# On Unimodality of Independence Polynomials of some Well-Covered Trees* 

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#### Abstract

The stability number $\alpha(G)$ of the graph $G$ is the size of a maximum stable set of $G$. If $s_{k}$ denotes the number of stable sets of cardinality $k$ in graph $G$, then $I(G ; x)=\sum_{k=0}^{\alpha(G)} s_{k} x^{k}$ is the independence polynomial of $G$ (I. Gutman and F. Harary, 1983). In 1990, Y. O. Hamidoune proved that for any claw-free graph $G$ (a graph having no induced subgraph isomorphic to $\left.K_{1,3}\right), I(G ; x)$ is unimodal, i.e., there exists some $k \in\{0,1, \ldots, \alpha(G)\}$ such that $$
s_{0} \leq s_{1} \leq \ldots \leq s_{k} \geq s_{k+1} \geq \ldots \geq s_{\alpha(G)} .
$$ Y. Alavi, P. J. Malde, A. J. Schwenk and P. Erdös (1987) asked whether for trees (or perhaps forests) the independence polynomial is unimodal. J. I. Brown, K. Dilcher and R. J. Nowakowski (2000) conjectured that $I(G ; x)$ is unimodal for any well-covered graph $G$ (a graph whose all maximal independent sets have the same size). V. E. Levit and E. Mandrescu (1999) demonstrated that every well-covered tree can be obtained as a join of a number of well-covered spiders, where a spider is a tree having at most one vertex of degree at least three.

In this paper we show that the independence polynomial of any well-covered spider is unimodal. In addition, we introduce some graph transformations respecting independence polynomials. They allow us to reduce several types of well-covered trees to claw-free graphs, and, consequently, to prove that their independence polynomials are unimodal.


key words: stable set, independence polynomial, unimodal sequence, wellcovered tree, claw-free graph.

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## 1 Introduction

Throughout this paper $G=(V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V=V(G)$ and edge set $E=E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G-W$ we mean the subgraph $G[V-W]$, if $W \subset V(G)$. We also denote by $G-F$ the partial subgraph of $G$ obtained by deleting the edges of $F$, for $F \subset E(G)$, and we write shortly $G-e$, whenever $F=\{e\}$. The neighborhood of a vertex $v \in V$ is the set $N_{G}(v)=\{w: w \in V$ and $v w \in E\}$, and $N_{G}[v]=N_{G}(v) \cup\{v\}$; if there is no ambiguity on $G$, we use $N(v)$ and $N[v]$, respectively. If $N(v)$ induces a complete graph in $G$, then $v$ is a simplicial vertex of $G$. A simplicial vertex is pendant if its neighborhood contains only one vertex, and an edge is pendant if at least one of its endpoints is a pendant vertex. $K_{n}, P_{n}, C_{n}, K_{n_{1}, n_{2}, \ldots, n_{p}}$ denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, the chordless cycle on $n \geq 3$ vertices, and the complete $p$-partite graph on $n_{1}+n_{2}+\ldots+n_{p}$ vertices.

The disjoint union of the graphs $G_{1}, G_{2}$ is the graph $G=G_{1} \amalg G_{2}$ having as a vertex set the disjoint union of $V\left(G_{1}\right), V\left(G_{2}\right)$, and as an edge set the disjoint union of $E\left(G_{1}\right), E\left(G_{2}\right)$. In particular, $\amalg n G$ denotes the disjoint union of $n>1$ copies of the graph $G$. If $G_{1}, G_{2}$ are disjoint graphs, then their Zykov sum, (Zykov, 18], 19]), is the graph $G_{1}+G_{2}$ with

$$
\begin{aligned}
V\left(G_{1}+G_{2}\right) & =V\left(G_{1}\right) \cup V\left(G_{2}\right) \\
E\left(G_{1}+G_{2}\right) & =E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\left\{v_{1} v_{2}: v_{1} \in V\left(G_{1}\right), v_{2} \in V\left(G_{2}\right)\right\}
\end{aligned}
$$

As usual, a tree is an acyclic connected graph. A tree having at most one vertex of degree $\geq 3$ is called a spider, (8), or an aster, (5).

A stable set in $G$ is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a maximum stable set of $G$, and the stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in $G$. Let $s_{k}$ be the number of stable sets in $G$ of cardinality $k, k \in\{1, \ldots, \alpha(G)\}$. The polynomial

$$
I(G ; x)=\sum_{k=0}^{\alpha(G)} s_{k} x^{k}, s_{0}=1
$$

is called the independence polynomial of $G$, (Gutman and Harary, [6]).
A number of general properties of the independence polynomial of a graph are presented in [6] and [2]. As important examples, we mention the following:

$$
\begin{aligned}
I\left(G_{1} \amalg G_{2} ; x\right) & =I\left(G_{1} ; x\right) \cdot I\left(G_{2} ; x\right), \\
I\left(G_{1}+G_{2} ; x\right) & =I\left(G_{1} ; x\right)+I\left(G_{2} ; x\right)-1 .
\end{aligned}
$$

A finite sequence of non-negative real numbers $\left\{a_{0}, a_{1}, a_{2}, \ldots, a_{n}\right\}$ is said to be unimodal if there is some $k \in\{0,1, \ldots, n\}$, called the mode of the sequence, such that

$$
0 \leq a_{0} \leq a_{1} \leq \ldots \leq a_{k} \geq a_{k+1} \geq \ldots \geq a_{n}
$$

The mode is unique if $a_{k-1}<a_{k}>a_{k+1}$.

Unimodal sequences occur in many areas of mathematics, including algebra, combinatorics, and geometry (see Brenti, [3], and Stanley, 17]). In the context of our paper, for instance, if $a_{i}$ denotes the number of ways to select a subset of $i$ independent edges (a matching of size $i$ ) in a graph, then the sequence of these numbers is unimodal (Schwenk, 16]). As another example, if $a_{i}$ denotes the number of dependent $i$-sets of a graph $G$ (sets of size $i$ that are not stable), then the sequence of $\left\{a_{i}\right\}_{i=0}^{n}$ is unimodal (Horrocks, 10).

A polynomial $P(x)=a_{0}+a_{1} x+a_{2} x^{2}+\ldots+a_{n} x^{n}$ is called unimodal if the sequence of its coefficients is unimodal. For instance, the independence polynomial of $K_{n}$ is unimodal, as $I\left(K_{n} ; x\right)=1+n x$. However, the independence polynomial of the graph $G=K_{100}+\amalg 3 K_{6}$ is not unimodal, since $I(G ; x)=1+\mathbf{1 1 8} x+108 x^{2}+\mathbf{2 0 6} x^{3}$ (for another examples, see Alavi et al [1]). Moreover, in [1] it is shown that for any permutation $\sigma$ of $\{1,2, \ldots, \alpha\}$ there exists a graph $G$, with $\alpha(G)=\alpha$, such that $s_{\sigma(1)}<s_{\sigma(2)}<\ldots<s_{\sigma(\alpha)}$, i.e., there are graphs for which $s_{1}, s_{2}, \ldots, s_{\alpha}$ is as "shuffled" as we like.

A graph $G$ is called well-covered if all its maximal stable sets have the same cardinality, (Plummer, 13]). In particular, a tree $T$ is well-covered if and only if $T=K_{1}$ or it has a perfect matching consisting of pendant edges (Ravindra, 14]).

The roots of the independence polynomial of well-covered graphs are investigated by Brown et al in (4). It is shown that for any well-covered graph $G$ there is a well-covered graph $H$ with $\alpha(G)=\alpha(H)$ such that $G$ is an induced subgraph of $H$, where all the roots of $I(H ; x)$ are simple and real. As it is also mentioned in [4] , a root of independence polynomial of a graph (not necessarily well-covered) of smallest modulus is real, and there are well-covered graphs whose independence polynomials have non-real roots. Moreover, it is easy to check that the complete $n$-partite graph $G=K_{\alpha, \alpha, \ldots, \alpha}$ is well-covered, $\alpha(G)=\alpha$, and its independence polynomial, namely $I(G ; x)=n(1+x)^{\alpha}-(n-1)$, has only one real root, whenever $\alpha$ is odd, and exactly two real roots, for any even $\alpha$. In other words, the theorem of Newton (stating that if a polynomial with positive coefficients has only real roots, then its coefficients form a unimodal sequence) does not help in proving the following conjecture.

Conjecture 1.1 , A/ The independence polynomial of any well-covered graph is unimodal.


