Petri Nets over Partial Algebra

Jörg Desel, Gabriel Juhás, and Robert Lorenz*

Lehrstuhl für Angewandte Informatik
Katholische Universität Eichstätt, 85071 Eichstätt, Germany
{joerg.desel,gabriel.juhas,robert.lorenz}@ku-eichstaett.de

Abstract. Partial algebra is a suitable tool to define sequential semantics for arbitrary restrictions of the occurrence rule, such as capacity or context restrictions. This paper focuses on non-sequential process semantics of Petri nets over partial algebras. It is shown that the concept of partial algebra is suitable as a basis for process construction of different classes of Petri nets taking dependencies between processes that restrict concurrent composition into consideration.
Thus, Petri nets over partial algebra provide a unifying framework for Petri net classes in which some processes cannot be executed concurrently, such as elementary nets with context. We will illustrate this claim proving a one-to-one correspondence between processes constructed using partial algebra and processes based on partial orders for elementary nets with context. Furthermore, we provide compositional process term semantics using the presented framework for place/transition nets with (both weak and strong) capacities and place/transition nets with inhibitor arcs.

1 Introduction

Petri nets are applied in an increasing number of areas. As a consequence, numerous different variants of Petri nets have been developed, many of them based on the same behavioral principles but with slightly different occurrence rules. Examples include Petri nets extended by capacities, inhibitor arcs, read arcs or asymmetric synchronization of transitions.

The restrictions of the occurrence rule can be expressed by restricting the set of legal markings in the case of nets with capacities or by means of different kinds of arcs in the case of nets with inhibitor arcs, read arcs or asymmetric synchronization. Whereas the definition of sequential semantics for these variants can be obtained in a straightforward way from the occurrence rule, partial order semantics providing an explicit representation of concurrent transition occurrences is usually constructed in an ad-hoc way. The aim of this paper is to present a unifying concept for generalized Petri nets, i.e. for Petri nets with restricted occurrence rule, to obtain non-sequential semantics in a systematic way.

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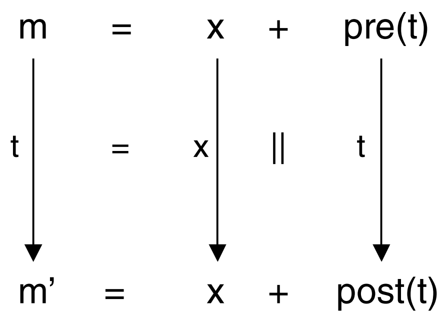

Fig. 1. Occurrence of a transition t from a marking m to a marking m' and its interpretation as a concurrent rewriting of the transition t and the marking x.

In [25,26] and in [18] the authors realized that non-sequential semantics of elementary nets and place/transitions nets can be expressed in terms of concurrent rewriting using partial monoids and total monoids, respectively. In such an algebraic approach, a transition t is understood to be an elementary rewrite term allowing to replace the marking $pre(t)$ by the marking $post(t)$. Moreover, any marking m is understood to be an elementary term, rewriting m by m itself. A single occurrence of a transition t leading from a marking m to a marking m' (in symbols $m \xrightarrow{t} m'$) can be understood as a concurrent composition of the elementary term t and the elementary term corresponding to the marking x, satisfying $m = x + pre(t)$ and $m' = x + post(t)$, where $+$ denotes a suitable operation on markings (see Figure 1). For example, in [18] $+$ is the addition of multi-sets of places, and hence this approach describes place/transition nets. The non-sequential behaviour of a net is given by a set of process terms, constructed from elementary terms using operators for sequential and for concurrent composition, denoted by ; and $\|$, respectively.

Now, assume that for some class of Petri nets a suitable operation $+$ over the set of markings is given such that for each transition occurrence $m \xrightarrow{t} m'$ there exists a marking x satisfying $x + pre(t) = m$ and $x + post(t) = m'$. Then the occurrence of t at m is expressed by the term $x \| t$. Conversely, t cannot necessarily occur at any marking $x + pre(t)$ but its enabledness might be restricted. Such restrictions of the occurrence rule will be encoded by a restriction of concurrent composition, i.e. if $x + pre(t)$ does not enable t, then x and t are not allowed to be composed by $\|$. To describe such a restriction, we use an abstract set I of information elements together with a symmetric independence relation on I. Every marking x as well as every transition t has attached an information element. A marking x and a transition t can be composed concurrently if and only if their respective information elements are independent. For independent information elements we define an operation called concurrent composition with the intended meaning that the information of the composed term is the composition of the information elements of its components. Because the operation of concurrent composition between elementary terms and information elements is defined only partially, i.e. partial algebra is employed, such nets are called Petri nets over partial algebra [14,15].

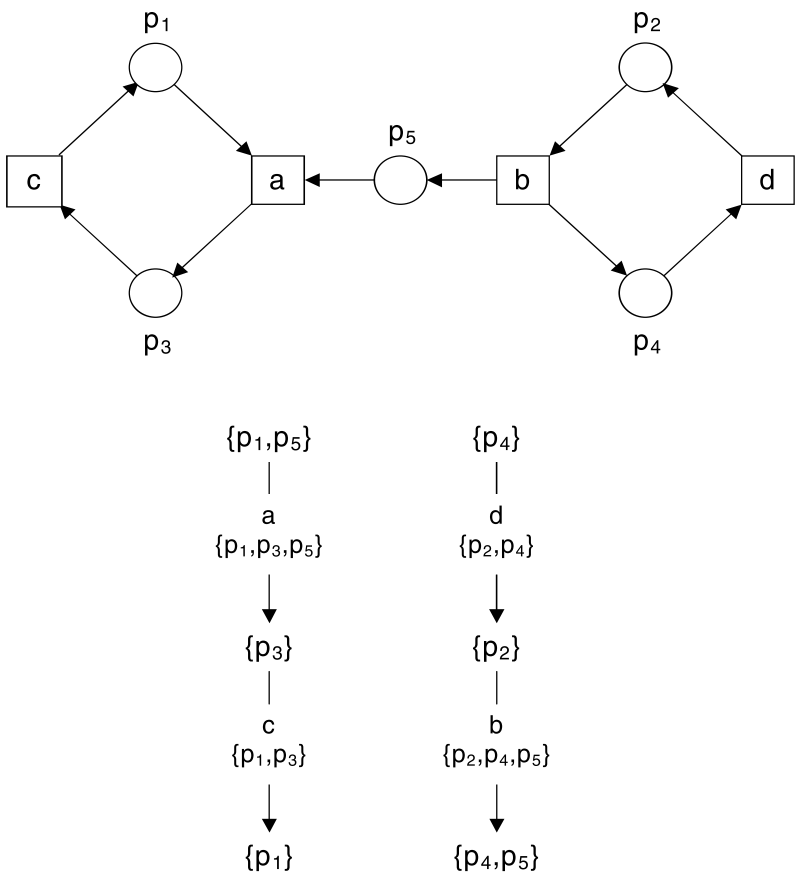

Fig. 2. An elementary net with places p_1, p_2, p_3, p_4, p_5 and the elementary terms corresponding to transitions. For example, transition a is enabled to occur if the places p_1 and p_5 are marked and the place p_3 is unmarked. Its occurrence removes a token from p_1 and p_5 and adds a token to p_3. In other words, transition a rewrites its pre-set $pre(a) = \{p_1, p_5\}$ by its post-set $post(a) = \{p_3\}$. It has attached the information element $\{p_1, p_3, p_5\}$, given by the union of its pre- and post-set.

For example, in the case of elementary nets, where markings are sets of places, we attach to a transition t as information element the union of $pre(t)$ and $post(t)$, while the information element for a marking m is the marking m itself. Two information elements are independent if they are disjoint. The concurrent composition of independent information elements is their union. For an illustrating example see Figure 2.

If a restriction of the occurrence rule is encoded by means of a partial algebra of information elements, one can build non-sequential semantics of nets over partial algebra. This semantics is given by process terms generated from the elementary terms (transitions and markings) using the partial operations sequential composition and concurrent composition.

Each process term has associated an initial marking, final marking and a set of information elements. For elementary process terms, the set of information elements is the one-element set containing the attached information element.

Initial and final markings are necessary for sequential composition: Two process terms can be composed sequentially only if the final marking of the first process term coincides with the initial marking of the second one. The set of information elements associated to the resulting process term is given by the union of the sets of information elements associated to the two composed terms.

Concurrent composition of two process terms is defined only if each information element associated to the first process term is independent from each information element associated to the second. Then the initial and final marking of the resulting term are given by concurrent composition of the initial markings and of the final markings of the two terms. The set of information elements of the resulting process term contains the concurrent composition of each information element associated to the first term with each information element associated to the second.

Thus, sets of information elements are employed for concurrent composition of terms. As already observed by Winkowski in [25,26], for a process term of an elementary net (where information elements are markings, i.e. sets of places), instead of considering the set of information elements, it is sufficient to consider just those places which appear in at least one of the markings being information elements. In [6] we generalize this idea: Two sets of information elements A and B do not have to be distinguished, if for each set of information elements C either both A and B are independent from C[1] or both A and B are not independent from C. Therefore, we can use any equivalence $\cong \in 2^I \times 2^I$ that is a congruence with respect to the operations concurrent composition and union (for sequential composition) and satisfies: If $A \cong B$ and A is independent from C, then B is independent from C. That means, we can use any equivalence $\cong \in 2^I \times 2^I$ which is a *closed congruence* with respect to the operations concurrent composition and union. Equivalence classes of the greatest closed congruence represent the minimal information assigned to process terms necessary for concurrent composition. Thus, instead of sets of information elements we associate to process terms equivalence classes with respect to the greatest closed congruence.

There is a strong connection between the process term semantics described above and the usual partial order based semantics. Consider, for example, the process given in Figure 3. It determines that transition a occurs before b and c, and that transition d occurs before b. This process can be decomposed into the sequence $a\,c$ occurring at the marking $\{p_1, p_4, p_5\}$ (described by the process term $(a; c) \parallel \{p_4\}$), followed by the sequence $d\,b$ occurring at the marking $\{p_1, p_4, \overline{p_5}\}$ (described by the process term $(d; b) \parallel \{p_1\}$). The resulting term is $((a; c) \parallel \{p_4\}); ((d; b) \parallel \{p_1\})$ (see Figure 4). Another interpretation of this process is the following: Transitions a and d occur concurrently at the marking $\{p_1, p_4, p_5\}$ replacing this marking by $\{p_2, p_3, \overline{p_5}\}$. At this marking transitions c and b occur concurrently. The corresponding term is $(a \parallel d); (c \parallel b)$ (see Figure 5). Each process term α defines a partially ordered set of events representing transition occurrences in an obvious way: an event e_2 *depends on* another event e_1 if the process term α contains a subterm $\alpha_1; \alpha_2$ such that e_1 occurs in α_1 and e_2 occurs in α_2. For example, the process term $\alpha = ((a; c) \parallel \{p_4\}); ((d; b) \parallel \{p_1\})$ generates

[1] Two sets of information elements X and Y are independent if and only if each information element of X is independent from each information element of Y.

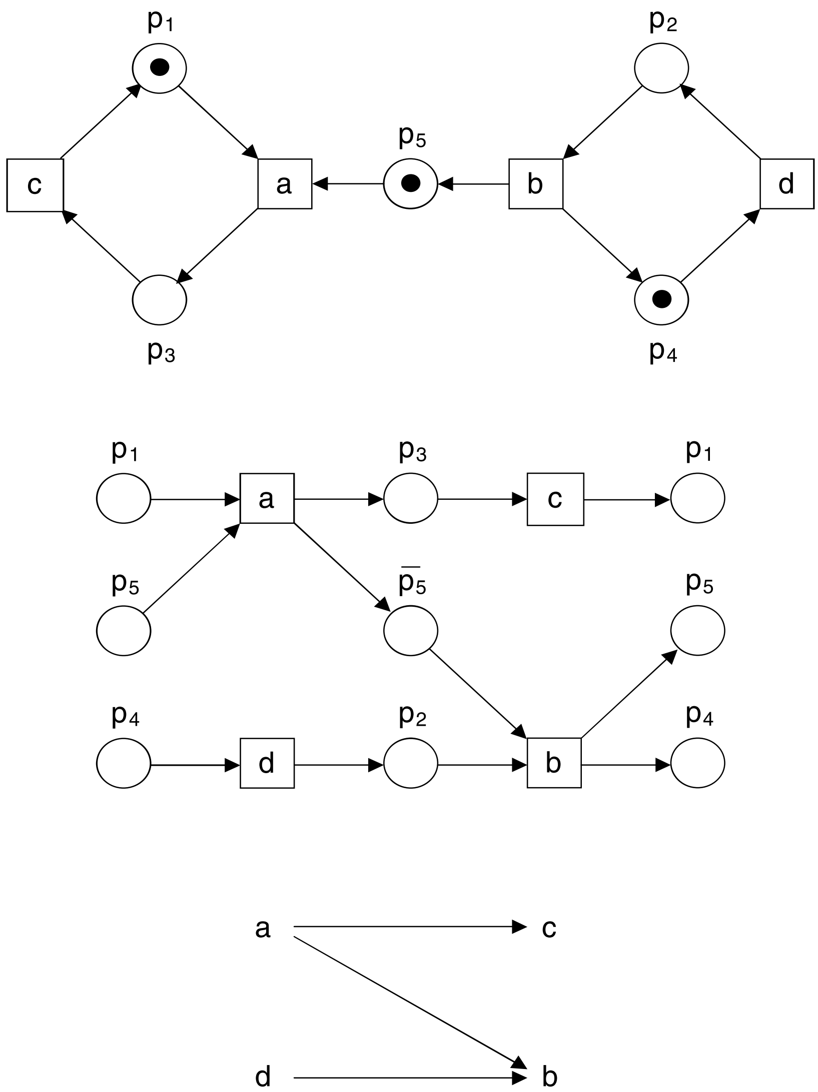

Fig. 3. The elementary net from Figure 2 with the initial marking $\{p_1, p_4, p_5\}$ together with a process and the corresponding partial order of the occurring transitions. The place annotated by $\overline{p_5}$ establishes an order between the occurrence of a and b, due to the contact situation at p_5 after the occurrence of d. For details how to construct processes of elementary nets with contacts see e.g. [23] or Subsection 8.1. The interpretation of $\overline{p_5}$ is that p_5 is not marked.

the partial order given in Figure 6, while the process term $\beta = (a \parallel d); (c \parallel b)$ generates the partial order given in Figure 7.

