^{3} \cdot (1 + 2x) \cdot (1 + 4x) \cdot I(H - N_{H}[w]; x).$

Figure 10 shows the graphs G and L.

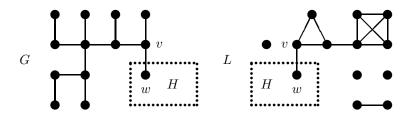


Figure 10: The graphs G and L in Proposition 4.4(ii).

Let us denote $Q_1 = 3K_1 \amalg K_2 \amalg K_4, Q_2 = 3K_1 \amalg K_2 \amalg (K_4 \ominus K_3), L = (Q_2; v) \ominus (H; w)$ and e = vw. Then, Proposition 2.2(iii) implies that

$$\begin{split} I(L;x) &= I(L-vw;x) - x^2 \cdot I(L-N(v) \cup N(w);x) \\ &= I(Q_2;x) \cdot I(H;x) - x^2 \cdot I(Q_1;x) \cdot I(H-N_H[w];x) \\ &= I(G_{2,4};x) \cdot I(H;x) - \\ &- x^2 \cdot (1+x)^3 \cdot (1+2x) \cdot (1+4x) \cdot I(H-N_H[w];x) \end{split}$$

Consequently, I(G; x) = I(L; x) holds.

In addition, if w is simplicial in H, and H is claw-free, then L is claw-free and, by Theorem 1.3, I(L; x) is unimodal. Hence, I(G; x) is unimodal, as well.

(iii) Let $e = uw_2 \in E(G)$. Then, according to Proposition 2.2(iii), we get that

$$\begin{split} I(G;x) &= I(G - uw_2;x) - x^2 \cdot I(G - N(u) \cup N(w_2);x) \\ &= I(H_2;x) \cdot I((G_{2,4};v) \ominus (H_1;w_1);x) - \\ &- x^2 \cdot (1+x)^2 \cdot (1+2x) \cdot I(H_2 - N_{H_2}[w_2];x) \cdot I((P_4;v) \ominus (H_1;w_1);x). \end{split}$$

Now, Lemma 2.5(i) implies that

$$I(G;x) = I(H_2;x) \cdot I((G_{2,4};v) \ominus (H_1;w_1);x) - -x^2 \cdot (1+x)^2 \cdot (1+2x) \cdot I(H_2 - N_{H_2}[w_2];x) \cdot I((K_1 \sqcup K_3;v) \ominus (H_1;w_1);x).$$

Figure 11 shows the graphs G and L.

Let $e = uw_2 \in E(L)$. Again by Proposition 2.2(iii), we infer that

$$\begin{split} I(L;x) &= I(L-uw_2;x) - x^2 \cdot I(L-N(u) \cup N(w_2);x) \\ &= I(H_2;x) \cdot I((3K_1 \amalg K_2 \amalg (K_4 \ominus K_3);v) \ominus (H_1;w_1);x) - \\ &- x^2 \cdot (1+x)^3 \cdot (1+2x) \cdot I(H_2 - N_{H_2} [w_2];x) \cdot I((K_3;v) \ominus (H_1;w_1);x). \end{split}$$

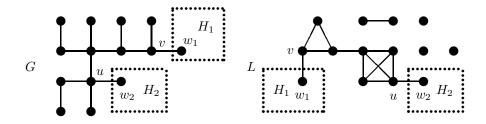


Figure 11: The graphs G and L in Proposition 4.4(iii).

Further, Proposition 4.4(ii) helps us to deduce that

 $I(H_2; x) \cdot I((G_{2,4}; v) \ominus (H_1; w_1); x) = I(H_2; x) \cdot I((3K_1 \amalg K_2 \amalg (K_4 \ominus K_3); v) \ominus (H_1; w_1); x).$

Eventually, since

$$I((K_1 \sqcup K_3; v) \ominus (H_1; w_1); x) = (1 + x) \cdot I((K_3; v) \ominus (H_1; w_1); x),$$

we obtain I(G; x) = I(L; x).

If w_1, w_2 are simplicial in H_1, H_2 , respectively, and H_1, H_2 are claw-free, then L is claw-free and, therefore, I(L; x) is unimodal, by Theorem 1.3. Hence, I(G; x) is unimodal, too.

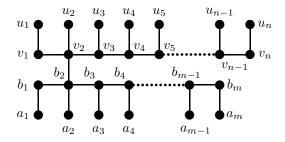


Figure 12: The graph $G_{m,n}, m \ge 2, n \ge 3$.

Theorem 4.5 The independence polynomial of $G_{m,n} = (W_m; b_2) \ominus (W_n; v_2)$ is unimodal, for any $m \ge 2, n \ge 2$.

Proof. Case 1. Suppose that m = 2, 3, n = 3, 4. The polynomial $I(G_{2,3}; x)$ is unimodal, because

$$I(G_{2,3};x) = (1+x)^2 \cdot (1+2x) \cdot (1+6x+7x^2)$$

= 1+10x+36x²+60x³+47x⁴+14x⁵.