Figure 1: Two well-covered trees.
The claw-graph $K_{1,3}$ (see Figure 3) is a non-well-covered tree and $I\left(K_{1,3} ; x\right)=$ $1+4 x+3 x^{2}+x^{3}$ is unimodal, but has non-real roots. The trees $T_{1}, T_{2}$ in Figure 1 are well-covered, and their independence polynomials are respectively

$$
I\left(T_{1} ; x\right)=(1+x)^{2} \cdot(1+2 x) \cdot\left(1+6 x+7 x^{2}\right)
$$

$$
\begin{aligned}
& =1+10 x+36 x^{2}+60 x^{3}+47 x^{4}+14 x^{5} \\
I\left(T_{2} ; x\right) & =\left(1+6 x+10 x^{2}+5 x^{3}\right)^{2}-x^{2}(1+x)^{3}(1+2 x) \\
& =1+12 x+55 x^{2}+125 x^{3}+151 x^{4}+93 x^{5}+23 x^{6}
\end{aligned}
$$

which are both unimodal, while only for the first is true that all its roots are real. Hence, Newton's theorem is not useful in verifying the following conjecture, even for the particular case of well-covered trees.

Conjecture 1.2 [1] Independence polynomials of trees are unimodal.
A graph is called claw-free if it has no induced subgraph isomorphic to $K_{1,3}$. There are non-claw-free graphs whose independence polynomials are unimodal, e.g., the $n$-star $K_{1, n}, n \geq 3$. The following result of Hamidoune will be used in the sequel.

Theorem 1.3 , The independence polynomial of a claw-free graph is unimodal.
As a simple application of this statement, one can easily see that independence polynomials of paths and cycles are unimodal. In (2], Arocha shows that

$$
I\left(P_{n} ; x\right)=F_{n+1}(x), \text { and } I\left(C_{n}, x\right)=F_{n-1}(x)+2 x F_{n-2}(x)
$$

where $F_{n}(x), n \geq 0$, are Fibonacci polynomials, i.e., the polynomials defined recursively by

$$
F_{0}(x)=1, F_{1}(x)=1, F_{n}(x)=F_{n-1}(x)+x F_{n-2}(x) .
$$

Based on this recurrence, one can deduce that

$$
F_{n}(x)=\binom{n}{0}+\binom{n-1}{1} x+\binom{n-2}{2} x^{2}+\ldots+\binom{\lceil n / 2\rceil}{\lfloor n / 2\rfloor} x^{\lfloor n / 2\rfloor}
$$

(for example, see Riordan, 15], where this polynomial is discussed as a special kind of rook polynomials). It is amusing that the unimodality of the polynomial $F_{n}(x)$, which may be not so trivial to establish directly, follows now immediately from Theorem 1.3, since any $P_{n}$ is claw-free. Let us notice that for $n \geq 5, P_{n}$ is not well-covered.


Figure 2: Two pairs of non-isomorphic graphs $G_{1}, G_{2}$ and $G_{3}, G_{4}$ satisfying $I\left(G_{1} ; x\right)=$ $I\left(G_{2} ; x\right)$ and $I\left(G_{3} ; x\right)=I\left(G_{4} ; x\right)$.

Clearly, any two isomorphic graphs have the same independence polynomial. The converse is not generally true. For instance, while $I\left(G_{1} ; x\right)=I\left(G_{2} ; x\right)=1+5 x+5 x^{2}$, the well-covered graphs $G_{1}$ and $G_{2}$ are non-isomorphic (see Figure 2).

In addition, the graphs $G_{3}, G_{4}$ in Figure 2, have identical independence polynomials $I\left(G_{3} ; x\right)=I\left(G_{4} ; x\right)=1+6 x+10 x^{2}+6 x^{3}+x^{4}$, while $G_{3}$ is a tree, and $G_{4}$ is not connected and has cycles.

However, if $I(G ; x)=1+n x, n \geq 1$, then $G$ is isomorphic to $K_{n}$. Figure 3 gives us a source of some more examples of such uniqueness. Namely, the figure presents all the graphs of size four with the stability number equal to three. A simple check shows that their independence polynomials are different:

$$
\begin{aligned}
I\left(K_{1,3} ; x\right) & =1+4 x+3 x^{2}+x^{3} \\
I\left(G_{1} ; x\right) & =1+4 x+5 x^{2}+2 x^{3} \\
I\left(G_{2} ; x\right) & =1+4 x+4 x^{2}+x^{3}
\end{aligned}
$$

In other words, if the independence polynomials of two graphs (one from Figure 3 and an arbitrary one) coincide, then these graphs are exactly the same up to isomorphism.


Figure 3: $\alpha\left(K_{1,3}\right)=\alpha\left(G_{1}\right)=\alpha\left(G_{2}\right)=3$.
Let us mention that the equality $I\left(G_{1} ; x\right)=I\left(G_{2} ; x\right)$ implies

$$
\left|V\left(G_{1}\right)\right|=s_{1}=\left|V\left(G_{2}\right)\right| \text { and }\left|E\left(G_{1}\right)\right|=\frac{s_{1}^{2}-s_{1}}{2}-s_{2}=\left|E\left(G_{2}\right)\right|
$$

Consequently, if $G_{1}, G_{2}$ are connected, $I\left(G_{1} ; x\right)=I\left(G_{2} ; x\right)$ and one of them is a tree, then the other must be a tree, as well.

In this paper we show that the independence polynomial of any well-covered spider is unimodal. In addition, we introduce some graph transformations respecting independence polynomials. They allow us to reduce several types of well-covered trees to claw-free graphs, and, consequently, to prove that their independence polynomials are unimodal.

## 2 Preliminary results

Let us notice that if the product of two polynomials is unimodal, this is not a guaranty for the unimodality of at least one of the factors. For instance, we have

$$
\begin{aligned}
& I\left(K_{100}+\amalg 3 K_{6} ; x\right) \cdot I\left(K_{100}+\amalg 3 K_{6} ; x\right)=\left(1+\mathbf{1 1 8} x+108 x^{2}+\mathbf{2 0 6} x^{3}\right)^{2} \\
& =1+236 x+14140 x^{2}+25900 x^{3}+\mathbf{6 0 2 8 0} x^{4}+44496 x^{5}+42436 x^{6} .
\end{aligned}
$$

The converse is also true: the product of two unimodal polynomials is not necessarily unimodal. As an example, we see that:

$$
\begin{aligned}
& I\left(K_{100}+\amalg 3 K_{7} ; x\right) \cdot I\left(K_{100}+\amalg 3 K_{7} ; x\right)=\left(1+121 x+147 x^{2}+\mathbf{3 4 3} x^{3}\right)^{2} \\
& =1+242 x+14935 x^{2}+36260 x^{3}+\mathbf{1 0 4 6 1 5} x^{4}+100842 x^{5}+\mathbf{1 1 7 6 4 9} x^{6} .
\end{aligned}
$$

However, if one of them is of degree one, we show that their product is still unimodal.

Lemma 2.1 If $R_{n}$ is a unimodal polynomial, then $R_{n} \cdot R_{1}$ is unimodal for any polynomial $R_{1}$.

Proof. Let $R_{n}(x)=a_{0}+a_{1} x+a_{2} x^{2}+\ldots+a_{n} x^{n}$ be a unimodal polynomial and $R_{1}(x)=b_{0}+b_{1} x$. Suppose that $a_{0} \leq a_{1} \leq \ldots \leq a_{k} \geq a_{k+1} \geq \ldots \geq a_{n}$ and $b_{0} \leq b_{1}$. Then, $R_{n}(x) \cdot R_{1}(x)=a_{0} b_{0}+\sum_{i=1}^{n}\left(a_{i} b_{0}+a_{i-1} b_{1}\right) \cdot x^{i}+a_{n} b_{1} \cdot x^{n+1}=\sum_{i=0}^{n+1} c_{i} \cdot x^{i}$ and we show that $R_{n} \cdot R_{1}$ is unimodal, with the mode $m$, where

$$
\begin{equation*}
c_{m}=\max \left\{c_{k}, c_{k+1}\right\}=\max \left\{a_{k} b_{0}+a_{k-1} b_{1}, a_{k+1} b_{0}+a_{k} b_{1}\right\} \tag{1}
\end{equation*}
$$

Firstly, $a_{0} b_{0} \leq a_{1} b_{0}+a_{0} b_{1}$ because $a_{0} b_{0} \leq \max \left\{a_{1} b_{0}, a_{0} b_{1}\right\}$ and $0 \leq \min \left\{a_{1} b_{0}, a_{0} b_{1}\right\}$. Secondly, $a_{i-1} \leq a_{i} \leq a_{i+1}$ are true for any $i \in\{1, \ldots, k-1\}$, and these assure that $a_{i} b_{0}+a_{i-1} b_{1} \leq a_{i+1} b_{0}+a_{i} b_{1}$. Further, $a_{i-1} \geq a_{i} \geq a_{i+1}$ are valid for any $i \in\{k+1, \ldots, n-1\}$, which imply that $a_{i} b_{0}+a_{i-1} b_{1} \geq a_{i+1} b_{0}+a_{i} b_{1}$. Finally, $a_{n} b_{0}+a_{n-1} b_{1} \geq a_{n} b_{1}$, since $a_{n-1} \geq a_{n}$.