Unfortunately not all reasonable partial orders can be generated in this way. For example, consider the partial order shown in Figure 3, which is determined by the process from Figure 3. It is easy to show by induction on the structure of process terms that this partial order cannot be generated by any process term. However, this partial order can be constructed from the partial orders generated by process terms α and β, i.e. by two possible decompositions of the process from Figure 3, removing the contradicting connections between c and d. We will define an equivalence of process terms identifying exactly those process terms

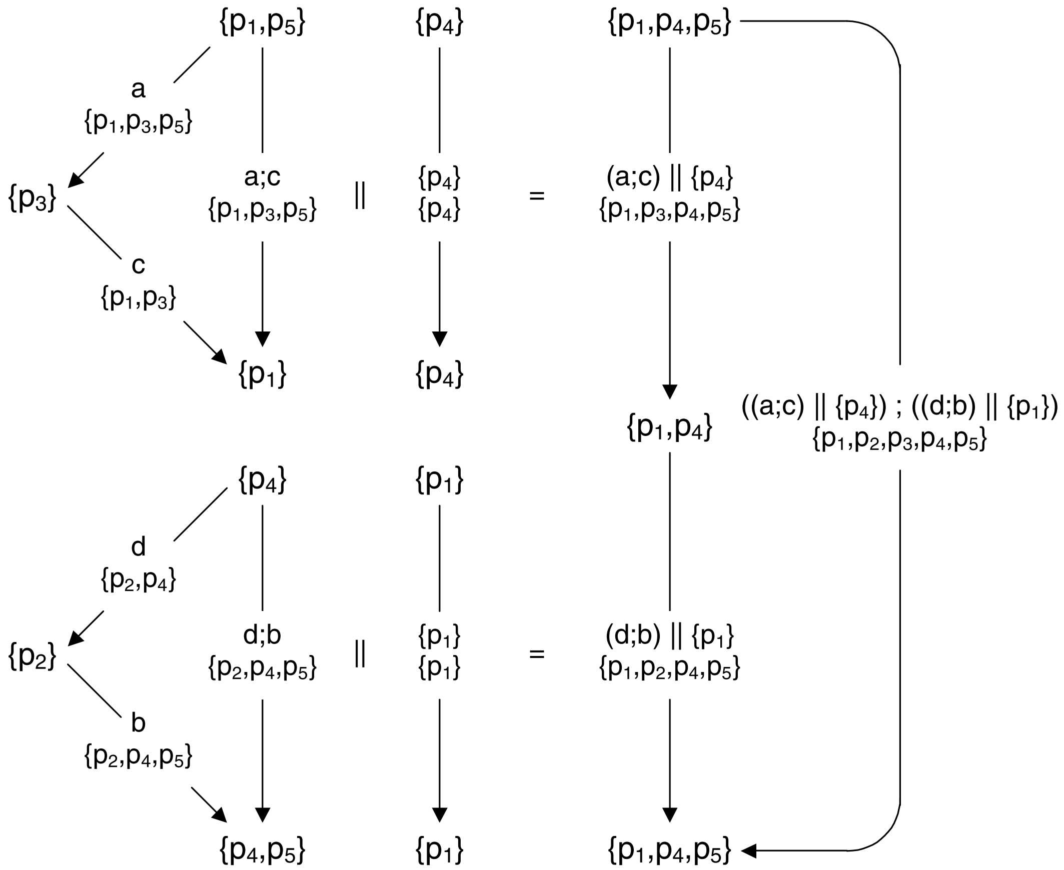

Fig. 4. Derivation of a process term of the elementary net from Figure 2. Instead of the whole set of information elements, each process term has attached only the set of all involved places, i.e. the set of places characterizing the greatest closed congruence class of the related set of information elements. For example, the process term $a;c$ has attached the information $\{p_1, p_3, p_5\}$ instead of the set of two information elements $\{\{p_1, p_3, p_5\}, \{p_1, p_3\}\}$.

representing the same run. Then each run is represented by an equivalence class of process terms.

The paper is organized as follows. Section 2 gives mathematical preliminaries. After introducing formally our concept in Section 3, we provide a couple of examples in Sections 4–9.

The first example given in Section 4 will re-formulate results achieved in [25,26,6] for elementary nets, showing that the information for concurrent composition used in [25,26] is in fact (isomorphic to) the equivalence class of the greatest closed congruence of the related partial algebra and therefore is the minimal information necessary for concurrent composition.

Usually, if a transition depends on the state of a place, then this state is changed by the transition's occurrence. We call this a write operation and the place a write place. Extensions with context requirements release this property: a transition can only occur if in addition the context places are in a certain state but this state remains unchanged by the transition's occurrence (read operation). In [19] elementary nets with context are defined, generalizing the notions of

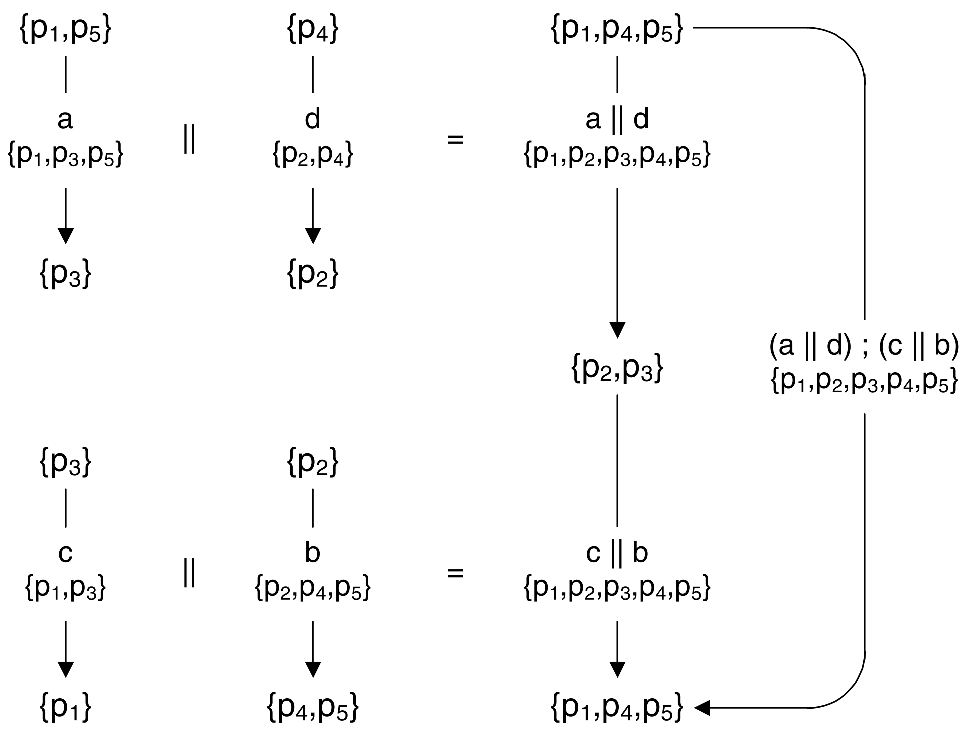

Fig. 5. Derivation of another process term of the elementary net from Figure 2.

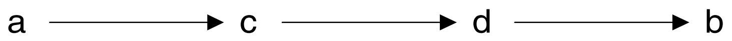

Fig. 6. Partial order generated by the process term $\alpha = ((a;c) \parallel \{p_4\});((d;b) \parallel \{p_1\})$

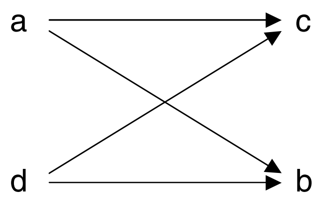

Fig. 7. Partial order generated by the process term $\beta = (a \parallel d);(c \parallel b)$

inhibitor arcs (negative context) and read arcs (positive context). In Sections 5–7 we apply our concept to elementary nets with context. For these nets, two enabled transitions using common places as (positive or negative) context can occur concurrently if their pre- and post-sets are disjoint. Accordingly, if two processes do not employ common places for the flow of tokens but partly use the same context, then the composition of these process terms should not be excluded. This means that read operations on a place can occur concurrently, whereas mixed read and write operations as well as two write operations are incompatible with respect to concurrent composition, just as concurrent access to a storage element is only possible for read access (this interpretation of context places was also chosen in [19] whereas [13,17] allows concurrent read and write operations). Hence we need information about the nature (write or read) of the access to places for each process term. Therefore, the necessary information is more complex as the simple collection of markings associated to a process. In each case, i.e. in case of elementary nets with positive context (Section 5), negative context (Section 6), and mixed context (Section 7), the minimal information for concurrent composition is computed. Further, in Section 8 we prove that the non-sequential semantics given by process terms coincides with the partial order semantics given by process nets of elementary nets with context introduced in [19].

Section 9 illustrates the generality of our approach by applying it to two more net classes, namely place/transition nets with inhibitor arcs (negative context) and place/transition nets with capacities. In Subsection 9.1 we show that for place/transition nets with inhibitor arcs, concurrent composition of two processes should only be excluded if a place is a common context and write place, and therefore it is enough to store the set of context and the set of write places. Thus, the set of information is less complex (it is particularly a finite set) than the set of markings (which in this case is the infinite set of multi-sets over the set of places). We conclude by showing that our approach fits well for place/transtion nets with strong and with weak capacities (Subsection 9.2).

2 Mathematical Preliminaries

We use $\mathbb{N}$ to denote the nonnegative integers and $\mathbb{N}^+$ to denote the positive integers. Given two arbitrary sets A and B, the symbol B^A denotes the set of all functions from A to B. Given a function f from A to B and a subset C of A we write $f|_C$ to denote the restriction of f to the set C. The symbol 2^A denotes the power set of a set A. Given a set A, the symbol $|A|$ denotes the cardinality of A and the symbol id_A the identity on the set A. We write id to denote id_A whenever A is clear from the context. The set of all multi-sets over a set A is denoted by $\mathbb{N}^A$. Given a binary relation $R \subseteq A \times A$ over a set A, the symbol R^+ denotes the transitive closure of R.

A partial groupoid is an ordered tuple $\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$ where I is a set called the carrier of $\mathcal{I}$, $dom_{\dot{+}} \subseteq I \times I$ is the domain of $\dot{+}$, and $\dot{+} : dom_{\dot{+}} \to I$ is the partial operation of $\mathcal{I}$. In the rest of the paper we will consider only partial groupoids $(I, dom_{\dot{+}}, \dot{+})$ which fulfil the following conditions:

- If $a \dot{+} b$ is defined then $b \dot{+} a$ is defined and $a \dot{+} b = b \dot{+} a$.
- If $(a \dot{+} b) \dot{+} c$ is defined then $a \dot{+} (b \dot{+} c)$ is defined and $(a \dot{+} b) \dot{+} c = a \dot{+} (b \dot{+} c)$.

We use the symbol I for a set of information elements associated to elementary terms and the operation $\dot{+}$ to express concurrent composition of information elements. Not each pair of process terms can be concurrently composed, hence $\dot{+}$ is a partial operation. The relation $dom_{\dot{+}}$ contains the pairs of elements which are independent and can be concurrently composed.

As explained in Introduction, generated terms have associated sets of information elements. So, the partial groupoid $(I, dom_{\dot{+}}, \dot{+})$ is extended to the partial groupoid $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\})$, where

- $dom_{\{\dot{+}\}} = \{(X, Y) \in 2^I \times 2^I \mid X \times Y \subseteq dom_{\dot{+}}\}$.

– $X\{\dot{+}\}Y = \{x \dot{+} y \mid x \in X \wedge y \in Y\}$.