According to Proposition 4.4(i), the independence polynomial of $G_{2,4}$ is unimodal and $I(G_{2,4};x) = I(3K_1 \amalg K_2 \amalg (K_4 \ominus K_3);x).$

Further, $I(G_{3,3}; x)$ is unimodal, since

$$I(G_{3,3};x) = I(G_{3,3} - v_2b_2;x) - x^2 \cdot I(G_{3,3} - N(v_2) \cup N(b_2);x)$$

= $I(W_3;x) \cdot I(W_3;x) - x^2 \cdot (1+x)^4$
= $1 + 12x + 55x^2 + 126x^3 + 154x^4 + 96x^5 + 24x^6.$

Finally, $I(G_{3,4}; x)$ is unimodal, because

$$\begin{split} I(G_{3,4};x) &= I(G_{3,4} - b_1b_2;x) - x^2 \cdot I(G_{3,4} - N(b_1) \cup N(b_2);x) \\ &= (1+2x) \cdot I(G_{2,4};x) - x^2 \cdot (1+x)^2 \cdot (1+2x) \cdot I(P_4;x) \\ &= 1 + 14x + 78x^2 + 227x^3 + \mathbf{376}x^4 + 357x^5 + 181x^6 + 38x^7. \end{split}$$

Case 2. Assume that $m = 2, n \geq 5$. According to Proposition 4.4 (ii), we infer that $I(G_{2,n}; x) = I(L_1; x)$, where $L_1 = Q \ominus W_{n-4}$ and $Q = 3K_1 \amalg K_2 \amalg K_4 \ominus K_3$ (see Figure 13). Applying Lemma 2.5(ii), $I(L_1; x) = I((mK_1 \amalg (Q \ominus mK_3); x))$, if

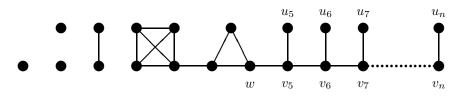


Figure 13: The graph $L_1 = 3K_1 \amalg K_2 \amalg K_4 \ominus K_3 \ominus W_{n-4}$.

n-4 = 2m, and $I(L_1; x) = I((mK_1 \amalg (Q \ominus mK_3 \ominus K_2); x))$, if n-4 = 2m+1. Since $mK_1 \amalg (Q \ominus mK_3 \ominus K_2)$ is claw-free, it follows that $I(L_1; x)$ is unimodal, and consequently, $I(G_{2,n}; x)$ is unimodal, too.

Case 3. Assume that $m \ge 3, n \ge 5$. According to Proposition 4.4(iii), we obtain

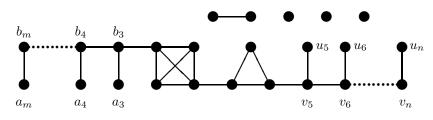


Figure 14: The graph $L_2 = W_{m-2} \ominus (3K_1 \amalg K_2 \amalg K_4 \ominus K_3) \ominus W_{n-4}$.

that $I(G_{m,n}; x) = I(L_2; x)$, where

$$L_2 = W_{m-2} \ominus Q \ominus W_{n-4} \text{ and } Q = 3K_1 \amalg K_2 \amalg K_4 \ominus K_3$$

(see Figure 14). Finally, by Theorem 1.3, we infer that $I(L_2; x)$ is unimodal, since by applying Lemma 2.5, W_{m-2} and W_{n-4} can be substituted by $pK_1 \amalg (\oplus pK_3 \oplus K_2)$ or $pK_1 \amalg (\oplus pK_3)$, depending on the parity of the numbers m-2, n-4. Consequently, the polynomial $I(G_{m,n}; x)$ is unimodal, as well.

5 Conclusions

In this paper we keep investigating the unimodality of independence polynomials of some well-covered trees started in [12]. Any such a tree is an edge-join of a number of "*atoms*", called well-covered spiders. We proved that the independence polynomial of any well-covered spider is unimodal, straightforwardly indicating the location of the mode. We also showed that the independence polynomial of some edge-join of well-covered spiders is unimodal. In the later case, our approach was indirect, via claw-free graphs.

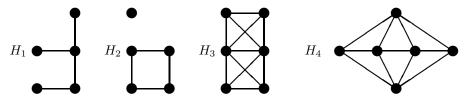


Figure 15: $I(H_1; x) = I(H_2; x)$ and $I(H_3; x) = I(H_4; x)$.