Similarly, we can show that $R_{n} \cdot R_{1}$ is unimodal, whenever $b_{0}>b_{1}$.
The following proposition constitutes an useful tool in computing independence polynomials of graphs and also in finding recursive formulae for independence polynomials of various classes of graphs.

Proposition 2.2 [G], 鸟 Let $G=(V, E)$ be a graph, $w \in V, u v \in E$ and $U \subset V$ be such that $G[U]$ is a complete subgraph of $G$. Then the following equalities hold:
(i) $I(G ; x)=I(G-w ; x)+x \cdot I(G-N[w] ; x)$;
(ii) $I(G ; x)=I(G-U ; x)+x \cdot \sum_{v \in U} I(G-N[v] ; x)$;
(iii) $I(G ; x)=I(G-u v ; x)-x^{2} \cdot I(G-N(u) \cup N(v) ; x)$.

The edge-join of two disjoint graphs $G_{1}, G_{2}$ is the graph $G_{1} \ominus G_{2}$ obtained by adding an edge joining two vertices belonging to $G_{1}, G_{2}$, respectively. If the two vertices are $v_{i} \in V\left(G_{i}\right), i=1,2$, then by $\left(G_{1} ; v_{1}\right) \ominus\left(G_{2} ; v_{2}\right)$ we mean the graph $G_{1} \ominus G_{2}$.

Lemma 2.3 Let $G_{i}=\left(V_{i}, E_{i}\right), i=1,2$, be two well-covered graphs and $v_{i} \in V_{i}, i=$ 1,2 , be simplicial vertices in $G_{1}, G_{2}$, respectively, such that $N_{G_{i}}\left[v_{i}\right], i=1,2$, contains at least another simplicial vertex. Then the following assertions are true:
(i) $G=\left(G_{1} ; v_{1}\right) \ominus\left(G_{2} ; v_{2}\right)$ is well-covered and $\alpha(G)=\alpha\left(G_{1}\right)+\alpha\left(G_{2}\right)$;
(ii) $I(G ; x)=I\left(G_{1} ; x\right) \cdot I\left(G_{2} ; x\right)-x^{2} \cdot I\left(G_{1}-N_{G_{1}}\left[v_{1}\right] ; x\right) \cdot I\left(G_{2}-N_{G_{2}}\left[v_{2}\right] ; x\right)$.

Proof. (i) Let $S_{1}, S_{2}$ be maximum stable sets in $G_{1}, G_{2}$, respectively. Since $G_{1}, G_{2}$ are well-covered, we may assume that $v_{i} \notin S_{i}, i=1,2$. Hence, $S_{1} \cup S_{2}$ is stable in $G$ and any maximum stable set $A$ of $G$ has $\left|A \cap V_{1}\right| \leq\left|S_{1}\right|,\left|A \cap V_{2}\right| \leq\left|S_{2}\right|$, and consequently we obtain:

$$
\left|S_{1}\right|+\left|S_{2}\right|=\left|S_{1} \cup S_{2}\right| \leq|A|=\left|A \cap V_{1}\right|+\left|A \cap V_{2}\right| \leq\left|S_{1}\right|+\left|S_{2}\right|
$$

i.e., $\alpha\left(G_{1}\right)+\alpha\left(G_{2}\right)=\alpha(G)$.

Let $B$ be a stable set in $G$ and $B_{i}=B \cap V_{i}, i=1,2$. Clearly, at most one of $v_{1}, v_{2}$ may belong to $B$. Since $G_{1}, G_{2}$ are well-covered, there exist $S_{1}, S_{2}$ maximum stable sets in $G_{1}, G_{2}$, respectively, such that $B_{1} \subseteq S_{1}, B_{2} \subseteq S_{2}$.

Case 1. $v_{1} \in B$ (similarly, if $v_{2} \in B$ ), i.e., $v_{1} \in B_{1}$. If $v_{2} \notin S_{2}$, then $S_{1} \cup S_{2}$ is a maximum stable set in $G$ such that $B \subset S_{1} \cup S_{2}$. Otherwise, let $w$ be the other simplicial vertex belonging to $N_{G_{2}}\left[v_{2}\right]$. Then $S_{3}=S_{2} \cup\{w\}-\left\{v_{2}\right\}$ is a maximum stable set in $G_{2}$, that includes $B_{2}$, because $B_{2} \subseteq S_{2}-\left\{v_{2}\right\}$. Hence, $S_{1} \cup S_{3}$ is a maximum stable set in $G$ such that $B \subset S_{1} \cup S_{3}$.

Case 2. $v_{1}, v_{2} \notin B$. If $v_{1}, v_{2} \in S_{1} \cup S_{2}$, then as above, $S_{1} \cup\left(S_{2} \cup\{w\}-\left\{v_{2}\right\}\right)$ is a maximum stable set in $G$ that includes $B$. Otherwise, $S_{1} \cup S_{2}$ is a maximum stable set in $G$ such that $B \subset S_{1} \cup S_{2}$.

Consequently, $G=\left(G_{1} ; v_{1}\right) \ominus\left(G_{2} ; v_{2}\right)$ is well-covered.
(ii) Using Proposition 2.2 (iii), we obtain that

$$
\begin{aligned}
I(G ; x) & =I\left(G-v_{1} v_{2} ; x\right)-x^{2} \cdot I\left(G-N_{G}\left(v_{1}\right) \cup N_{G}\left(v_{2}\right) ; x\right) \\
& =I\left(G_{1} ; x\right) \cdot I\left(G_{2} ; x\right)-x^{2} \cdot I\left(G_{1}-N_{G_{1}}\left(v_{1}\right) ; x\right) \cdot I\left(G_{2}-N_{G_{2}}\left(v_{2}\right) ; x\right),
\end{aligned}
$$

which completes the proof.
By $\triangle_{n}$ we denote the graph $\ominus n K_{3}$ defined as $\triangle_{n}=K_{3} \ominus(n-1) K_{3}, n \geq 1$, (see Figure (4). $\triangle_{0}$ denotes the empty graph.


Figure 4: The graph $\triangle_{n}=K_{3} \ominus(n-1) K_{3}$.

Proposition 2.4 The following assertions are true:
(i) for any $n \geq 1$, the graphs $\triangle_{n}, K_{2} \ominus \triangle_{n}$ are well-covered;
(ii) $I\left(\triangle_{n} ; x\right)$ is unimodal for any $n \geq 1$, and

$$
I\left(\triangle_{n} ; x\right)=(1+3 x) \cdot I\left(\triangle_{n-1} ; x\right)-x^{2} \cdot I\left(\triangle_{n-2} ; x\right), n \geq 2,
$$

where $I\left(\triangle_{0} ; x\right)=1, I\left(\triangle_{1} ; x\right)=1+3 x$;
(iii) $I\left(K_{2} \ominus \triangle_{n} ; x\right)$ is unimodal for any $n \geq 1$, and

$$
I\left(K_{2} \ominus \triangle_{n} ; x\right)=(1+2 x) \cdot I\left(\triangle_{n} ; x\right)-x^{2} \cdot I\left(\triangle_{n-1} ; x\right) .
$$

Proof. (i) We show, by induction on $n$, that $\triangle_{n}$ is well-covered. Clearly, $\triangle_{1}=K_{3}$ is well-covered. For $n \geq 2$ we have $\triangle_{n}=\left(\triangle_{1} ; v_{2}\right) \ominus\left(\triangle_{n-1} ; v_{4}\right)$, (see Figure (1). Hence, according to Lemma 2.3, $\triangle_{n}$ is well-covered, because $v_{2}, v_{3}$ and $v_{4}, v_{6}$ are simplicial vertices in $\triangle_{1}, \triangle_{n-1}$, respectively.