We will use more than one partial operation on the same carrier. A partial algebra is a set (called carrier) together with a couple of partial operations on this set (with possibly different arity). Given a partial algebra with carrier X, an equivalence $\sim$ on X satisfying the following conditions is a *congruence*: If op is an n-ary partial operation, $a_1 \sim b_1, \ldots, a_n \sim b_n$, $(a_1, \ldots, a_n) \in dom_{op}$ and $(b_1, \ldots, b_n) \in dom_{op}$, then $op(a_1, \ldots, a_n) \sim op(b_1, \ldots, b_n)$. If moreover $a_1 \sim b_1, \ldots, a_n \sim b_n$ and $(a_1, \ldots, a_n) \in dom_{op}$ imply $(b_1, \ldots, b_n) \in dom_{op}$ for each n-ary partial operation then the congruence $\sim$ is said to be *closed*. Thus, a congruence is an equivalence preserving all operations of a partial algebra, while a closed congruence moreover preserves the domains of the operations. For a given partial algebra there always exists a unique greatest closed congruence. The intersection of two congruences is again a congruence. Given a binary relation on X, there always exists a unique least congruence containing this relation. In general, the same does not hold for closed congruences. Given a partial algebra $\mathcal{X}$ with carrier X and a congruence $\sim$ on $\mathcal{X}$, we write $[x]_\sim = \{y \in X \mid x \sim y\}$ and $X/_\sim = \bigcup_{x \in X} [x]_\sim$. A closed congruence $\sim$ defines the partial algebra $\mathcal{X}/_\sim$ with carrier $X/_\sim$, and with n-ary partial operation $op/_\sim$ defined for each n-ary partial operation $op : dom_{op} \to X$ of $\mathcal{X}$ as follows: $dom_{op/_\sim} = \{([a_1]_\sim, \ldots, [a_n]_\sim) \mid (a_1, \ldots, a_n) \in dom_{op}\}$ and, for each $(a_1, \ldots, a_n) \in dom_{op}$, $op/_\sim([a_1]_\sim, \ldots, [a_n]_\sim) = [op(a_1, \ldots, a_n)]_\sim$. The partial algebra $\mathcal{X}/_\sim$ is called factor algebra of $\mathcal{X}$ with respect to the congruence $\sim$.

Let $\mathcal{X}$ be a partial algebra with k operations $op_i^{\mathcal{X}}, i \in \{1, \ldots, k\}$, and let $\mathcal{Y}$ be a partial algebra with k operations $op_i^{\mathcal{Y}}, i \in \{1, \ldots, k\}$ such that the arity $n_i^{\mathcal{X}}$ of $op_i^{\mathcal{X}}$ equals the arity $n_i^{\mathcal{Y}}$ of $op_i^{\mathcal{Y}}$ for every $i \in \{1, \ldots, k\}$. Denote by X the carrier of $\mathcal{X}$ and by Y the carrier of $\mathcal{Y}$. Then a function $f : X \to Y$ is called homomorphism if for every $i \in \{1, \ldots, k\}$ and $x_1, \ldots x_{n_i^{\mathcal{X}}} \in X$ we have: if $op_i^{\mathcal{X}}(x_1, \ldots, x_{n_i^{\mathcal{X}}})$ is defined then $op_i^{\mathcal{Y}}(f(x_1), \ldots, f(x_{n_i^{\mathcal{X}}}))$ is also defined and $f(op_i^{\mathcal{X}}(x_1, \ldots, x_{n_i^{\mathcal{X}}})) = op_i^{\mathcal{Y}}(f(x_1), \ldots, f(x_{n_i^{\mathcal{X}}}))$. A homomorphism $f : X \to Y$ is called closed if for every $i \in \{1, \ldots, k\}$ and $x_1, \ldots x_{n_i^{\mathcal{X}}} \in X$ we have: if $op_i^{\mathcal{Y}}(f(x_1), \ldots, f(x_{n_i^{\mathcal{X}}}))$ is defined then $op_i^{\mathcal{X}}(x_1, \ldots, x_{n_i^{\mathcal{X}}})$ is also defined. If f is a bijection, then it is called an isomorphism, and the partial algebras $\mathcal{X}$ and $\mathcal{Y}$ are called isomorphic.

There is a strong connection between the concepts of homomorphism and congruence in partial algebras: If f is a surjective (closed) homomorphism from $\mathcal{X}$ to $\mathcal{Y}$, then the relation $\sim \subseteq X \times X$ defined by $a \sim b \iff f(a) = f(b)$ is a (closed) congruence and $\mathcal{Y}$ is isomorphic to $\mathcal{X}/_\sim$. Conversely, given a (closed) congruence $\sim$ of $\mathcal{X}$, the mapping $h : X \to X/_\sim$ given by $h(x) = [x]_\sim$ is a surjective (closed) homomorphism. This homomorphism is called the *natural homomorphism w.r.t.* $\sim$. For more details on partial algebras see e.g. [4].

3 The General Approach

An algebraic Petri net as introduced in [18] is based on a graph with vertices representing markings and edges labeled by transitions representing steps between markings. Moreover, an operator $+$ adds markings. The set of markings together with addition of markings denotes a commutative monoid $\mathcal{M} = (M, +)$ with neutral element e (the empty marking). To obtain the process term semantics of an algebraic Petri net, we assign to every marking and to every transition an information element used for concurrent composition. Two elementary process terms can be concurrently composed only if their associated information elements are independent. The set of all possible information elements is denoted by a partial groupoid $\mathcal{I} = (I, \dot{+}, dom_{\dot{+}})$, where $\dot{+}$ denotes the composition of independent information elements, and independence is given by the symmetric relation $dom_{\dot{+}} \subseteq I \times I$.

Since we will compose process terms concurrently and process terms have associated sets of information elements, we lift the partial groupoid $(I, \dot{+}, dom_{\dot{+}})$ to the partial groupoid $(2^I, \{\dot{+}\}, dom_{\{\dot{+}\}})$.

A process term $\alpha\colon m_1 \to m_2$ represents a process transforming marking m_1 to marking m_2. Process terms $\alpha\colon m_1 \to m_2$ and $\beta\colon m_3 \to m_4$ can be sequentially composed, provided $m_2 = m_3$, resulting in $\alpha;\beta : m_1 \to m_4$. This notation illustrates the occurrence of β after the occurrence of α. The set of information elements of the sequentially composed process term is the union of the sets of information elements of the single process terms. The process terms can also be composed concurrently to $\alpha \parallel \beta : m_1 + m_3 \to m_2 + m_4$, provided the set of information elements of α is independent from the set of information elements of β. The set of information elements of $\alpha \parallel \beta$ contains the concurrent composition of each element of the set of information elements of α with each element of the set of information elements of β.

For sequential composition of process terms we need information about the start and the end of a process term, which are both single markings. For concurrent composition, we require that the associated sets of information elements are independent.

Two sets of information elements A and B do not have to be distinguished, if for each set of information elements C either both A and B are independent from C or both A and B are not independent from C. Therefore, we can use any equivalence $\cong \in 2^I \times 2^I$ that is a congruence with respect to the operations $\{\dot{+}\}$ (concurrent composition) and $\cup$ (sequential composition) and satisfies $(A \cong B \wedge (A, C) \in dom_{\{\dot{+}\}}) \implies (B, C) \in dom_{\{\dot{+}\}}$, i.e. which is a *closed congruence* of the partial algebra $\mathcal{X} = (2^I, \{\dot{+}\}, dom_{\{\dot{+}\}}, \cup)$. The equivalence classes of the greatest (and hence coarsest) closed congruence represent the minimal information assigned to process terms necessary for concurrent composition. This congruence is unique ([4]).

Definition 1 (Algebraic $(\mathcal{M}, \mathcal{I})$-net and its process term semantics). *Let $\mathcal{M} = (M, +)$ be a commutative monoid and $\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$ be a partial groupoid satisfying the properties defined in the previous section. Let $\cong \in 2^I \times 2^I$ be the greatest closed congruence of the partial algebra $\mathcal{X} = (2^I, \{\dot{+}\}, dom_{\{\dot{+}\}}, \cup)$.*

An algebraic $(\mathcal{M}, \mathcal{I})$-net is a quadruple

$$\mathcal{A} = (M, T, pre\colon T \to M, post\colon T \to M)$$

together with a mapping $\mathit{inf}\colon M \cup T \to I$ *satisfying*

(a) $\forall x, y \in M: \quad (\mathit{inf}(x), \mathit{inf}(y)) \in \mathit{dom}_{\dot{+}} \Longrightarrow \mathit{inf}(x+y) = \mathit{inf}(x) \dot{+} \mathit{inf}(y).$

(b) $\forall t \in T: \{\mathit{inf}(t)\} \cong \{\mathit{inf}(t), \mathit{inf}(\mathit{pre}(t)), \mathit{inf}(\mathit{post}(t))\}.$

Out of an algebraic net $\mathcal{A}$ *we can build process terms that represent all abstract concurrent computations of* $\mathcal{A}$*. Every process term* α *has associated an initial marking* $\mathit{pre}(\alpha) \in M$*, a final marking* $\mathit{post}(\alpha) \in M$*, and an information for concurrent composition* $\mathit{Inf}(\alpha) \in 2^I/_{\cong}$*. In the following, for a process term* α *we write* $\alpha : a \longrightarrow b$ *to denote that* a *is the initial marking and* b *is the final marking of* α*.*

The elementary process terms are

$$id_a : a \longrightarrow a$$

with associated information $\mathit{Inf}(id_a) = [\{\mathit{inf}(a)\}]_{\cong}$ *for each* $a \in M$*, and*

$$t : \mathit{pre}(t) \longrightarrow \mathit{post}(t)$$

with associated information $\mathit{Inf}(t) = [\{\mathit{inf}(t)\}]_{\cong}$ *for each* $t \in T$*.*

If $\alpha : a_1 \longrightarrow a_2$ *and* $\beta : b_1 \longrightarrow b_2$ *are process terms satisfying* $(\mathit{Inf}(\alpha), \mathit{Inf}(\beta)) \in \mathit{dom}_{\{\dot{+}\}}/_{\cong}$*, their concurrent composition yields the process term*

$$\alpha \parallel \beta\colon a_1 + b_1 \longrightarrow a_2 + b_2$$

with associated information $\mathit{Inf}(\alpha \parallel \beta) = \mathit{Inf}(\alpha)\ \{\dot{+}\}/_{\cong}\ \mathit{Inf}(\beta)$*.*

If $\alpha : a_1 \longrightarrow a_2$ *and* $\beta : b_1 \longrightarrow b_2$ *are process terms satisfying* $a_2 = b_1$*, their sequential composition yields the process term*

$$\alpha;\beta\ : a_1 \longrightarrow b_2$$

with associated information $\mathit{Inf}(\alpha;\beta) = \mathit{Inf}(\alpha) \cup/_{\cong} \mathit{Inf}(\beta)$*.*

The partial algebra of all process terms with the partial operations concurrent composition and sequential composition as defined above will be denoted by $\mathcal{P}(\mathcal{A})$*.*

We consider the used factor algebra $\mathcal{X}/_{\cong}$ up to isomorphism. Hence one can freely use any partial algebra isomorphic to $\mathcal{X}/_{\cong}$.

Requirement (a) in the previous definition means that the concurrent composition of information elements attached to markings respects the concurrent composition of the markings. Requirement (b) means that the information about the initial and the final marking of a transition is already included in the information associated to the transition.

As mentioned in Introduction, we now define an equivalence of process terms identifying exactly those process terms representing the same run. Then each run is represented by an equivalence class of process terms. We require this equivalence to preserve the concurrent composition and sequential composition of process terms, i.e. to be a congruence with respect to these operations.

Definition 2 (Congruence of process terms). *The congruence relation $\sim$ on the set of process terms of an algebraic $(\mathcal{M}, \mathcal{I})$-net is the least congruence on process terms with respect to the partial operations $\|$ and ; given by the following axioms for process terms $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and markings $x, y \in M$:*

1. $(\alpha_1 \| \alpha_2) \sim (\alpha_2 \| \alpha_1)$, *whenever* $\|$ *is defined for* α_1 *and* α_2.
2. $((\alpha_1 \| \alpha_2) \| \alpha_3) \sim (\alpha_1 \| (\alpha_2 \| \alpha_3))$, *whenever these terms are defined.*
3. $((\alpha_1; \alpha_2); \alpha_3) \sim (\alpha_1; (\alpha_2; \alpha_3))$, *whenever these terms are defined.*
4. $\alpha = ((\alpha_1 \| \alpha_2); (\alpha_3 \| \alpha_4)) \sim \beta = ((\alpha_1; \alpha_3) \| (\alpha_2; \alpha_4))$, *whenever these terms are defined and* $\mathit{Inf}(\alpha) = \mathit{Inf}(\beta)$.
5. $(\alpha; id_{post(\alpha)}) \sim \alpha \sim (id_{pre(\alpha)}; \alpha)$.
6. $id_{(x+y)} \sim (id_x \| id_y)$, *whenever these terms are defined.*
7. $\alpha \| id_x \sim \alpha$ *whenever the left term is defined,* $pre(\alpha) + x = pre(\alpha)$ *and* $post(\alpha) + x = post(\alpha)$.

In the sequel we will write x to denote the elementary term id_x.

Proposition 1. *By construction,* $\alpha \sim \beta$ *implies* $pre(\alpha) = pre(\beta)$, $post(\alpha) = post(\beta)$ *and* $\mathit{Inf}(\alpha) = \mathit{Inf}(\beta)$.