Let us notice that $I(H_1; x) = I(H_2; x) = 1 + 5x + 6x^2 + 2x^3$, and also $I(H_3; x) = I(H_4; x) = 1 + 6x + 4x^2$, where H_1, H_2, H_3, H_4 are depicted in Figure 15. In other words, there exist a well-covered graph whose independence polynomial equals the independence polynomial of a non-well-covered tree (e.g., H_2 and H_1), and also a well-covered graph, different from a tree, namely H_4 , satisfying $I(H_3; x) = I(H_4; x)$, where H_3 is not a well-covered graph. Moreover, we can show that for any $\alpha \ge 2$ there are two connected graphs G_1, G_2 such that $\alpha(G_1) = \alpha(G_2) = \alpha$ and $I(G_1; x) = I(G_2; x)$, but only one of them is well-covered. To see this, let us consider the following two graphs:

$$G_1 = L + (H_1 \amalg H_2 \amalg 2K_1), G_2 = (L_1 \amalg L_2) + (H_1 \amalg H_2 \amalg K_2),$$

where L, L_1, L_2 are well-covered graphs, and

$$L = (L_1, v_1) \ominus (L_2, v_2), H_1 = L_1 - N[v_1], H_2 = L_2 - N[v_2], \alpha(L) = \alpha(L_1) + \alpha(L_2).$$

It follows that $\alpha(H_1) = \alpha(L_1) - 1, \alpha(H_2) = \alpha(L_2) - 1$, and therefore, we obtain $\alpha(G_1) = \alpha(G_2) = \alpha(L)$. It is easy to check that G_1 is well-covered, while G_2 is not well-covered. According to Proposition 2.2(iii), we infer that

$$I(L;x) = I(L - v_1v_2;x) - x^2 \cdot I(L - N_L(v_1) \cup N_L(v_2);x)$$

= $I(L_1;x) \cdot I(L_2;x) - x^2 \cdot I(H_1;x) \cdot I(H_2;x)$

which we can write as follows:

$$I(L;x) + (1+x)^2 \cdot I(H_1;x) \cdot I(H_2;x) = I(L_1;x) \cdot I(L_2;x) + (1+2x) \cdot I(H_1;x) \cdot I(H_2;x)$$

or, equivalently, as

$$I(L;x) + I(2K_1;x) \cdot I(H_1;x) \cdot I(H_2;x) = I(L_1;x) \cdot I(L_2;x) + I(K_2;x) \cdot I(H_1;x) \cdot I(H_2;x).$$

In other words, we get:

$$I(L + (2K_1 \amalg H_1 \amalg H_2); x) = I(L_1 \amalg L_2 + (K_2 \amalg H_1 \amalg H_2); x),$$

i.e., $I(G_1; x) = I(G_2; x)$.

However, in some of our findings we defined claw-free graphs that simultaneously are well-covered and have the same independence polynomials as the well-covered trees under investigation. These results give an evidence for the following conjecture.

Conjecture 5.1 If T is a well-covered tree and I(T;x) = I(G;x), then G is well-covered.

References

- Y. Alavi, P. J. Malde, A. J. Schwenk, P. Erdös, *The vertex independence sequence of a graph is not constrained*, Congressus Numerantium 58 (1987) 15-23.
- [2] J. L. Arocha, Propriedades del polinomio independiente de un grafo, Revista Ciencias Matematicas, vol. V (1984) 103-110.
- [3] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, in "Jerusalem Combinatorics '93", Contemporary Mathematics 178 (1994) 71-89.
- [4] J. I. Brown, K. Dilcher, R. J. Nowakowski, Roots of independence polynomials of well-covered graphs, Journal of Algebraic Combinatorics 11 (2000) 197-210.
- [5] G. Gordon, E. McDonnel, D. Orloff and N. Yung, On the Tutte polynomial of a tree, Congressus Numerantium 108 (1995) 141-151.
- [6] I. Gutman, F. Harary, Generalizations of the matching polynomial, Utilitas Mathematica 24 (1983) 97-106.
- [7] Y. O. Hamidoune, On the number of independent k-sets in a claw-free graph, Journal of Combinatorial Theory B 50 (1990) 241-244.
- [8] S. T. Hedetniemi and R. Laskar, Connected domination in graphs, in Graph Theory and Combinatorics, Eds. B. Bollobas, Academic Press, London (1984) 209-218.
- C. Hoede, X. Li, Clique polynomials and independent set polynomials of graphs, Discrete Mathematics 125 (1994) 219-228.
- [10] D. G. C. Horrocks, The numbers of dependent k-sets in a graph is log concave, Journal of Combinatorial Theory B 84 (2002) 180-185.
- [11] V. E. Levit, E. Mandrescu, Well-covered trees, Congressus Numerantium 139 (1999) 101-112.

- [12] V. E. Levit, E. Mandrescu, On well-covered trees with unimodal independence polynomials, Congressus Numerantium (2002) (accepted).
- [13] M. D. Plummer, Some covering concepts in graphs, Journal of Combinatorial Theory 8 (1970) 91-98.
- [14] G. Ravindra, Well-covered graphs, J. Combin. Inform. System Sci. 2 (1977) 20-21.
- [15] J. Riordan, An introduction to combinatorial analysis, John Wiley and Sons, New York, 1958.
- [16] A. J. Schwenk, On unimodal sequences of graphical invariants, Journal of Combinatorial Theory B 30 (1981) 247-250.
- [17] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Annals of the New York Academy of Sciences 576 (1989) 500-535.
- [18] A. A. Zykov, On some properties of linear complexes, Math. Sb. 24 (1949) 163-188 (in Russian).
- [19] A. A. Zykov, Fundamentals of graph theory, BCS Associates, Moscow, 1990.