Therefore, $\triangle_{n}$ is well-covered for any $n \geq 1$.
(ii) If $e=v_{2} v_{4}$ and $n \geq 2$, then according to Proposition 2.2(iii), we obtain that

$$
\begin{aligned}
I\left(\triangle_{n} ; x\right) & =I\left(\triangle_{n}-e ; x\right)-x^{2} \cdot I\left(\triangle_{n}-N\left(v_{2}\right) \cup N\left(v_{4}\right) ; x\right) \\
& =I\left(K_{3} ; x\right) \cdot I\left(\triangle_{n-1} ; x\right)-x^{2} \cdot I\left(\triangle_{n-2} ; x\right) \\
& =(1+3 x) \cdot I\left(\triangle_{n-1} ; x\right)-x^{2} \cdot I\left(\triangle_{n-2} ; x\right) .
\end{aligned}
$$

In addition, $I\left(\triangle_{n} ; x\right)$ is unimodal by Theorem 1.3 , because $\triangle_{n}$ is claw-free.
(iii) Let us notice that both $K_{2}$ and $\triangle_{n}$ are well-covered. The graph $K_{2} \ominus \triangle_{n}=$ $\left(K_{2} ; u_{2}\right) \ominus\left(\triangle_{n} ; v_{1}\right)$ is well-covered according to Lemma 2.3, and $I\left(K_{2} \ominus \triangle_{n} ; x\right)$ is unimodal for any $n \geq 1$, by Theorem 1.3, since $K_{2} \ominus \triangle_{n}$ is claw-free (see Figure 5).


Figure 5: The graph $K_{2} \ominus \triangle_{n}$.
In addition, applying Proposition 2.2(iii), we infer that

$$
\begin{aligned}
I\left(K_{2} \ominus \triangle_{n} ; x\right) & =I\left(K_{2} \ominus \triangle_{n}-u_{2} v_{1} ; x\right)-x^{2} \cdot I\left(K_{2} \ominus \triangle_{n}-N\left(u_{2}\right) \cup N\left(v_{1}\right) ; x\right) \\
& =I\left(K_{2} ; x\right) \cdot I\left(\triangle_{n} ; x\right)-x^{2} \cdot I\left(\triangle_{n-1} ; x\right) \\
& =(1+2 x) \cdot I\left(\triangle_{n} ; x\right)-x^{2} \cdot I\left(\triangle_{n-1} ; x\right)
\end{aligned}
$$

that completes the proof.
Lemma 2.5 Let $G_{i}=\left(V_{i}, E_{i}\right), v_{i} \in V_{i}, i=1,2$, and $P_{4}=(\{a, b, c, d\},\{a b, b c, c d\})$. Then the following assertions are true:
(i) $I\left(L_{1} ; x\right)=I\left(L_{2} ; x\right)$, where $L_{1}=\left(P_{4} ; b\right) \ominus\left(G_{1} ; v\right)$, while $L_{2}$ has $V\left(L_{2}\right)=V\left(L_{1}\right), E\left(L_{2}\right)=E\left(L_{1}\right) \cup\{a c\}-\{c d\}$. If $G_{1}$ is claw-free and $v$ is simplicial in $G_{1}$, then $I\left(L_{1} ; x\right)$ is unimodal.
(ii) $I(G ; x)=I(H ; x)$, where $G=\left(G_{3} ; c\right) \ominus\left(G_{2} ; v_{2}\right)$ and $G_{3}=\left(G_{1} ; v_{1}\right) \ominus\left(P_{4} ; b\right)$, while $H$ has $V(H)=V(G), E(H)=E(G) \cup\{a c\}-\{c d\}$.
If $G_{1}, G_{2}$ are claw-free and $v_{1}, v_{2}$ are simplicial in $G_{1}, G_{2}$, respectively, then $I(G ; x)$ is unimodal.

Proof. (i) The graphs $L_{1}=\left(P_{4} ; b\right) \ominus\left(G_{1} ; v\right)$ and $L_{2}=\left(K_{1} \amalg K_{3} ; b\right) \ominus\left(G_{1} ; v\right)$ are depicted in Figure 6 .
Clearly, $I\left(P_{4} ; x\right)=I\left(K_{3} \amalg K_{1} ; x\right)=1+4 x+3 x^{2}$. By Proposition 2.2(iii), we obtain:

$$
\begin{aligned}
I\left(L_{1} ; x\right) & =I\left(L_{1}-v b ; x\right)-x^{2} \cdot I\left(L_{1}-N(v) \cup N(b) ; x\right) \\
& =I\left(G_{1} ; x\right) \cdot I\left(P_{4} ; x\right)-x^{2} \cdot I\left(G_{1}-N_{G_{1}}[v] ; x\right) \cdot I(\{d\} ; x)
\end{aligned}
$$



Figure 6: The graphs $L_{1}=\left(P_{4} ; b\right) \ominus\left(G_{1} ; v\right)$ and $L_{2}=\left(K_{1} \amalg K_{3} ; b\right) \ominus\left(G_{1} ; v\right)$.

On the other hand, we get:

$$
\begin{aligned}
I\left(L_{2} ; x\right) & =I\left(L_{2}-v b ; x\right)-x^{2} \cdot I\left(L_{2}-N(v) \cup N(b) ; x\right) \\
& =I\left(G_{1} ; x\right) \cdot I\left(K_{3} \amalg K_{1} ; x\right)-x^{2} \cdot I\left(G_{1}-N_{G_{1}}[v] ; x\right) \cdot I(\{d\} ; x) .
\end{aligned}
$$

Consequently, the equality $I\left(L_{1} ; x\right)=I\left(L_{2} ; x\right)$ holds. If, in addition, $v$ is simplicial in $G_{1}$, and $G_{1}$ is claw-free, then $L_{2}$ is claw-free, too. Theorem 1.3 implies that $I\left(L_{2} ; x\right)$ is unimodal, and, hence, $I\left(L_{1} ; x\right)$ is unimodal, as well.
(ii) Figure 7 shows the graphs $G$ and $H$.


Figure 7: The graphs $G$ and $H$ from Lemma 2.5(ii).
According to Proposition 2.2 (iii), we obtain:

$$
\begin{aligned}
I(G ; x) & =I\left(G-v_{1} b ; x\right)-x^{2} \cdot I\left(G-N\left(v_{1}\right) \cup N(b) ; x\right) \\
& =I\left(G_{1} ; x\right) \cdot I\left(G-G_{1} ; x\right)-x^{2} \cdot I\left(G_{1}-N_{G_{1}}\left[v_{1}\right] ; x\right) \cdot I\left(G_{2} ; x\right) \cdot I(\{d\} ; x) .
\end{aligned}
$$

On the other hand, using again Proposition 2.2(iii), we get:

$$
\begin{aligned}
I(H ; x) & =I\left(H-v_{1} b ; x\right)-x^{2} \cdot I\left(H-N\left(v_{1}\right) \cup N(b) ; x\right) \\
& =I\left(G_{1} ; x\right) \cdot I\left(H-G_{1} ; x\right)-x^{2} \cdot I\left(G_{1}-N_{G_{1}}\left[v_{1}\right] ; x\right) \cdot I\left(G_{2} ; x\right) \cdot I(\{d\} ; x) .
\end{aligned}
$$

Finally, let us observe that the equality $I\left(G-G_{1} ; x\right)=I\left(H-G_{1} ; x\right)$ holds according to part (i).

Now, if $G_{1}, G_{2}$ are claw-free and $v_{1}, v_{2}$ are simplicial in $G_{1}, G_{2}$, respectively, then $H$ is claw-free, and by Theorem 1.3, its independence polynomial is unimodal. Consequently, $I(G ; x)$ is also unimodal.

## 3 Independence polynomials of well-covered spiders

The well-covered spider $S_{n}, n \geq 2$ has one vertex of degree $n+1, n$ vertices of degree 2 , and $n+1$ vertices of degree 1 (see Figure 8).


Figure 8: Well-covered spiders.

Theorem 3.1 The independence polynomial of any well-covered spider is unimodal, moreover,

$$
I\left(S_{n} ; x\right)=(1+x) \cdot\left\{1+\sum_{k=1}^{n}\left[\binom{n}{k} \cdot 2^{k}+\binom{n-1}{k-1}\right] \cdot x^{k}\right\}, n \geq 2
$$

and its mode is unique and equals $1+(n-1) \bmod 3+2(\lceil n / 3\rceil-1)$.
Proof. Well-covered spiders comprise $K_{1}, K_{2}, P_{4}$ and $S_{n}, n \geq 2$. Clearly, the independence polynomials of $K_{1}, K_{2}, P_{4}$ are unimodal.