Axiom (1) represents commutativity of concurrent composition, axioms (2) and (3) associativity of concurrent and sequential composition. Axiom (4) states distributivity whenever both terms have the same information. It is also used in related approaches such as [18]. Notice that the partial order induced by β is a subset of the partial order induced by α. Therefore, the partial order induced by α can be understood as a partial sequentialization of the partial order induced by β, i.e. it is a partial sequentialization of the run represented by the corresponding equivalence class of process terms. Axiom (5) states that elementary terms corresponding to elements of M are partial neutral elements with respect to sequential composition. Axiom (6) expresses that composition of these neutral elements is congruent to the neutral element constructed from their composition. Finally, axiom (7) states that elements of M which are neutral to the initial and final marking of a term are neutral to the term itself.

For example, the process term $((a; c) \| \{p_4\}); ((d; b) \| \{p_1\})$ of the elementary net from Figure 2 generated in Figure 4 and the process term $(a \| d); (c \| b)$ of the elementary net from Figure 2 generated in Figure 5 are congruent:

$$\begin{aligned}
((a;c) \| \{p_4\}); ((d;b) \| \{p_1\}) &\overset{(4),(5)}{\sim} ((a \| \{p_4\}); (c \| \{p_4\})); ((d \| \{p_1\}); (b \| \{p_1\})) \\
&\overset{(1),(3),(4)}{\sim} (a \| \{p_4\}); ((c; \{p_1\}) \| (\{p_4\}; d)); (b \| \{p_1\}) \\
&\overset{(1),(5)}{\sim} (a \| \{p_4\}); (d \| c); (b \| \{p_1\}) \\
&\overset{(5)}{\sim} (a \| \{p_4\}); ((d; \{p_2\}) \| (\{p_3\}; c)); (b \| \{p_1\}) \\
&\overset{(4)}{\sim} (a \| \{p_4\}); ((d \| \{p_3\}); (c \| \{p_2\})); (b \| \{p_1\}) \\
&\overset{(1),(3),(4),(5)}{\sim} (a \| d); (c \| b).
\end{aligned}$$

Note that given a transition t of a $(\mathcal{M},\mathcal{I})$-net, the elementary term t represents the single occurrence of the transition t leading from the marking $m = pre(t)$ to the marking $m' = post(t)$, and any term in the form $t \parallel x$, where $x \in M$, represents the single occurrence of the transition t leading from the marking $m = x + pre(t)$ to the marking $m' = x + post(t)$.

Despite the differences between different classes of Petri nets, there are some common features of almost all net classes, such as the notions of marking (state), transition, and occurrence rule (see our contribution [8]).

Thus, in the following definition we suppose a Petri net with a set of markings, a set of transitions and an occurrence rule characterizing whether a transition is enabled to occur at a given marking and if yes determining the follower marking. We suppose that the considered Petri net has no fixed initial marking.

Definition 3 (Corresponding algebraic $(\mathcal{M},\mathcal{I})$-net). *Let N be a Petri net with a set of markings M_N, and a set of transitions T_N. Let $m \xrightarrow{t} m'$ denote that a transition t is enabled to occur in m and that its occurrence leads to the follower marking m'.*

Let $\mathcal{M} = (M,+)$ and $\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$. Then an algebraic $(\mathcal{M},\mathcal{I})$-net

$$\mathcal{A} = (M, T, pre\colon T \to M, post\colon T \to M)$$

together with a mapping $inf : M \cup T \to I$ is called a corresponding algebraic $(\mathcal{M},\mathcal{I})$-net to the net N iff:

- *$\mathcal{A}$ has the same domain for markings as N, i.e. $M = M_N$*
- *transitions of $\mathcal{A}$ are those transitions of N which are enabled to occur in some marking, i.e. $T = \{t \in T_N \mid \exists m, m' \in M : m \xrightarrow{t} m'\}$, and*
- *the occurrence rule is preserved, i.e. $\forall m, m' \in M, t \in T : m \xrightarrow{t} m' \iff ((m = pre(t) \wedge m' = post(t)) \vee (\exists x \in M : (inf(x), inf(t)) \in dom_{\dot{+}} \wedge x + pre(t) = m \wedge x + post(t) = m'))$.*

In the following sections we construct corresponding algebraic $(\mathcal{M},\mathcal{I})$-nets for several classes of Petri nets using the following scenario:

- We give a classical definition of the considered net class including the occurrence rule.
- We identify $\mathcal{M}$ and construct $\mathcal{I}$ such that the requirements from Section 2 are satisfied.
- We construct functions $pre, post, inf$ in such a way that condition (a) from Definition 1 is valid and that $dom_{\dot{+}}$, the independence relation of $\mathcal{I}$, encodes the restriction of the occurrence rule.
- We construct the greatest closed congruence $\cong$ of the partial algebra $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$. Then, we construct a partial algebra isomorphic to $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)/_{\cong}$.
- We show that property (b) from Definition 1 is satisfied.

4 Elementary Nets

In this section we represent elementary nets as algebraic $(\mathcal{M}, \mathcal{I})$-nets.

An elementary net consists of a set of places P, a set of transitions T and relations between them. Places can be in different *states*. Transitions can occur, depending on the state of some places. The occurrence of a transition can change the state of some places.

Definition 4 (Elementary nets). *An elementary net is a triple $N = (P, T, F)$, where P (places) and T (transitions) are disjoint finite sets, and $F \subseteq (P \times T) \cup (T \times P)$ is a relation (flow relation). For a transition $t \in T$, ${}^\bullet t = \{p \in P \mid (p, t) \in F\}$ is the* pre-set *of t and $t^\bullet = \{p \in P \mid (t, p) \in F\}$ is the* post-set *of t. Throughout the paper we assume that each transition has nonempty pre- and post-sets.*

Each subset of P is called a marking. *A transition $t \in T$ is enabled to occur in a marking $m \subseteq P$ iff ${}^\bullet t \subseteq m \wedge (m \setminus {}^\bullet t) \cap t^\bullet = \emptyset$. In this case, its occurrence leads to the marking $m' = (m \setminus {}^\bullet t) \cup t^\bullet$.*

As usual, places are graphically expressed by circles, transitions by boxes and elements of the flow relation by directed arcs. A marking of the net is represented by tokens in places. For an example of an elementary net see Introduction.

The union of markings represents concurrent composition. Hence the approach of the previous section looks as follows: $\mathcal{M} = (M, +) = (2^P, \cup)$.

The information element associated to an elementary process term consists of the set of used places. An information element is independent from another information element, if they are disjoint. Hence we define the set of information elements $I = M = 2^P$ together with the independence relation $dom_{\dot{+}} = \{(w, w') \in M \times M \mid w \cap w' = \emptyset\}$ and the operation $w \dot{+} w' = w \cup w'$. The partial groupoid $\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$ respects the requirements of Section 2.

To find a $(\mathcal{M}, \mathcal{I})$-net corresponding to an elementary net $N = (P, T, F)$, we need to define mappings $pre, post : T \to M$ which assign an initial and final marking to every transition, and a function $inf : M \cup T \to I$ which assigns an information element to every marking $m \in M$ and every transition $t \in T$:

- For a transition $t \in T$, $pre(t) = {}^\bullet t$ and $post(t) = t^\bullet$.
- For a marking $m \in M$, $inf(m) = m$.
- For a transition $t \in T$, $inf(t) = {}^\bullet t \cup t^\bullet$.

It is easy to observe that the mapping inf satisfies the property (a) from Definition 1.

The following lemma shows that the occurrence rule is encoded by inf and $dom_{\dot{+}}$, as described in Introduction.

Lemma 1. *A transition $t \in T$ is enabled to occur in a marking m and its occurrence leads to the marking m' iff there exists a marking x such that $(inf(x), inf(t)) \in dom_{\dot{+}}$, $x + pre(t) = m$ and $x + post(t) = m'$.*

Proof. $\Rightarrow$: Choose $x = m \setminus {}^\bullet t$.

$\Leftarrow$: Assume an x with $x \cap ({}^\bullet t \cup t^\bullet) = \emptyset$. Obviously, ${}^\bullet t \subseteq (x \cup {}^\bullet t)$. Furthermore we have $x = (x \cup {}^\bullet t) \setminus {}^\bullet t$ and $x \cap t^\bullet = \emptyset$. Therefore t is enabled to occur in $x \cup {}^\bullet t = x + pre(t)$ and its occurrence leads to $x \cup t^\bullet = x + post(t)$. □

To define a corresponding algebraic net and its process terms we have to find the greatest closed congruence on $(2^I, \{\dotplus\}, dom_{\{\dotplus\}}, \cup)$. Actually, instead of considering the set of all information elements associated with a process term, it will be enough to consider the information about all involved places of a process term. We define the mapping $supp : 2^I \to I$, $supp\,(A) = \bigcup_{w \in A} w$ and show that $supp$ is the natural homomorphism w.r.t. the greatest closed congruence $\cong$ on $(2^I, \{\dotplus\}, dom_{\{\dotplus\}}, \cup)$.

Lemma 2. *The relation $\cong \subseteq 2^I \times 2^I$ defined by $A \cong B \iff supp(A) = supp(B)$ is a closed congruence on $(2^I, \{\dotplus\}, dom_{\{\dotplus\}}, \cup)$.*

Proof. Straightforward observation.

Lemma 3. *The closed congruence $\cong \subseteq 2^I \times 2^I$ is the greatest closed congruence on $(2^I, \{\dotplus\}, dom_{\{\dotplus\}}, \cup)$.*

Proof. We show that any congruence $\approx$ such that $\cong$ is a proper subset of $\approx$ is not closed. Assume there are $A, A' \in 2^I$ such that $A \approx A'$ but $A \ncong A'$. Then $supp(A) \neq supp(A')$.

We define a set $C \in 2^I$ such that $(A, C) \in dom_{\{\dotplus\}}$ but $(A', C) \notin dom_{\{\dotplus\}}$ or vice versa (which implies that $\approx$ is not closed). Denoting $supp(A) = \overline{w}$ and $supp(A') = \overline{w}'$ we have that $\overline{w} \neq \overline{w}'$.

Without loss of generality we assume $\overline{w}' \setminus \overline{w} \neq \emptyset$. Set $C = \{c\}$ with $c = \overline{w}' \setminus \overline{w}$. Then $c \cap \overline{w} = \emptyset$, but $c \cap \overline{w}' \neq \emptyset$, i.e. $(A, C) \in dom_{\{\dotplus\}}$, but $(A', C) \notin dom_{\{\dotplus\}}$. □

Taking $pre, post, inf$ defined above, we have $supp(\{inf(t)\}) = {}^\bullet t \cup t^\bullet = ({}^\bullet t \cup t^\bullet) \cup {}^\bullet t \cup t^\bullet = supp(\{inf(t), inf(pre(t)), inf(post(t))\})$, and therefore the property (b) from Definition 1 is satisfied. Thus, we can formulate the following theorem.

Theorem 1. *Given an elementary net $N = (P, T, F)$ with $\mathcal{M}, \mathcal{I}, pre, post, inf$ as defined in this section, the quadruple $\mathcal{A}_N = (2^P, T, pre, post)$ together with the mapping inf is an algebraic $(\mathcal{M}, \mathcal{I})$-net. Moreover, it is a corresponding algebraic $(\mathcal{M}, \mathcal{I})$-net to the net N.*

Remark 1. In our definition of elementary nets we use an occurrence rule which slightly differs from the standard occurrence rule as given in [23]. Our main motivation of using the presented occurrence rule is to have a definition which is compatible with [19]. The only difference is that the occurrence of a transition with non-disjoint pre- and post-set is allowed in our definition, while using the standard occurrence rule for elementary nets such transitions are never enabled to occur and therefore, according to Definition 3, are irrelevant for a corresponding $(\mathcal{M}, \mathcal{I})$-net. In other words, the corresponding $(\mathcal{M}, \mathcal{I})$-net for the standard occurrence rule of an elementary net would differ from the one we presented in Theorem 1 only in the absence of transitions with non-disjoint pre- and post-set. In general, there is a more substantial difference between both occurrence rules. Namely, the occurrence rule used for elementary nets in this section corresponds in general to the occurrence rule of place/transition nets with weak capacity restrictions, while the standard occurrence rule for elementary nets corresponds in general to the occurrence rule of place/transition nets with the strong capacity restrictions. For more details on this difference we refer to the Section 9.2 and to [10,9,15].

5 Elementary Nets with Positive Context

In this section we represent elementary nets with positive context as algebraic $(\mathcal{M}, \mathcal{I})$-nets.

Definition 5 (Elementary nets with positive context). *An elementary net with positive context is a quadruple $N = (P, T, F, C_+)$, where (P, T, F) is an elementary net and $C_+ \subseteq P \times T$ is a positive context relation satisfying $(F \cup F^{-1}) \cap C_+ = \emptyset$. For a transition t, ${}^+t = \{p \in P \mid (p, t) \in C_+\}$ is the* positive context *of t.*

A transition t is enabled to occur in a marking m iff $({}^\bullet t \cup {}^+t) \subseteq m \wedge (m \setminus {}^\bullet t) \cap t^\bullet = \emptyset$. Its occurrence leads to the marking $m' = (m \setminus {}^\bullet t) \cup t^\bullet$.

The positive context of a transition is the set of places which are tested on presence of a token as a neccessary condition for the possible occurrence of the transition. As usual, elements of the positive context relation are graphically expressed by arcs ending with a black bullet (so called read arcs). An elementary net with positive context is shown in Figure 8.

In comparison to elementary nets without context, an information element consists of two disjoint components: the set of write places and the set of positive context places. Information elements are independent, if each component of the first element is disjoint with each component of the second element except positive contexts, which may be overlapping. This reflects the fact that concurrent testing on presence of a token is allowed.