Using Proposition 2.2(i), we obtain the following formula for $S_{n}$ :

$$
\begin{aligned}
I\left(S_{n} ; x\right) & =I\left(S_{n}-b_{0} ; x\right)+x \cdot I\left(S_{n}-N\left[b_{0}\right] ; x\right) \\
& =(1+x) \cdot(1+2 x)^{n}+x \cdot(1+x)^{n}=(1+x) \cdot R_{n}(x)
\end{aligned}
$$

where $R_{n}(x)=(1+2 x)^{n}+x \cdot(1+x)^{n-1}$. By Lemma 2.1, to prove that $I\left(S_{n} ; x\right)$ is unimodal, it is sufficient to show that $R_{n}(x)$ is unimodal. It is easy to see that

$$
\begin{aligned}
R_{n}(x) & =1+\sum_{k=1}^{n}\left[\binom{n}{k} \cdot 2^{k}+\binom{n-1}{k-1}\right] \cdot x^{k}=1+\sum_{k=1}^{n} A(k) \cdot x^{k} \\
A(k) & =\binom{n}{k} \cdot 2^{k}+\binom{n-1}{k-1}, 1 \leq k \leq n, A(0)=1
\end{aligned}
$$

To start with, we show that $R_{n}$ is unimodal with the mode $k=n-1-\lfloor(n-2) / 3\rfloor$. Taking into account the proof of Lemma 2.1, namely, the equality 1, the mode of the polynomial

$$
I\left(S_{n} ; x\right)=(1+x) \cdot R_{n}(x)=1+\sum_{k=1}^{n} c_{k} \cdot x^{k}
$$

is the index $m$, with $c_{m}=\max \left\{c_{k}, c_{k+1}\right\}=\max \{A(k)+A(k-1), A(k+1)+A(k)\}$. In other words, we will give evidence for

$$
m= \begin{cases}k, & \text { if } A(k-1)>A(k+1)  \tag{2}\\ k+1, & \text { if } A(k-1) \leq A(k+1)\end{cases}
$$

- Claim 1. If $n=3 m+1$, then $R_{n}$ is unimodal with the mode $2 m+1, I\left(S_{n} ; x\right)$ is also unimodal, and its unique mode equals $2 m+1$.

We show that

$$
\begin{aligned}
& A(2 m+i+1) \geq A(2 m+i+2), 0 \leq i \leq m-1, \text { and } \\
& A(2 m+1-j) \geq A(2 m-j), 0 \leq j \leq 2 m
\end{aligned}
$$

We have successively (for $2 m+i+1=h$ ):

$$
\begin{gathered}
A(h)-A(h+1)=\left[\binom{3 m+1}{h} \cdot 2^{h}+\binom{3 m}{h-1}\right]-\left[\binom{3 m+1}{h+1} \cdot 2^{h+1}+\binom{3 m}{h}\right] \\
=\frac{(3 m+1)!\cdot 2^{h} \cdot[(h+1)-2 \cdot(m-i)]}{(h+1)!\cdot(m-i)!}+\frac{(3 m)!\cdot[h-(m-i)]}{h!\cdot(m-i)!} \\
=\frac{(3 m+1)!\cdot(3 i+2) \cdot 2^{h}}{(h+1)!\cdot(m-i)!}+\frac{(3 m)!\cdot(m+2 i+1)}{h!\cdot(m-i)!} \geq 0
\end{gathered}
$$

Further, we get (for $2 m-j=h$ ):

$$
\begin{aligned}
& A(h+1)-A(h)=\left[\binom{3 m+1}{h+1} \cdot 2^{h+1}+\binom{3 m}{h}\right]-\left[\binom{3 m+1}{h} \cdot 2^{h}+\binom{3 m}{h-1}\right] \\
& =\frac{(3 m+1)!\cdot 2^{h} \cdot[2 \cdot(m+j+1)-(h+1)]}{(h+1)!\cdot(m+j+1)!}+\frac{(3 m)!\cdot[(m+j+1)-h]}{h!\cdot(m+j+1)!} \\
& =\frac{(3 m)!}{(h+1)!\cdot(m+j+1)!}\left[(3 m+1) \cdot(3 j+1) \cdot 2^{h}-(m-2 j-1) \cdot(h+1)\right] \geq 0
\end{aligned}
$$

because $3 m+1>m \geq m-2 j-1$ and $2^{h} \geq h+1$.
To find the location of the mode of $I\left(S_{n} ; x\right)$, we obtain

$$
\begin{gathered}
A(2 m)-A(2 m+2)= \\
=\binom{3 m+1}{2 m} \cdot 2^{2 m}+\binom{3 m}{2 m-1}-\binom{3 m+1}{2 m+2} \cdot 2^{2 m+2}+\binom{3 m}{2 m+1} \\
=\frac{(3 m)!\cdot 2^{2 m}}{(2 m)!\cdot(m-1)!} \cdot\left[\frac{3 m+1}{m \cdot(m+1)} \cdot\left(\frac{1}{2 m}-\frac{1}{2 m+1}\right]+\right. \\
+\frac{(3 m)!}{(2 m-1)!\cdot(m-1)!} \cdot\left[\frac{1}{m \cdot(m+1)}-\frac{1}{2 m \cdot(2 m+1)}\right]>0
\end{gathered}
$$

Consequently, in accordance with equality (2), the mode of $I\left(S_{n} ; x\right)$ equals $2 m+1$. Since $A(2 m)-A(2 m+2)>0$, the mode is unique.

- Claim 2. If $n=3 m$, then $R_{n}$ is unimodal with the mode $2 m, I\left(S_{n} ; x\right)$ is also unimodal, and its unique mode equals $2 m+1$.

We show that

$$
\begin{aligned}
& A(2 m+i) \geq A(2 m+i+1), 0 \leq i \leq m-1, \text { and } \\
& A(2 m-j) \geq A(2 m-j-1), 0 \leq j \leq 2 m-1
\end{aligned}
$$

We have successively (for $2 m+i=h$ ):

$$
\begin{gathered}
A(h)-A(h+1)=\left[\binom{3 m}{h} 2^{h}+\binom{3 m-1}{h-1}\right]-\left[\binom{3 m}{h+1} 2^{h+1}+\binom{3 m-1}{h}\right] \\
=\frac{(3 m)!\cdot 2^{h} \cdot[(h+1)-2 \cdot(m-i)]}{(h+1)!\cdot(m-i)!}+\frac{(3 m-1)!\cdot[h-(m-i)]}{h!\cdot(m-i)!} \\
=\frac{(3 m)!\cdot(3 i+1) \cdot 2^{h}}{(h+1)!\cdot(m-i)!} \cdot+\frac{(3 m-1)!\cdot(m+2 i)}{h!\cdot(m-i)!} \geq 0
\end{gathered}
$$

Further, we get (for $2 m-j=h$ ):

$$
\begin{gathered}
A(h)-A(h-1)=\left[\binom{3 m}{h} \cdot 2^{h}+\binom{3 m-1}{h-1}\right]-\left[\binom{3 m}{h-1} \cdot 2^{h-1}+\binom{3 m-1}{h-2}\right] \\
=\frac{(3 m)!\cdot 2^{h-1} \cdot[2(m+j+1)-h]}{h!\cdot(m+j+1)!}+\frac{(3 m-1)!\cdot[(m+j+1)-(h-1)]}{(h-1)!\cdot(m+j+1)!} \\
=\frac{(3 m-1)!}{h!\cdot(m+j+1)!} \cdot\left[3 m \cdot \frac{3 j+2}{2} \cdot 2^{h}-(m-2 j-2) \cdot h\right] \geq 0
\end{gathered}
$$

since $3 m>m \geq m-2 j-2$ and $2^{h} \geq h$.
To determine the mode of $I\left(S_{n} ; x\right)$, we obtain

$$
\begin{gathered}
A(2 m-1)-A(2 m+1)= \\
=\binom{3 m}{2 m-1} \cdot 2^{2 m-1}+\binom{3 m-1}{2 m-2}-\binom{3 m}{2 m+1} \cdot 2^{2 m+1}+\binom{3 m-1}{2 m} \\
=\frac{3}{2} \cdot \frac{(3 m-1)!}{(2 m+1)!\cdot(m+1)!} \cdot\left[(m-1) \cdot(2 m+1)-m \cdot 2^{2 m}\right]<0
\end{gathered}
$$

Consequently, in accordance with equality (2), the mode of $I\left(S_{n} ; x\right)$ equals $2 m+1$. Since $A(2 m-1)-A(2 m+1)<0$, the mode is unique.

- Claim 3. If $n=3 m-1$, then $R_{n}$ is unimodal with the mode $2 m-1, I\left(S_{n} ; x\right)$ is also unimodal, and its unique mode equals $2 m$.

We show that

$$
\begin{aligned}
& A(2 m+i-1) \geq A(2 m+i), 0 \leq i \leq m-1, \text { and } \\
& A(2 m-j-1) \geq A(2 m-j-2), 0 \leq j \leq 2 m-2
\end{aligned}
$$

We have successively (for $2 m+i=h$ ):

$$
\begin{gathered}
A(h-1)-A(h)=\left[\binom{3 m-1}{h-1} \cdot 2^{h-1}+\binom{3 m-2}{h-2}\right]-\left[\binom{3 m-1}{h} \cdot 2^{h}+\binom{3 m-2}{h-1}\right] \\
=\frac{(3 m-1)!\cdot 2^{h-1} \cdot[h-2 \cdot(m-i)]}{h!\cdot(m-i)!}+\frac{(3 m-2)!\cdot[(h-1)-(m-i)]}{(h-1)!\cdot(m-i)!} \\
=\frac{(3 m-1)!\cdot 3 i \cdot 2^{h-1}}{h!\cdot(m-i)!}+\frac{(3 m-2)!\cdot(m+2 i-1)}{(h-1)!\cdot(m-i)!} \geq 0
\end{gathered}
$$

Further, we get (for $2 m-j-1=h$ ):

$$
\begin{aligned}
A(h)- & A(h-1)=\left[\binom{3 m-1}{h} \cdot 2^{h}+\binom{3 m-2}{h-1}\right]-\left[\binom{3 m-1}{h-1} \cdot 2^{h-1}+\binom{3 m-2}{h-2}\right] \\
& =\frac{(3 m-1)!\cdot 2^{h-1} \cdot[2 \cdot(3 m-h)-h]}{h!\cdot(3 m-h)!}+\frac{(3 m-2)!\cdot[(3 m-h)-(h-1)]}{(h-1)!\cdot(3 m-h)!} \\
& =\frac{(3 m-2)!}{h!\cdot(m+j+1)!} \cdot\left[(3 m-1) \cdot \frac{3 j+3}{2} \cdot 2^{h}-h \cdot(m-2 j-3)\right] \geq 0
\end{aligned}
$$

because $3 m-1>m \geq m-2 j-1$ and $2^{h} \geq h$.
Finally, we obtain that

$$
\begin{gathered}
A(2 m-2)-A(2 m)= \\
=\binom{3 m}{2 m-2} \cdot 2^{2 m-2}+\binom{3 m-1}{2 m-3}-\binom{3 m-1}{2 m} \cdot 2^{2 m}+\binom{3 m-2}{2 m-1} \\
=\frac{(3 m-1)!}{(2 m-1)!\cdot(m+1)!} \cdot\left[m-2-3 \cdot 2^{2 m-2}\right]<0
\end{gathered}
$$

Consequently, in accordance with equality (2), the mode of $I\left(S_{n} ; x\right)$ equals $2 m$. Since $A(2 m-1)-A(2 m+1)<0$, the mode is unique.

## 4 Transformations of some well-covered trees

If both $v_{1}$ and $v_{2}$ are vertices of degree at least two in $G_{1}, G_{2}$, respectively, then $\left(G_{1} ; v_{1}\right) \ominus\left(G_{2}, v_{2}\right)$ is an internal edge-join of $G_{1}, G_{2}$. Notice that the edge-join of two trees is a tree, and also that two trees can be internal edge-joined provided each one is of order at least three. An alternative characterization of well-covered trees is the following:

Theorem 4.1 11] A tree $T$ is well-covered if and only if either $T$ is a well-covered spider, or $T$ is the internal edge-join of a number of well-covered spiders.

As examples, $W_{n}, n \geq 4$, and $G_{m, n}, m \geq 2, n \geq 3$, (see Figures 9 and 12 ) are internal edge-join of a number of well-covered spiders, and consequently, they are wellcovered trees. The aim of this section is to show that the independence polynomials of $W_{n}$ and of $G_{m, n}$ are unimodal. The idea is to construct, for these trees, some claw-free graphs having the same independence polynomial, and then to use Theorem 1.3. We leave open the question whether the procedure we use is helpful to define a claw-free graph with the same independence polynomial as a general well-covered tree.

A centipede is a tree denoted by $W_{n}=(A, B, E), n \geq 1$, (see Figure 9), where $A \cup B$ is its vertex set, $A=\left\{a_{1}, \ldots, a_{n}\right\}, B=\left\{b_{1}, \ldots, b_{n}\right\}, A \cap B=\emptyset$, and the edge set $E=\left\{a_{i} b_{i}: 1 \leq i \leq n\right\} \cup\left\{b_{i} b_{i+1}: 1 \leq i \leq n-1\right\}$.


Figure 9: The centipede $W_{n}$.
The following result was proved in 12], but for the sake of self-consistency of this paper and to illustrate the idea of a hidden correspondence between well-covered trees and claw-free graphs, we give here its proof.

Theorem 4.2 1G For any $n \geq 1$ the following assertions hold:
(i) $I\left(W_{2 n} ; x\right)=(1+x)^{n} Q_{n}(x)=I\left(\triangle_{n} \amalg n K_{1} ; x\right)$, where $Q_{n}(x)=I\left(\triangle_{n} ; x\right)$;
$I\left(W_{2 n+1} ; x\right)=(1+x)^{n} Q_{n+1}(x)=I\left(\left(K_{2} \ominus \triangle_{n}\right) \amalg n K_{1} ; x\right)$,
where $Q_{n+1}(x)=I\left(K_{2} \ominus \triangle_{n} ; x\right)$;
(ii) $I\left(W_{n} ; x\right)$ is unimodal and

$$
I\left(W_{n} ; x\right)=(1+x) \cdot\left(I\left(W_{n-1} ; x\right)+x \cdot I\left(W_{n-2} ; x\right)\right), n \geq 2
$$

where $I\left(W_{0} ; x\right)=1, I\left(W_{1} ; x\right)=1+2 x$.
Proof. (i) Evidently, the polynomials $I\left(W_{n} ; x\right), 1 \leq n \leq 3$, are unimodal, since

$$
I\left(W_{1} ; x\right)=1+\mathbf{2} x, I\left(W_{2} ; x\right)=1+\mathbf{4} x+3 x^{2}, I\left(W_{3} ; x\right)=1+6 x+\mathbf{1 0} x^{2}+5 x^{3} .
$$

Applying $\lfloor n / 2\rfloor$ times Lemma 2.5(ii), we obtain that for $n=2 m \geq 4$,

$$
I\left(W_{n} ; x\right)=I\left(\triangle_{m} \amalg m K_{1} ; x\right)=I\left(\triangle_{m} ; x\right) \cdot(1+x)^{m},
$$

while for $n=2 m+1 \geq 5$,

$$
I\left(W_{n} ; x\right)=I\left(K_{2} \ominus \triangle_{m} \amalg m K_{1} ; x\right)=I\left(K_{2} \ominus \triangle_{m} ; x\right) \cdot(1+x)^{m} .
$$

(iii) According to Proposition 2.4 and Lemma 2.1, it follows that $I\left(W_{n} ; x\right)$ is unimodal. Further, taking $U=\left\{a_{n}, b_{n}\right\}$ and applying Proposition 2.2(ii), we obtain:

$$
\begin{aligned}
I\left(W_{n} ; x\right) & =I\left(W_{n}-U ; x\right)+x \cdot\left(I\left(W_{n}-N\left[a_{n}\right] ; x\right)+I\left(W_{n}-N\left[b_{n}\right] ; x\right)\right) \\
& \left.=I\left(W_{n-1} ; x\right)+x \cdot I\left(W_{n-1} ; x\right)+x \cdot(1+x) \cdot I\left(W_{n-2} ; x\right)\right) \\
& =(1+x) \cdot\left(I\left(W_{n-1} ; x\right)+x \cdot I\left(W_{n-2} ; x\right)\right)
\end{aligned}
$$

which completes the proof.
It is worth mentioning that the problem of finding the mode of the centipede is still unsolved.