For the rest of this section, let $N = (P, T, F, C_+)$ be an elementary net with positive context.

Formally, we have $\mathcal{M} = (M, +) = (2^P, \cup)$. The set of information elements is given by $I = \{(w, p) \in 2^P \times 2^P \mid w \cap p = \emptyset\}$. The independence relation is defined by $dom_{\dot{+}} = \{((w, p), (w', p')) \mid w \cap w' = w \cap p' = w' \cap p = \emptyset\}$, and the operation $\dot{+}$ by $(w, p) \dot{+} (w', p') = (w \cup w', p \cup p')$.

$\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$ satisfies the properties defined in Section 2.

To define a $(\mathcal{M}, \mathcal{I})$-net corresponding to the elementary net with positive context $N = (P, T, F, C_+)$, we need to define the mappings $pre, post : T \to M$ which attach an initial and final marking to every transition, and the mapping $inf : M \cup T \to I$ which assigns an information element to every marking $m \in M$ and every transition $t \in T$:

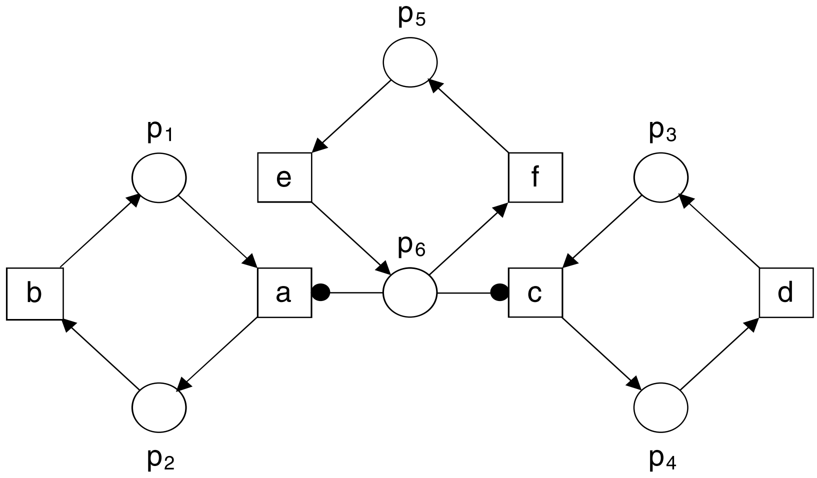

Fig. 8. An example of an elementary net with positive context. Observe that ${}^{\bullet}a = \{p_1\}$, $a^{\bullet} = \{p_2\}$ and ${}^{+}a = \{p_6\}$. Therefore, transition a is enabled to occur if the places p_1 and p_6 are marked and the place p_2 is unmarked. Its occurrence removes a token from p_1 and adds a token to p_2.

- A transition t has the initial marking $pre(t) = {}^{\bullet}t \cup {}^{+}t$ and the final marking $post(t) = t^{\bullet} \cup {}^{+}t$.
- A marking m carries the information $\mathit{inf}(m) = (\emptyset, m)$.
- A transition t carries the information about the places which are contained in the pre- or post-set and about its positive context places, i.e. $\mathit{inf}(t) = ({}^{\bullet}t \cup t^{\bullet}, {}^{+}t)$.

For example, transition a from the net in Figure 8 has attached $pre(a) = \{p_1, p_6\}$, $post(a) = \{p_2, p_6\}$ and the information element $\mathit{inf}(a) = (w, p) = (\{p_1, p_2\}, \{p_6\})$. Transition c has attached the information element $\mathit{inf}(c) = (w', p') = (\{p_3, p_4\}, \{p_6\})$. These information elements are independent. They have the common positive context place p_6, but concurrent testing on presence of a token is allowed. On the other hand, transition e with information element $\mathit{inf}(e) = (w'', p'') = (\{p_5, p_6\}, \emptyset)$ is independent neither with a nor with c, because the write place p_6 of e is the positive context place of both a and c, i.e. $w'' \cap p \neq \emptyset$ as well as $w'' \cap p' \neq \emptyset$.

Property (a) from Definition 1 is valid for $(M, +)$, $\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$ and inf defined above.

The following lemma shows that taking such mappings $pre, post, \mathit{inf}$, the partial groupoid $\mathcal{I}$ encodes the occurrence rule.

Lemma 4. *Given an elementary net with positive context, a transition t is enabled to occur in a marking m and its occurrence leads to the marking m' iff there exists a marking x such that $(\mathit{inf}(x), \mathit{inf}(t)) \in dom_{\dot{+}}$, $x + pre(t) = m$ and $x + post(t) = m'$.*

Proof. $\Rightarrow$: Choosing $x = m \setminus ({}^{\bullet}t \cup {}^{+}t)$ we have that $(\mathit{inf}(x), \mathit{inf}(t)) \in dom_{\dot{+}}$ and $m = x + pre(t) = x \cup ({}^{\bullet}t \cup {}^{+}t)$. We have to show that $x + post(t)$ equals m', i.e.

$x \cup (t^\bullet \cup {}^+t) = ((x \cup ({}^\bullet t \cup {}^+t)) \setminus {}^\bullet t) \cup t^\bullet$. This follows from the fact that by definition of elementary nets with positive context ${}^\bullet t \cap {}^+t = \emptyset$.

$\Leftarrow$: Taking any x such that $x \cap ({}^\bullet t \cup t^\bullet) = \emptyset$, we have $({}^\bullet t \cup {}^+t) \subseteq x \cup ({}^\bullet t \cup {}^+t)$. Furthermore (because of ${}^+t \cap {}^\bullet t = {}^+t \cap t^\bullet = \emptyset$) we have $x \cup {}^+t = (x \cup {}^+t \cup {}^\bullet t) \setminus {}^\bullet t$ and $(x \cup {}^+t) \cap t^\bullet = \emptyset$. Therefore t is enabled to occur in $x \cup ({}^\bullet t \cup {}^+t) = x + pre(t)$ and its occurence leads to $x \cup (t^\bullet \cup {}^+t) = x + post(t)$. □

Finally, we construct the greatest closed congruence $\cong$ of $(2^I, \{\dot{+}\}, dom_{\{\dot{+}\}}, \cup)$. Again we define a mapping *supp* which turns out to be the natural homomorphism of this greatest closed congruence. Define two mappings $s_1, s_2 : 2^I \to 2^P$ by

$$s_1(A) = \bigcup_{(w,p) \in A} w \quad \text{and} \quad s_2(A) = \bigcup_{(w,p) \in A} p,$$

and $supp : 2^I \to I$ by $supp\,(A) = (s_1(A), s_2(A) \setminus s_1(A))$.

Lemma 5. *Let $\circ$ be the binary operation on I defined by $(w,p) \circ (w',p') = (w \cup w', (p \cup p') \setminus (w \cup w'))$. Then the mapping supp : $(2^I, \{\dot{+}\}, dom_{\{\dot{+}\}}, \cup) \to (I, \dot{+}, dom_{\dot{+}}, \circ)$ is a surjective closed homomorphism.*

Proof. The operation $\circ$ is well-defined because for any $x, y \in I$, we have $x \circ y \in I$.

(a) *supp* is a homomorphism for the operations $\{\dot{+}\}$ and $\cup$ on 2^I, because both equations $supp\,(A \{\dot{+}\} A') = supp\,(A) \dot{+} supp\,(A')$ (whenever both sides are defined) and $supp\,(A \cup A') = supp\,(A) \circ supp\,(A')$ follow directly from the properties of $\cup$.

(b) We show the closedness of *supp*, that is

$$(A, A') \in dom_{\{\dot{+}\}} \iff (supp\,(A), supp\,(A')) \in dom_{\dot{+}}$$

for any two $A, A' \subseteq I$. Denote $s_1(A) = \overline{w}$, $s_2(A) = \overline{p}$, $s_1(A') = \overline{w}'$ and $s_2(A') = \overline{p}'$. Then

$$\begin{aligned} &\forall (w,p) \in A,\ \forall (w',p') \in A' : \quad w \cap w' = (w \cup w') \cap (p \cup p') = \emptyset \\ \iff &\overline{w} \cap \overline{w}' = \emptyset \wedge (\overline{w} \cup \overline{w}') \cap ((\overline{p} \setminus \overline{w}) \cup (\overline{p}' \setminus \overline{w}')) = \emptyset. \end{aligned}$$

(c) The mapping *supp* is surjective, because, for any $(w,p) \in I$, we have $supp\,(\{(w,p)\}) = (w,p)$.

□

Lemma 6. *The closed congruence $\cong \subseteq 2^I \times 2^I$ defined by*

$$A \cong B \iff supp(A) = supp(B)$$

is the greatest closed congruence on $(2^I, \{\dot{+}\}, dom_{\{\dot{+}\}}, \cup)$.

Proof. We will show that any congruence $\approx$ such that $\cong$ is a proper subset of $\approx$ is not closed. Assume there are $A, A' \in 2^I$ such that $A \approx A'$ but $A \ncong A'$. Then $supp(A) \neq supp(A')$.

We define a set $C \in 2^I$ such that $(A, C) \in dom_{\{+\}}$ but $(A', C) \notin dom_{\{+\}}$ or vice versa (which implies that $\approx$ is not closed). If $supp(A) = (\overline{w}, \overline{p})$ and $supp(A') = (\overline{w}', \overline{p}')$, then $\overline{w} \cap \overline{p} = \overline{w}' \cap \overline{p}' = \emptyset$ (by definition of I) and $\overline{p} \neq \overline{p}' \vee \overline{w} \neq \overline{w}'$ (since $supp(A) \neq supp(A')$).

Let $\overline{w} \neq \overline{w}'$. Without loss of generality we assume $\overline{w}' \setminus \overline{w} \neq \emptyset$. Set $C = \{(c_w, c_p)\}$ with $c_w = \emptyset$ and $c_p = \overline{w}' \setminus \overline{w}$. Then $c_w \cap \overline{w} = c_w \cap \overline{p} = c_p \cap \overline{w} = \emptyset$, but $c_p \cap \overline{w}' \neq \emptyset$, i.e. $(A, C) \in dom_{\{+\}}$, but $(A', C) \notin dom_{\{+\}}$.

Now let $\overline{w} = \overline{w}'$ and $\overline{p} \neq \overline{p}'$. Without loss of generality we assume $\overline{p}' \setminus \overline{p} \neq \emptyset$. Set $C = \{(c_w, c_p)\}$ with $c_w = (\overline{p}' \setminus \overline{p})$ and $c_p = \emptyset$. Then $c_w \neq \emptyset$, $c_w \cap \overline{w} = c_w \cap \overline{p} = \overline{w} \cap c_p = \emptyset$ and $c_w \cap \overline{p}' \neq \emptyset$, and we are finished. □

Easy computation, using $({}^\bullet t \cup t^\bullet) \cap {}^+t = \emptyset$, proves condition (b) from Definition 1, i.e. $supp(\{inf(t)\}) = supp(\{inf(t), inf(pre(t)), inf(post(t))\})$.

Now we are able to represent an elementary net with positive context as an algebraic $(\mathcal{M}, \mathcal{I})$-net.

Theorem 2. *Let $N = (P, T, F, C_+)$ be an elementary net with positive context, together with $\mathcal{M}, \mathcal{I}, pre, post, inf$ defined throughout this section. Then the quadruple $\mathcal{A}_N = (2^P, T, pre, post)$ together with the mapping inf is an algebraic $(\mathcal{M}, \mathcal{I})$-net. Moreover, it is a corresponding algebraic $(\mathcal{M}, \mathcal{I})$-net to N.*

Remark 2. Taking an elementary net with empty positive context, Theorem 1 and Theorem 2 define algebraic nets $\mathcal{A}_1$ and $\mathcal{A}_2$ generating different sets of process terms: the set of the process terms $\mathcal{P}(\mathcal{A}_1)$ obtained using Theorem 1 is a subset of the set of process terms $\mathcal{P}(\mathcal{A}_2)$ obtained using Theorem 2. By Theorem 2 terms of the form $id_a \parallel id_a$ are allowed for any marking a. However, because we have $id_a \parallel id_a \sim id_a$, and id_a belongs to $\mathcal{P}(\mathcal{A}_1)$, the partial algebra $\mathcal{P}(\mathcal{A}_1)/_\sim$ according to Theorem 1 and the partial algebra $\mathcal{P}(\mathcal{A}_2)/_\sim$ according to Theorem 2 are isomorphic.

A possible process term of the net from Figure 8 is $\alpha = (e \parallel \{p_1, p_3\}); (a \parallel c); (b \parallel f \parallel d) : \{p_1, p_3, p_5\} \to \{p_1, p_3, p_5\}$ with the information $Inf(\alpha) = (\{p_1, p_2, p_3, p_4, p_5, p_6\}, \emptyset)$. Observe, that the place p_6, which is a write place of e and f but the positive context place of a and c appears as a write place of α.

In the case of elementary nets without context we have

$$\{inf(t)\} \cong \{inf(pre(t)), inf(post(t))\}.$$

That means that the information of a transition can be derived from the information of its initial and final marking. However, as it is illustrated by elementary nets with positive context, this is not the general case. For elementary nets with positive context $inf(t)$ contains more detailed information. This information about the nature of places distinguishes places whose state is changed by the occurrence of a transition and those places which are only tested.

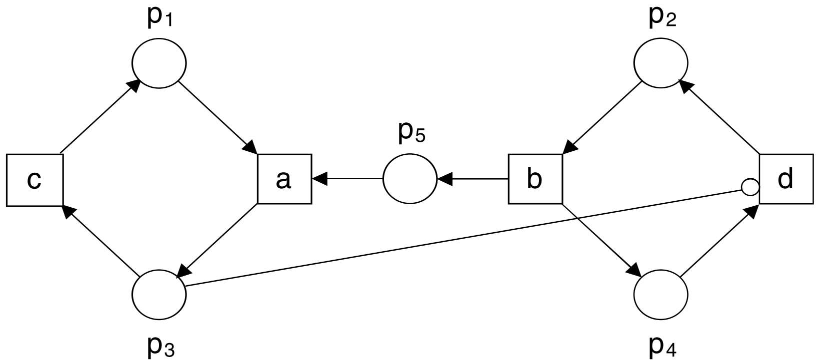

Fig. 9. An example of an elementary net with negative context. Observe that ${}^\bullet d = \{p_4\}$, $d^\bullet = \{p_2\}$ and ${}^- d = \{p_3\}$. Therefore, transition d is enabled to occur if p_4 is marked and both p_2 and p_3 are unmarked. Its occurrence removes a token from p_4 and adds a token to p_2.

6 Elementary Nets with Negative Context

In this section we represent elementary nets with negative context as algebraic $(\mathcal{M}, \mathcal{I})$-nets.

Definition 6 (Elementary net with negative context). *An elementary net with negative context is a quadruple $N = (P, T, F, C_-)$, where (P, T, F) is an elementary net and $C_- \subseteq P \times T$ is a negative context relation satisfying $(F \cup F^{-1}) \cap C_- = \emptyset$. For a transition t, ${}^- t = \{p \in P \mid (p, t) \in C_-\}$ is the* negative context *of t.*

A transition t is enabled in a marking m iff ${}^\bullet t \subseteq m \wedge (m \setminus {}^\bullet t) \cap ({}^- t \cup t^\bullet) = \emptyset$[2]. Its occurrence leads to the marking $m' = (m \setminus {}^\bullet t) \cup t^\bullet$.

The negative context of a transition t is the set of places which are tested on absence of a token for the possible occurrence of a transition. Elements of the negative context relation are graphically expressed by arcs ending with a circle (so called inhibitor arcs). Figure 9 shows an elementary net with negative context.

Similarly to elementary nets with positive context, we need information elements which consist of two disjoint components: the set of write places, and the set of negative context places. Concurrent composition of information elements is allowed if each component of the first element is disjoint with each component of the second element except negative contexts, which may be overlapping. This reflects the fact that concurrent testing on absence of a token is allowed.

Formally, we have the same algebra $\mathcal{M}$ for markings and the same partial algebra $\mathcal{I}$ for information elements as for elementary nets with positive context, and therefore requirements from Section 2 are fulfilled.

We define $pre, post$ by $pre(t) = {}^\bullet t$, $post(t) = t^\bullet$ for each $t \in T$ and inf by $\mathit{inf}(m) = (m, \emptyset)$ for each $m \in 2^P$ and $\mathit{inf}(t) = ({}^\bullet t \cup t^\bullet, {}^- t)$ for each $t \in T$.

[2] Remember that ${}^- t \cap {}^\bullet t = {}^- t \cap t^\bullet = \emptyset$ but ${}^\bullet t \cap t^\bullet$ can be nonempty.

For example, transition d from the net in Figure 9 has attached $pre(d) = \{p_4\}$, $post(d) = \{p_2\}$ and the information element $\mathit{inf}(d) = (w, p) = (\{p_2, p_4\}, \{p_3\})$.

Property (a) from Definition 1 is fulfilled and $\mathcal{I}$ encodes the occurrence rule of the net with negative context. Moreover, property (b) from Definition 1 is preserved. So, we can formulate the following theorem.

Theorem 3. *Given an elementary net with negative context $N = (P, T, F, C_-)$ together with $\mathcal{M}, \mathcal{I}, pre, post, \mathit{inf}$ defined in this section, the quadruple $\mathcal{A}_N = (2^P, T, pre, post)$ together with the mapping inf is a $(\mathcal{M}, \mathcal{I})$-net. Moreover, it is a corresponding $(\mathcal{M}, \mathcal{I})$-net to N.*

Remark 3. For nets with positive context, idle tokens generated by an elementary process term m can be concurrently composed with each other. Hence the respective places belong to the second component representing the context. However, for nets with negative context, an additional token can spoil the enabledness of a transition. So, for this class places carrying tokens generated by elementary process terms m belong to the first component representing write places. This way, a concurrent composition of a process term using a place for inhibition with a process term using the same place for an (idle or moving) token is prevented.

A possible process term of the net from Figure 9 is

$$(d \parallel \{p_1, p_5\}); (a \parallel \{p_2\}); (b \parallel c) : \{p_1, p_4, p_5\} \to \{p_1, p_4, p_5\}$$

with information $(\{p_1, p_2, p_3, p_4, p_5\}, \emptyset)$.

7 Elementary Nets with Mixed Context

In this section we associate to an elementary net with (mixed) context an algebraic $(\mathcal{M}, \mathcal{I})$-net.

Definition 7 (Elementary net with (mixed) context). *An elementary net with (mixed) context is a five-tuple $N = (P, T, F, C_+, C_-)$, where (P, T, F) is an elementary net, and $C_+, C_- \subseteq P \times T$ are positive and negative context relations satisfying $(F \cup F^{-1}) \cap (C_+ \cup C_-) = C_+ \cap C_- = \emptyset$. For a transition t, ${}^\bullet t, t^\bullet$, ${}^+t$ and ${}^-t$ are defined as in the previous sections.*

A transition t is enabled to occur in a marking m iff $({}^\bullet t \cup {}^+t) \subseteq m \wedge (m \setminus {}^\bullet t) \cap ({}^-t \cup t^\bullet) = \emptyset$. Its occurrence leads to the marking $m' = (m \setminus {}^\bullet t) \cup t^\bullet$, in symbols $m \xrightarrow{t} m'$.

Figure 10 shows an elementary net with (mixed) context.

Again we have $\mathcal{M} = (M, +) = (2^P, \cup)$. An information element consists of three disjoint components: the set of write places, the set of positive context places and the set of negative context places. Information elements are independent if each component of the first element is disjoint from each component

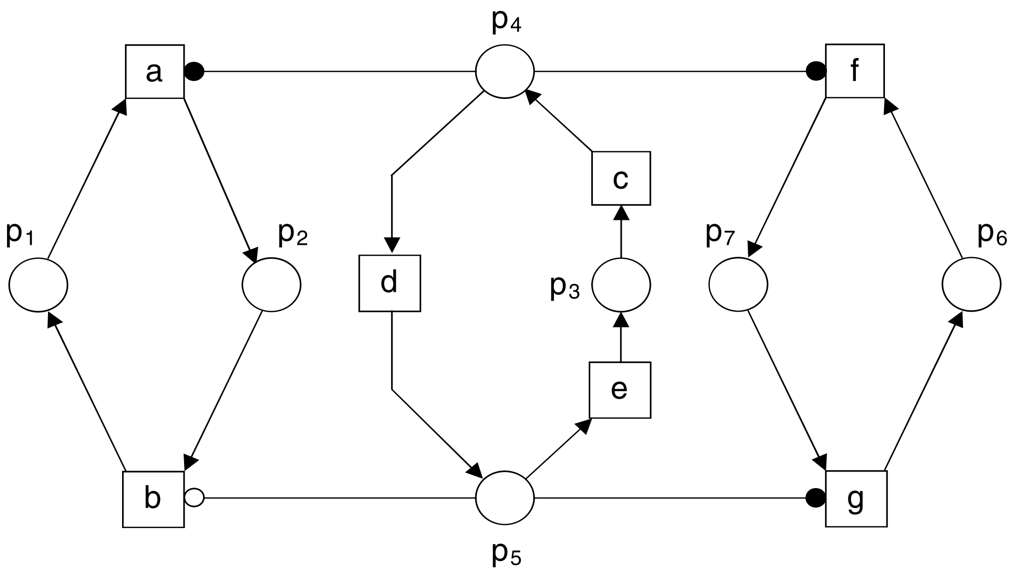

Fig. 10. An elementary net with (mixed) context.

of the second element, except positive contexts (the second components) and negative contexts (the third components). This reflects the fact that concurrent testing on presence of a token as well as concurrent testing on absence of a token is allowed.

Formally, we define the set of information elements

$$I = \{(w, p, n) \in 2^P \times 2^P \times 2^P \mid w \cap p = w \cap n = p \cap n = \emptyset\},$$

together with the independence relation

$$\begin{aligned} dom_{\dot{+}} = \{((w, p, n), (w', p', n')) \mid w \cap w' &= w \cap (p' \cup n') \\ &= w' \cap (p \cup n) = p \cap n' = p' \cap n = \emptyset\}, \end{aligned}$$

and the operation

$$(w, p, n) \dot{+} (w', p', n') = (w \cup w', p \cup p', n \cup n').$$

For $\mathcal{I} = (I, dom_{\dot{+}}, \dot{+})$ the requirements from Section 2 are fulfilled.

To define a $(\mathcal{M}, \mathcal{I})$-net corresponding to an elementary net with context $N = (P, T, F, C_+, C_-)$, we need to define the mappings $pre, post : T \to M$ attaching an initial and final marking to every transition t, and the function $\mathit{inf} : M \cup T \to I$ assigning an information to every marking m and every transition t:

– A transition t has the initial marking $pre(t) = {}^\bullet t \cup {}^+t$ and the final marking $post(t) = t^\bullet \cup {}^+t$.
– A marking m carries the information $\mathit{inf}(m) = (\emptyset, m, \emptyset)$.
– A transition t carries information about write places and extra information about positive and negative context places, i.e. $\mathit{inf}(t) = ({}^\bullet t \cup t^\bullet, {}^+t, {}^-t)$.

For example, transition b from the net in Figure 10 has attached $pre(b) = \{p_2\}$, $post(b) = \{p_1\}$ and the information element $\mathit{inf}(b) = (w, p, n) = (\{p_1, p_2\}, \emptyset, \{p_5\})$, while transition g has attached the information element $\mathit{inf}(g) = (w', p', n') = (\{p_6, p_7\}, \{p_5\}, \emptyset)$. These information elements are not independent, because the negative context place p_5 of b is the positive context place of g, i.e. $n \cap p' \neq \emptyset$.

The mapping inf satisfies property (a) from Definition 1.

Similarly to Lemma 4 one can show for the functions $pre, post, \mathit{inf}$ that the partial algebra $\mathcal{I}$ encodes the occurrence rule of the net with mixed context.

Again, we have to find the greatest closed congruence $\cong$ of $(2^I, \{\dot{+}\}, dom_{\{\dot{+}\}}, \cup)$. We define a mapping $supp$ which turns out to be the natural homomorphism of this greatest closed congruence.

Define three mappings $s_1, s_2, s_3 : 2^I \to 2^P$ by

$$s_1(A) = \bigcup_{(w,p,n)\in A} w, \quad s_2(A) = \bigcup_{(w,p,n)\in A} p \quad \text{and} \quad s_3(A) = \bigcup_{(w,p,n)\in A} n.$$

Define $s : 2^I \to 2^P$ by $s(A) = s_1(A) \cup (s_2(A) \cap s_3(A))$.

Finally, define $supp : 2^I \to I$ by $supp(A) = (s(A), s_2(A)\backslash s(A), s_3(A)\backslash s(A))$.

Lemma 7. *Let $\circ$ be the binary operation on I defined by*

$$(w, p, n) \circ (w', p', n') = supp(\{(w, p, n), (w', p', n')\}).$$

Then the mapping $supp : (2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup) \to (I, dom_{\dot{+}}, \dot{+}, \circ)$ is a surjective closed homomorphism.

Proof. First we show the closedness of $supp$, i.e.

$$(A, A') \in dom_{\{\dot{+}\}} \iff (supp(A), supp(A')) \in dom_{\dot{+}}.$$

We write shortly s_1, s_2, s_3 and s to denote $s_1(A)$, $s_2(A)$, $s_3(A)$ and $s(A)$ resp. s_1', s_2', s_3' and s' to denote $s_1(A')$, $s_2(A')$, $s_3(A')$ and $s(A')$.

$\Rightarrow$: Suppose that $(A, A') \in dom_{\{\dot{+}\}}$ but $(supp(A), supp(A')) \notin dom_{\dot{+}}$.

Case 1: $s \cap s' \neq \emptyset$, i.e. $(s_1 \cup (s_2 \cap s_3)) \cap (s_1' \cup (s_2' \cap s_3')) \neq \emptyset$.

- $s_1 \cap s_1' \neq \emptyset$ contradicts $\forall (w, p, n) \in A, (w', p', n') \in A' : w \cap w' = \emptyset$,
- $s_1 \cap (s_2' \cap s_3') \neq \emptyset$ contradicts $\forall (w, p, n) \in A, (w', p', n') \in A' : w \cap (p \cup n) = \emptyset$,
- $(s_2 \cap s_3) \cap (s_2' \cap s_3') \neq \emptyset$ contradicts $\forall (w, p, n) \in A, (w', p', n') \in A' : p \cap n' = \emptyset$.