Conjecture 4.3 1G The mode of $I\left(W_{n} ; x\right)$ is $k=n-f(n)$ and $f(n)$ is given by

$$
\begin{aligned}
& f(n)=1+\lfloor n / 5\rfloor, 2 \leq n \leq 6 \\
& f(n)=f(2+(n-2) \bmod 5)+2\lfloor(n-2) / 5\rfloor, n \geq 7
\end{aligned}
$$

Proposition 4.4 The following assertions are true:
(i) $I\left(G_{2,4} ; x\right)$ is unimodal, moreover $I\left(G_{2,4} ; x\right)=I\left(3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right) ; x\right)$ (see Figure 18);
(ii) $I(G ; x)=I(\overline{L ; x})$, where $G=\left(G_{2,4} ; v_{4}\right) \ominus(H ; w)$ and $L=3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right) \ominus H$ (see Figure 10); if $w$ is simplicial in $H$, and $H$ is claw-free, then $I(G ; x)$ is unimodal;
(iii) $I(G ; x)=I(L ; x)$, where $G=\left(H_{1} ; w_{1}\right) \ominus\left(v ; G_{2,4} ; u\right) \ominus\left(H_{2} ; w_{2}\right)$ and $L=3 K_{1} \amalg K_{2} \amalg\left(\left(H_{1} ; w_{1}\right) \ominus\left(v ; K_{3}\right) \ominus\left(K_{4} ; u\right) \ominus\left(w_{2} ; H_{2}\right)\right.$ (see Figure 11); if $w_{1}, w_{2}$ are simplicial in $H_{1}, H_{2}$, respectively, and $H_{1}, H_{2}$ are claw-free, then $I(G ; x)$ is unimodal.

Proof. (i) Using Proposition 2.2 (iii) and the fact that $I\left(W_{4} ; x\right)=1+8 x+21 x^{2}+$ $22 x^{3}+8 x^{4}=(1+x)^{2}(1+2 x)(1+4 x)$, we get that

$$
\begin{aligned}
I\left(G_{2,4} ; x\right) & =I\left(G_{2,4}-b_{2} v_{2} ; x\right)-x^{2} \cdot I\left(G_{2,4}-N\left(b_{2}\right) \cup N\left(v_{2}\right) ; x\right) \\
& =1+12 x+55 x^{2}+125 x^{3}+\mathbf{1 5 0} x^{4}+91 x^{5}+22 x^{6}
\end{aligned}
$$

which is clearly unimodal. On the other hand, it is easy to check that

$$
I\left(G_{2,4} ; x\right)=(1+x)^{3}(1+2 x)\left(1+7 x+11 x^{2}\right)=I\left(3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right) ; x\right)
$$

(ii) According to Proposition 2.2(iii), we infer that

$$
\begin{aligned}
I(G ; x)= & I(G-v w ; x)-x^{2} \cdot I(G-N(v) \cup N(w) ; x) \\
= & I\left(G_{2,4} ; x\right) \cdot I(H ; x)-x^{2} \cdot(1+x) \cdot I\left(H-N_{H}[w] ; x\right) \cdot I\left(W_{4} ; x\right) \\
= & I\left(G_{2,4} ; x\right) \cdot I(H ; x)- \\
& -x^{2} \cdot(1+x)^{3} \cdot(1+2 x) \cdot(1+4 x) \cdot I\left(H-N_{H}[w] ; x\right) .
\end{aligned}
$$

Figure 10 shows the graphs $G$ and $L$.


Figure 10: The graphs $G$ and $L$ in Proposition 4.4(ii).
Let us denote $Q_{1}=3 K_{1} \amalg K_{2} \amalg K_{4}, Q_{2}=3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right), L=\left(Q_{2} ; v\right) \ominus(H ; w)$ and $e=v w$. Then, Proposition 2.2(iii) implies that

$$
\begin{aligned}
I(L ; x)= & I(L-v w ; x)-x^{2} \cdot I(L-N(v) \cup N(w) ; x) \\
= & I\left(Q_{2} ; x\right) \cdot I(H ; x)-x^{2} \cdot I\left(Q_{1} ; x\right) \cdot I\left(H-N_{H}[w] ; x\right) \\
= & I\left(G_{2,4} ; x\right) \cdot I(H ; x)- \\
& -x^{2} \cdot(1+x)^{3} \cdot(1+2 x) \cdot(1+4 x) \cdot I\left(H-N_{H}[w] ; x\right) .
\end{aligned}
$$

Consequently, $I(G ; x)=I(L ; x)$ holds.
In addition, if $w$ is simplicial in $H$, and $H$ is claw-free, then $L$ is claw-free and, by Theorem 1.3, $I(L ; x)$ is unimodal. Hence, $I(G ; x)$ is unimodal, as well.
(iii) Let $e=u w_{2} \in E(G)$. Then, according to Proposition 2.2(iii), we get that

$$
\begin{aligned}
I(G ; x)= & I\left(G-u w_{2} ; x\right)-x^{2} \cdot I\left(G-N(u) \cup N\left(w_{2}\right) ; x\right) \\
= & I\left(H_{2} ; x\right) \cdot I\left(\left(G_{2,4} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)- \\
& -x^{2} \cdot(1+x)^{2} \cdot(1+2 x) \cdot I\left(H_{2}-N_{H_{2}}\left[w_{2}\right] ; x\right) \cdot I\left(\left(P_{4} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right) .
\end{aligned}
$$

Now, Lemma 2.5(i) implies that

$$
\begin{gathered}
I(G ; x)=I\left(H_{2} ; x\right) \cdot I\left(\left(G_{2,4} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)- \\
-x^{2} \cdot(1+x)^{2} \cdot(1+2 x) \cdot I\left(H_{2}-N_{H_{2}}\left[w_{2}\right] ; x\right) \cdot I\left(\left(K_{1} \sqcup K_{3} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right) .
\end{gathered}
$$

Figure 11 shows the graphs $G$ and $L$.
Let $e=u w_{2} \in E(L)$. Again by Proposition 2.2(iii), we infer that

$$
\begin{aligned}
I(L ; x)= & I\left(L-u w_{2} ; x\right)-x^{2} \cdot I\left(L-N(u) \cup N\left(w_{2}\right) ; x\right) \\
& =I\left(H_{2} ; x\right) \cdot I\left(\left(3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right) ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)- \\
& -x^{2} \cdot(1+x)^{3} \cdot(1+2 x) \cdot I\left(H_{2}-N_{H_{2}}\left[w_{2}\right] ; x\right) \cdot I\left(\left(K_{3} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right) .
\end{aligned}
$$



Figure 11: The graphs $G$ and $L$ in Proposition 4.4(iii).

Further, Proposition 4.4(ii) helps us to deduce that

$$
I\left(H_{2} ; x\right) \cdot I\left(\left(G_{2,4} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)=I\left(H_{2} ; x\right) \cdot I\left(\left(3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right) ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)
$$

Eventually, since

$$
I\left(\left(K_{1} \sqcup K_{3} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)=(1+x) \cdot I\left(\left(K_{3} ; v\right) \ominus\left(H_{1} ; w_{1}\right) ; x\right)
$$

we obtain $I(G ; x)=I(L ; x)$.
If $w_{1}, w_{2}$ are simplicial in $H_{1}, H_{2}$, respectively, and $H_{1}, H_{2}$ are claw-free, then $L$ is claw-free and, therefore, $I(L ; x)$ is unimodal, by Theorem 1.3. Hence, $I(G ; x)$ is unimodal, too.


Figure 12: The graph $G_{m, n}, m \geq 2, n \geq 3$.

Theorem 4.5 The independence polynomial of $G_{m, n}=\left(W_{m} ; b_{2}\right) \ominus\left(W_{n} ; v_{2}\right)$ is
unimodal, for any $m \geq 2, n \geq 2$.
Proof. Case 1. Suppose that $m=2,3, n=3,4$. The polynomial $I\left(G_{2,3} ; x\right)$ is unimodal, because

$$
\begin{aligned}
I\left(G_{2,3} ; x\right) & =(1+x)^{2} \cdot(1+2 x) \cdot\left(1+6 x+7 x^{2}\right) \\
& =1+10 x+36 x^{2}+\mathbf{6 0} x^{3}+47 x^{4}+14 x^{5}
\end{aligned}
$$

According to Proposition $4.4(\mathrm{i})$, the independence polynomial of $G_{2,4}$ is unimodal and $I\left(G_{2,4} ; x\right)=I\left(3 K_{1} \amalg K_{2} \amalg\left(K_{4} \ominus K_{3}\right) ; x\right)$.