Case 2: $(s_2 \setminus s) \cap s' \neq \emptyset$, i.e. $(s_2 \setminus (s_1 \cup (s_2 \cap s_3))) \cap (s_1' \cup (s_2' \cap s_3')) \neq \emptyset$.

- $(s_2 \setminus (s_1 \cup (s_2 \cap s_3))) \cap s_1' \neq \emptyset$ contradicts $\forall (w, p, n) \in A, (w', p', n') \in A' : p \cap w' = \emptyset$.
- $(s_2 \setminus (s_1 \cup (s_2 \cap s_3))) \cap (s_2' \cap s_3') \neq \emptyset$ contradicts $\forall (w, p, n) \in A, (w', p', n') \in A' : p \cap n' = \emptyset$.

All remaining cases are similar.

$\Leftarrow$: Suppose that $(A, A') \notin dom_{\{\dot{+}\}}$ but $(\mathit{supp}\,(A), \mathit{supp}\,(A')) \in dom_{\dot{+}}$.

Case 1: $\exists (w, p, n) \in A, (w', p', n') \in A' : w \cap w' \neq \emptyset$ contradicts $s \cap s' = \emptyset$.

Case 2: $\exists (w, p, n) \in A, (w', p', n') \in A' : p \cap w' \neq \emptyset$:

$$-(p \cap w') \cap ((\bigcup_{(x,y,z)\in A} x) \cup (\bigcup_{(x,y,z)\in A} z)) \neq \emptyset \text{ contradicts } s \cap s' = \emptyset,$$
$$-(p \cap w') \cap ((\bigcup_{(x,y,z)\in A} x) \cup (\bigcup_{(x,y,z)\in A} z)) = \emptyset \text{ contradicts } (s_2 \setminus s) \cap s' = \emptyset.$$

All remaining cases are similar.

Now we show that $\mathit{supp}\,(A\{\dot{+}\}A') = \mathit{supp}\,(A) \dot{+} \mathit{supp}\,(A')$, whenever defined. Let $\mathit{supp}\,(A\{\dot{+}\}A') = (w, p, n)$, where $w = s_1 \cup s_1' \cup ((s_2 \cup s_2') \cap (s_3 \cup s_3'))$, $p = (s_2 \cup s_2') \setminus w$ and $n = (s_3 \cup s_3') \setminus w$. Since $(\mathit{supp}\,(A), \mathit{supp}\,(A')) \in dom_{\dot{+}}$, we have

$$(s_2 \setminus s) \cap (s_3' \setminus s') = s \cap s' = (s_2' \setminus s') \cap (s_3 \setminus s) = \emptyset, \tag{1}$$
$$(s_2 \setminus s) \cap s' = s \cap s' = (s_2' \setminus s') \cap s = \emptyset. \tag{2}$$

Equations (1) and (2) imply $(s_2 \cap s_3') = (s_2' \cap s_3) = \emptyset$. This gives $w = s_1 \cup (s_2 \cap s_3) \cup s_1' \cup (s_2' \cap s_3') = s \cup s'$. Together with equation (2) this gives $s_2 \cap s' = s_2' \cap s = \emptyset$. Then $p = (s_2 \setminus s) \cup (s_2' \setminus s')$. Similarly, $n = (s_3 \setminus s) \cup (s_3' \setminus s')$.

Finally, we have to show that

$$\mathit{supp}\,(A \cup A') = \mathit{supp}\,(A) \circ \mathit{supp}\,(A') = \mathit{supp}\,(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\}).$$

We have $s = s_1 \cup (s_2 \cap s_3)$ and $s' = s_1' \cup (s_2' \cap s_3')$, and therefore

$$s_1 \cup s_1' \subseteq s \cup s' \subseteq s_1 \cup s_1' \cup ((s_2 \cup s_2') \cap (s_3 \cup s_3')) = s(A \cup A').$$

Since $s(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\}) = s \cup s' \cup (((s_2 \setminus s) \cup (s_2' \setminus s')) \cap ((s_3 \setminus s) \cup (s_3' \setminus s')))$, we have $s(A \cup A') = s(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\})$. Similarly

$$s_2(A \cup A') \setminus s(A \cup A') = s_2(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\}) \setminus s(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\})$$

and

$$s_3(A \cup A') \setminus s(A \cup A') = s_3(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\}) \setminus s(\{\mathit{supp}\,(A), \mathit{supp}\,(A')\}).$$

To show surjectivity, let $(w, p, n) \in I$. Then $\mathit{supp}\,(\{(w, p, n)\}) = (w, p, n)$. □

Lemma 8. *The closed congruence $\cong \subseteq 2^I \times 2^I$ defined by*

$$A \cong B \iff \mathit{supp}\,(A) = \mathit{supp}\,(B)$$

is the greatest closed congruence on the partial algebra $\mathcal{X} = (2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$.

Proof. Assume there is a closed congruence $\approx$ on $\mathcal{X}$ with $\cong \subsetneq \approx$. Let $A, A' \in 2^I$ with $A \approx A'$ but $A \not\cong A'$. This means $supp\,(A) \neq supp\,(A')$. We will define a set $C \in 2^I$ with $(A, C) \in dom_{\{\dotplus\}}$ and $(A', C) \notin dom_{\{\dotplus\}}$ or vice versa, what contradicts the closedness of $\approx$.

Let $supp\,(A) = (\overline{w}, \overline{p}, \overline{n})$ and $supp\,(A') = (\overline{w}', \overline{p}', \overline{n}')$. Then $\overline{w} \neq \overline{w}'$ or $\overline{p} \neq \overline{p}'$ or $\overline{n} \neq \overline{n}\,'$.

Assume first that $\overline{w}\,' \setminus \overline{w} \neq \emptyset$. Set $C = \{(\emptyset, \overline{w}\,' \setminus (\overline{w} \cup \overline{n}), \overline{n})\}$. Clearly, $(A, C) \in dom_{\{\dotplus\}}$. If $\overline{w}\,' \setminus \overline{w} \subseteq \overline{n}$ then $\overline{w}\,' \cap \overline{n} \neq \emptyset$ and therefore $(A', C) \notin dom_{\{\dotplus\}}$. If $\overline{w}\,' \setminus \overline{w} \not\subseteq \overline{n}$ then $\overline{w}\,' \cap (\overline{w}\,' \setminus (\overline{w} \cup \overline{n})) \neq \emptyset$ and therefore $(A', C) \notin dom_{\{\dotplus\}}$.

Now assume $\overline{w} = \overline{w}\,'$ and $\overline{p}\,' \setminus \overline{p} \neq \emptyset$. Set $C = \{(\emptyset, \emptyset, \overline{p}\,' \setminus \overline{p})\}$. Assume finally $\overline{w} = \overline{w}\,'$ and $\overline{n}\,' \setminus \overline{n} \neq \emptyset$. Set $C = \{(\emptyset, \overline{n}\,' \setminus \overline{n}, \emptyset)\}$. In both previous cases $(A, C) \in dom_{\{\dotplus\}}$ but $(A', C) \notin dom_{\{\dotplus\}}$. □

The partial algebra $(2^I, dom_{\{\dotplus\}}, \{\dotplus\}, \cup)/_{\cong}$ is isomorphic to the partial algebra $(I, dom_{\dotplus}, \dotplus, \circ)$. For elementary nets with context we only have to use one element of the set I as the information of a process term. This element consists of three sets of places - the set of write places, the set of positive context places which are not write places, and the set of negative context places which are not write places.

For example, the process term $\alpha = a; (b \parallel \{p_4\}) : \{p_1, p_4\} \to \{p_1, p_4\}$ of the net in Figure 10 has the information $\mathit{Inf}(\alpha) = (\{p_1, p_2\}, \{p_4\}, \{p_5\})$ and the process term $\beta = f : \{p_4, p_6\} \to \{p_4, p_7\}$ has the information $\mathit{Inf}(\beta) = (\{p_6, p_7\}, \{p_4\}, \emptyset)$. Observe that they can be concurrently composed yielding the process term $\gamma = \alpha \parallel \beta = (a; (b \parallel \{p_4\})) \parallel f : \{p_1, p_4, p_6\} \to \{p_1, p_4, p_7\}$ with $\mathit{Inf}(\gamma) = (\{p_1, p_2, p_6, p_7\}, \{p_4\}, \{p_5\})$.

Property (b) from Definition 1 is valid, and therefore we can give the theorem:

Theorem 4. *Given an elementary net with (mixed) context $N = (P, T, F, C_+, C_-)$ together with $\mathcal{M}, \mathcal{I}, pre, post, \mathit{inf}$ defined in this section, the quadruple $\mathcal{A}_N = (2^P, T, pre, post)$ together with the mapping inf is an algebraic $(\mathcal{M}, \mathcal{I})$-net. Moreover, it is a corresponding algebraic $(\mathcal{M}, \mathcal{I})$-net to the net N.*

Remark 4. Similarly to Remark 2, given an elementary net with negative context, the equivalence classes of process terms obtained using Theorems 3 and 4 are isomorphic.

8 Relationship between Process Terms and Processes of Elementary Nets with Context

In this section we prove for elementary nets with mixed context a one-to-one correspondence between the obtained non-sequential semantics and the partial-order based semantics obtained in the usual way using process nets. Analogous results hold for elementary nets without context, for elementary nets with (only) positive context, and for elementary nets with (only) negative context.

8.1 Process Semantics of Elementary Nets with Context

In this subsection we give the definition of partial-order based process semantics of elementary nets with context as introduced in [19].

We say that a marking m' is reachable from a marking m, if $m = m'$ or if there is a finite sequence of transitions $t_1, \ldots, t_n$ such that

$$m \xrightarrow{t_1} m_1 \ldots m_{n-1} \xrightarrow{t_n} m'.$$

An elementary net with positive context is said to be contact-free w.r.t. an initial marking m_0, if for each marking m reachable from m_0 and each transition $t : ({}^\bullet t \cup {}^+t) \subseteq m \Rightarrow t^\bullet \cap m = \emptyset$.

As it is shown in [19], an elementary net with mixed context can be transformed via complementation into a contact-free elementary net with positive context exhibiting the same behaviour. For technical reasons we assign complement-places (co-places) to every place. The complementation is defined as follows:

Definition 8 (Complementation). *Given an elementary net with context $N = (P, T, F, C_+, C_-)$, let P' be a set satisfying $|P'| = |P|$ and $P' \cap (P \cup T) = \emptyset$, and let $c : P \to P'$ be a bijection.*

The complementation $\overline{N} = (\overline{P}, \overline{T}, \overline{F}, \overline{C_+})$ of N is defined by

$$\begin{aligned} \overline{P} &= P \cup P', \\ \overline{T} &= T, \\ \overline{F} &= F \cup \{(t, c(p)) \mid (p,t) \in F \wedge (t,p) \notin F\} \\ &\quad \cup \{(c(p), t) \mid (t,p) \in F \wedge (p,t) \notin F\}, \\ \overline{C_+} &= C_+ \cup \{(c(p), t) \mid (p,t) \in C_-\}. \end{aligned}$$

Given an initial marking m_0 of N, its complementation $\overline{m_0}$ is defined by

$$\overline{m_0} = m_0 \cup \{c(p) \mid p \in P,\, p \notin m_0\}.$$

Given an elementary net with context N, the construction of $\overline{N}$ is unique up to isomorphism.

Proposition 2 ([19]). *Given an elementary net with context N and an initial marking m_0 of N, its complementation $\overline{N}$ is contact-free w.r.t. $\overline{m_0}$.*

Figure 11 shows a complementation of the net from Figure 10 w.r.t. the initial marking $\{p_1, p_3, p_6\}$. We only draw the co-places, which are necessary to express negative context places using positive context places and to obtain a contact-free net according to the given initial marking. In Figure 11 the only co-place we need to draw is $\overline{p_5}$.

Definition 9 (The causality relation $\leqslant$ of an elementary net with positive context). *Let $N = (P, T, F, C_+)$ be a net with positive context. Then $\leqslant_N$ denotes the minimal transitive and reflexive binary relation on $P \cup T$ satisfying the following conditions:*

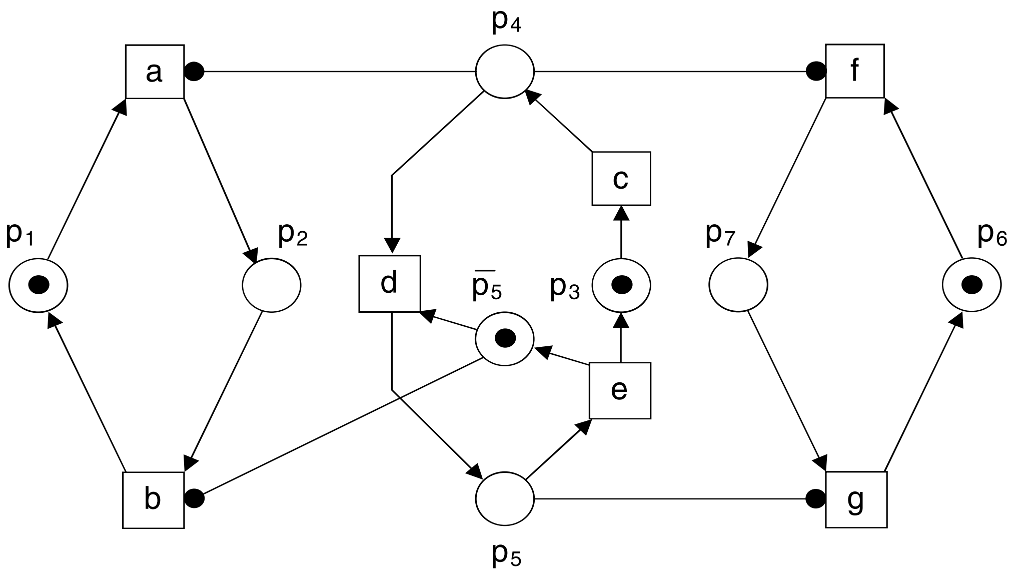

Fig. 11. The complementation of the net from Figure 10 w.r.t. the initial marking $\{p_1, p_3, p_6\}$.