Further, $I\left(G_{3,3} ; x\right)$ is unimodal, since

$$
\begin{aligned}
I\left(G_{3,3} ; x\right) & =I\left(G_{3,3}-v_{2} b_{2} ; x\right)-x^{2} \cdot I\left(G_{3,3}-N\left(v_{2}\right) \cup N\left(b_{2}\right) ; x\right) \\
& =I\left(W_{3} ; x\right) \cdot I\left(W_{3} ; x\right)-x^{2} \cdot(1+x)^{4} \\
& =1+12 x+55 x^{2}+126 x^{3}+\mathbf{1 5 4} x^{4}+96 x^{5}+24 x^{6} .
\end{aligned}
$$

Finally, $I\left(G_{3,4} ; x\right)$ is unimodal, because

$$
\begin{aligned}
I\left(G_{3,4} ; x\right) & =I\left(G_{3,4}-b_{1} b_{2} ; x\right)-x^{2} \cdot I\left(G_{3,4}-N\left(b_{1}\right) \cup N\left(b_{2}\right) ; x\right) \\
& =(1+2 x) \cdot I\left(G_{2,4} ; x\right)-x^{2} \cdot(1+x)^{2} \cdot(1+2 x) \cdot I\left(P_{4} ; x\right) \\
& =1+14 x+78 x^{2}+227 x^{3}+\mathbf{3 7 6} x^{4}+357 x^{5}+181 x^{6}+38 x^{7} .
\end{aligned}
$$

Case 2. Assume that $m=2, n \geq 5$. According to Proposition 4.4 (ii), we infer that $I\left(G_{2, n} ; x\right)=I\left(L_{1} ; x\right)$, where $L_{1}=Q \ominus W_{n-4}$ and $Q=3 K_{1} \amalg K_{2} \amalg K_{4} \ominus K_{3}$ (see Figure 13). Applying Lemma 2.5)(ii), $I\left(L_{1} ; x\right)=I\left(\left(m K_{1} \amalg\left(Q \ominus m K_{3}\right) ; x\right)\right.$, if


Figure 13: The graph $L_{1}=3 K_{1} \amalg K_{2} \amalg K_{4} \ominus K_{3} \ominus W_{n-4}$.
$n-4=2 m$, and $I\left(L_{1} ; x\right)=I\left(\left(m K_{1} \amalg\left(Q \ominus m K_{3} \ominus K_{2}\right) ; x\right)\right.$, if $n-4=2 m+1$. Since $m K_{1} \amalg\left(Q \ominus m K_{3} \ominus K_{2}\right)$ is claw-free, it follows that $I\left(L_{1} ; x\right)$ is unimodal, and consequently, $I\left(G_{2, n} ; x\right)$ is unimodal, too.

Case 3. Assume that $m \geq 3, n \geq 5$. According to Proposition 4.4(iii), we obtain


Figure 14: The graph $L_{2}=W_{m-2} \ominus\left(3 K_{1} \amalg K_{2} \amalg K_{4} \ominus K_{3}\right) \ominus W_{n-4}$.
that $I\left(G_{m, n} ; x\right)=I\left(L_{2} ; x\right)$, where

$$
L_{2}=W_{m-2} \ominus Q \ominus W_{n-4} \text { and } Q=3 K_{1} \amalg K_{2} \amalg K_{4} \ominus K_{3}
$$

(see Figure 14). Finally, by Theorem 1.3, we infer that $I\left(L_{2} ; x\right)$ is unimodal, since by applying Lemma 2.5, $W_{m-2}$ and $W_{n-4}$ can be substituted by $p K_{1} \amalg\left(\ominus p K_{3} \ominus K_{2}\right)$ or $p K_{1} \amalg\left(\ominus p K_{3}\right)$, depending on the parity of the numbers $m-2, n-4$. Consequently, the polynomial $I\left(G_{m, n} ; x\right)$ is unimodal, as well.

## 5 Conclusions

In this paper we keep investigating the unimodality of independence polynomials of some well-covered trees started in 12]. Any such a tree is an edge-join of a number of "atoms", called well-covered spiders. We proved that the independence polynomial of any well-covered spider is unimodal, straightforwardly indicating the location of the mode. We also showed that the independence polynomial of some edge-join of well-covered spiders is unimodal. In the later case, our approach was indirect, via claw-free graphs.


Figure 15: $I\left(H_{1} ; x\right)=I\left(H_{2} ; x\right)$ and $I\left(H_{3} ; x\right)=I\left(H_{4} ; x\right)$.
Let us notice that $I\left(H_{1} ; x\right)=I\left(H_{2} ; x\right)=1+5 x+6 x^{2}+2 x^{3}$, and also $I\left(H_{3} ; x\right)=$ $I\left(H_{4} ; x\right)=1+6 x+4 x^{2}$, where $H_{1}, H_{2}, H_{3}, H_{4}$ are depicted in Figure 15. In other words, there exist a well-covered graph whose independence polynomial equals the independence polynomial of a non-well-covered tree (e.g., $H_{2}$ and $H_{1}$ ), and also a wellcovered graph, different from a tree, namely $H_{4}$, satisfying $I\left(H_{3} ; x\right)=I\left(H_{4} ; x\right)$, where $H_{3}$ is not a well-covered graph. Moreover, we can show that for any $\alpha \geq 2$ there are two connected graphs $G_{1}, G_{2}$ such that $\alpha\left(G_{1}\right)=\alpha\left(G_{2}\right)=\alpha$ and $I\left(G_{1} ; x\right)=I\left(G_{2} ; x\right)$, but only one of them is well-covered. To see this, let us consider the following two graphs:

$$
G_{1}=L+\left(H_{1} \amalg H_{2} \amalg 2 K_{1}\right), G_{2}=\left(L_{1} \amalg L_{2}\right)+\left(H_{1} \amalg H_{2} \amalg K_{2}\right),
$$

where $L, L_{1}, L_{2}$ are well-covered graphs, and

$$
L=\left(L_{1}, v_{1}\right) \ominus\left(L_{2}, v_{2}\right), H_{1}=L_{1}-N\left[v_{1}\right], H_{2}=L_{2}-N\left[v_{2}\right], \alpha(L)=\alpha\left(L_{1}\right)+\alpha\left(L_{2}\right)
$$

It follows that $\alpha\left(H_{1}\right)=\alpha\left(L_{1}\right)-1, \alpha\left(H_{2}\right)=\alpha\left(L_{2}\right)-1$, and therefore, we obtain $\alpha\left(G_{1}\right)=\alpha\left(G_{2}\right)=\alpha(L)$. It is easy to check that $G_{1}$ is well-covered, while $G_{2}$ is not well-covered. According to Proposition 2.2(iii), we infer that

$$
\begin{aligned}
I(L ; x) & =I\left(L-v_{1} v_{2} ; x\right)-x^{2} \cdot I\left(L-N_{L}\left(v_{1}\right) \cup N_{L}\left(v_{2}\right) ; x\right) \\
& =I\left(L_{1} ; x\right) \cdot I\left(L_{2} ; x\right)-x^{2} \cdot I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)
\end{aligned}
$$

which we can write as follows:

$$
I(L ; x)+(1+x)^{2} \cdot I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)=I\left(L_{1} ; x\right) \cdot I\left(L_{2} ; x\right)+(1+2 x) \cdot I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)
$$

or, equivalently, as

$$
I(L ; x)+I\left(2 K_{1} ; x\right) \cdot I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)=I\left(L_{1} ; x\right) \cdot I\left(L_{2} ; x\right)+I\left(K_{2} ; x\right) \cdot I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)
$$

In other words, we get:

$$
I\left(L+\left(2 K_{1} \amalg H_{1} \amalg H_{2}\right) ; x\right)=I\left(L_{1} \amalg L_{2}+\left(K_{2} \amalg H_{1} \amalg H_{2}\right) ; x\right),
$$

i.e., $I\left(G_{1} ; x\right)=I\left(G_{2} ; x\right)$.

However, in some of our findings we defined claw-free graphs that simultaneously are well-covered and have the same independence polynomials as the well-covered trees under investigation. These results give an evidence for the following conjecture.

Conjecture 5.1 If $T$ is a well-covered tree and $I(T ; x)=I(G ; x)$, then $G$ is wellcovered.

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