(a) $(x, y) \in F$ *implies* $x \leqslant_N y$.
(b) $(t, p) \in F$ *and* $(p, s) \in C_+$ *implies* $t \leqslant_N s$.
(c) $(p, t) \in C_+$ *and* $(p, s) \in F$ *implies* $t \leqslant_N s$.

Furthermore we define $<_N = \leqslant_N \setminus \{(x, x) \mid x \in P \cup T\}$. *Whenever the net* N *is clear from the context we simply write* $\leqslant$ *instead of* $\leqslant_N$ *and* $<$ *instead of* $<_N$.

The intuition behind the definition of the causality relation is that the flow relation defines causality between transitions in the usual way, i.e.:

– If a place of the post-set of a transition t belongs to the pre-set of a transition s than t causally precedes s,

while the positive context relation defines causality in the following two ways:

– If an occurrence of a transition t produces a token in a place p and a transition s tests the place p on presence of a token, then transition t causally precedes transition s.
– If a transition t tests a place p on the presence of a token and an occurrence of a transition s removes a token from the place p then transition t causally precedes transition s.

Definition 10 (Contextual occurrence net). *A contextual occurrence net is an elementary net with positive context* $K = (B_K, E_K, F_K, C_K)$ *such that*

(a) $\leqslant_K$ *is a partial order,*
(b) $|{}^\bullet b|, |b^\bullet| \leqslant 1$ *for all* $b \in B_K$[3] *(places are unbranched).*

[3] where ${}^\bullet b = \{e \in E_K \mid (e, b) \in F_K\}$ is the *pre-set* of b and $b^\bullet = \{e \in E_K \mid (b, e) \in F_K\}$ is the *post-set* of b

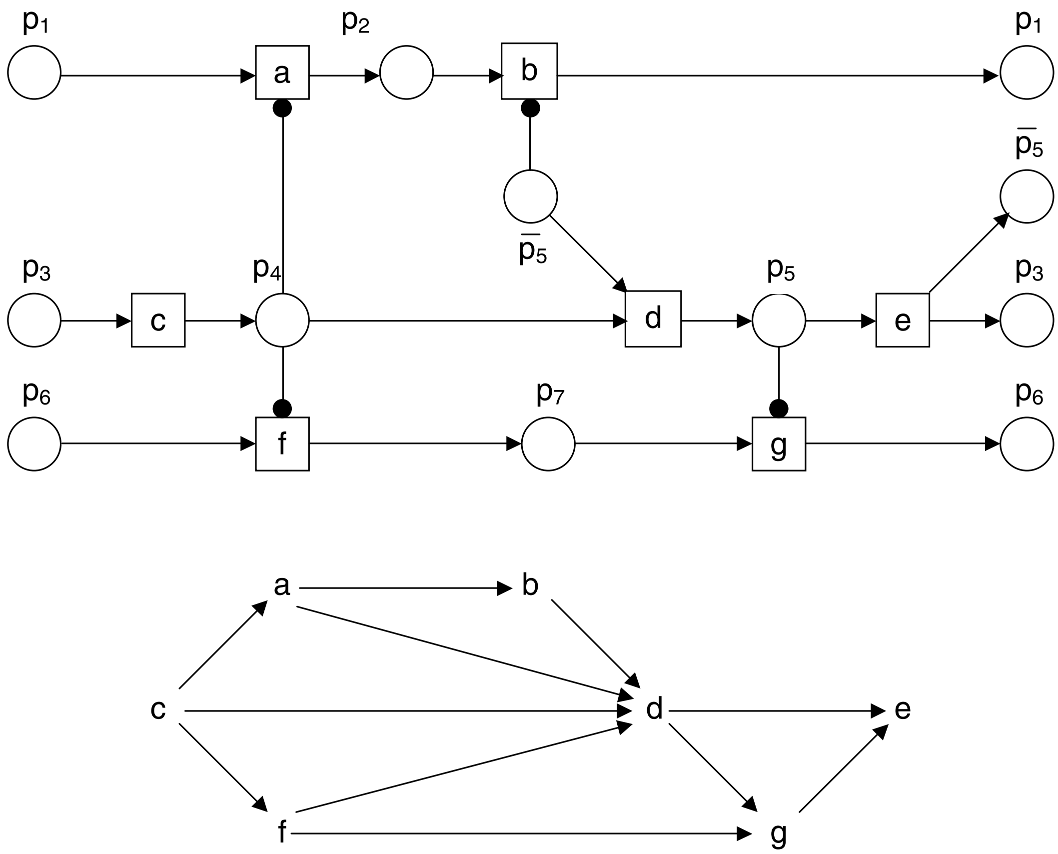

Fig. 12. An example of a contextual occurrence net and the underlying partial order. The contextual occurrence net together with the identity function is a process of the elementary net with context from Figure 10.

Graphically, a contextual occurrence net might have read arcs (arcs for positive context), but each place has at most one ingoing and one outgoing proper arc. Two ordered transitions are connected by a sequence of directed proper arcs (at least one) and undirected read arcs.

An example of a contextual occurrence net and the underlying partial order is shown in Figure 12.

Definition 11 (Co-set, slice). *A co-set of a contextual occurrence net K is a subset $S \subseteq B_K$ such that for no $a, b \in S$: $a <_K b$. A slice is a maximal co-set.*

Denote ${}^\bullet K = \{b \in B_K \mid |{}^\bullet b| = 0\}$ and $K^\bullet = \{b \in B_K \mid |b^\bullet| = 0\}$.

Definition 12 (Process of a contact-free elementary net with positive context). *Let $N = (P, T, F, C_+)$ be an elementary net with positive context and let $m_0 \subseteq P$ be an initial marking of N, such that N is contact-free w.r.t. m_0. A process K of N w.r.t. m_0 is a five-tuple $K = (B_K, E_K, F_K, C_K, \rho_K)$, where (B_K, E_K, F_K, C_K) is a contextual occurrence net and $\rho_K : B_K \cup E_K \to P \cup T$ is a mapping satisfying*

(a) $\rho_K({}^\bullet K) = m_0$,
(b) $\rho_K|_D$ *is injective for every slice D of K,*

(c) $\rho_K(D)$ *is reachable from* m_0 *for every slice* D *of* K*,*
(d) For each $e \in E_K$*:* $\rho({}^\bullet e) = {}^\bullet(\rho(e))$*,* $\rho(e^\bullet) = (\rho(e))^\bullet$ *and* $\rho({}^+e) = {}^+(\rho(e))$*.*

Given a process $K = (B_K, E_K, F_K, C_K, \rho_K)$ of a contact-free elementary net with positive context N (w.r.t. an initial marking m_0) and a set A of isolated places of K (i.e. $\forall b \in A, e \in E_K : b \notin {}^\bullet e \cup e^\bullet \cup {}^+e$) we have that $(B_K \setminus A, E_K, F_K, C_K, \rho_K|_{(B_K\setminus A)\cup E_K})$ is a process of N (w.r.t. the initial marking $m_0' = m \setminus \rho_K(A)$). In other words, after removing isolated places from a process of N we still have a process of N.

For technical reason, we assume that processes contain no isolated places which are mapped to co-places.

Definition 13 (Process of an elementary net with (mixed) context). *Let* $N = (P, T, F, C_+, C_-)$ *be an elementary net with context,* m_0 *be an initial marking of* N*,* $\overline{N} = (\overline{P}, \overline{T}, \overline{F}, \overline{C_+})$ *be the complementation of* N *and* $K = (B_K, E_K, F_K, C_K, \rho_K)$ *be a process of* $\overline{N}$ *w.r.t. the initial marking* $\overline{m_0}$*. Denote by* $B^K_{ICO} = \{b \in B_K \mid \rho_K(b) \notin P \wedge (\forall e \in E_K : b \notin {}^\bullet e \cup e^\bullet \cup {}^+e)\}$ *the set of all isolated places of* K *which are mapped to co-places of* N*. Then* $(B_K \setminus B^K_{ICO}, E_K, F_K, C_K, \rho_K|_{(B_K\setminus B^K_{ICO})\cup E_K})$ *is called a process of* N *w.r.t. an initial marking* m_0*.*

Let $\mathcal{P}(N, m)$ *be the set of all processes of* N *w.r.t. an initial marking* m*. By* $\mathcal{P}(N) = \bigcup_{m\subseteq P} \mathcal{P}(N, m)$ *we denote the set of all processes of* N*.*

The contextual occurrence net in Figure 12 together with the identity function is a process of the elementary net with context from Figure 10.

Processes $K_1 = (B_1, E_1, F_1, C_1, \rho_1)$ and $K_2 = (B_2, E_2, F_2, C_2, \rho_2)$ are isomorphic (in symbols $K_1 \simeq K_2$) iff there exist bijections $\gamma : B_1 \to B_2, \delta : E_1 \to E_2$ such that $\forall b \in B_1, e \in E_1$:

$$\begin{aligned}(b, e) \in F_1 &\iff (\gamma(b), \delta(e)) \in F_2,\\ (e, b) \in F_1 &\iff (\delta(e), \gamma(b)) \in F_2,\\ (b, e) \in C_1 &\iff (\gamma(b), \delta(e)) \in C_2,\\ \rho_1(b) = \rho_2(\gamma(b)), &\ \rho_1(e) = \rho_2(\delta(e)).\end{aligned}$$

8.2 Compositionality of Processes

In this section we show how processes of elementary nets with context can be concurrently and sequentially composed. The results are similar to those given for elementary nets without context in [25,26] (sequential composition is due to [19]).

Let $N = (P, T, F, C_+, C_-)$ be an elementary net with context, let $\overline{N} = (\overline{P}, \overline{T}, \overline{F}, \overline{C_+})$ be its complementation, and let c denote the bijection associating co-places to places from P.

For a process $K = (B_K, E_K, F_K, C_K, \rho_K)$ of N, we define:

– ${}^{\diamond}K$ as the set of write places of K mapped by ρ to places of N, formally

$$^{\diamond}K = \{b \in B_K \mid \rho_K(b) \in P \wedge (\exists e \in E_K : \, b \in {}^{\bullet}e \cup e^{\bullet})\},$$

– ${}^{+}K$ as the set of positive context places of K, which do not correspond to negative context places of N and are not write places of K, formally

$$^{+}K = \{b \in B_K \mid \rho_K(b) \in P \wedge (\forall e \in E_K : \, b \notin {}^{\bullet}e \cup e^{\bullet})\},$$

– and ${}^{-}K$ as the set of places, which correspond to negative context places of N and are not write places of K, formally

$$^{-}K = \{b \in B_K \mid \rho_K(b) \notin P \wedge (\forall e \in E_K : \, b \notin {}^{\bullet}e \cup e^{\bullet})\}.$$

We now define *elementary processes* w.r.t. markings and transitions of N.

Definition 14 (Elementary process associated to a marking). *Let $m \subseteq P$ be a marking of N. Then the process*

$$K(m) = (m, \emptyset, \emptyset, \emptyset, id_m)$$

of N is called elementary process associated to m.

Definition 15 (Elementary process associated to a transition). *Let $t \in T$ be a transition of N. Then the process $K(t)$ of net N defined by*

$$K(t) = ({}^{\bullet}t \cup t^{\bullet} \cup {}^{+}t, \{t\}, ({}^{\bullet}t \times \{t\}) \cup (\{t\} \times t^{\bullet}), {}^{+}t \times \{t\}, id_{{}^{\bullet}t \cup t^{\bullet} \cup {}^{+}t \cup \{t\}}),$$

where ${}^{\bullet}t$, $t^{\bullet}$ and ${}^{+}t$ are defined w.r.t. $\overline{N}$, is called elementary process associated to t.

Processes can be composed concurrently and sequentially:

Proposition 3. *Let c be the bijection associating co-places to places of N and c^{-1} its inverse. Let $K_1 = (B_1, E_1, F_1, C_1, \rho_1)$ and $K_2 = (B_2, E_2, F_2, C_2, \rho_2)$ be two processes of N w.r.t. initial markings m_1 and m_2 with disjoint sets of transitions such that $\forall b_1 \in B_1, b_2 \in B_2$:*

$$b_1 = b_2 \Longleftrightarrow (b_1 \in {}^{+}K_1 \cup {}^{-}K_1 \wedge b_2 \in {}^{+}K_2 \cup {}^{-}K_2 \wedge \rho_1(b_1) = \rho_2(b_2)), \tag{3}$$

and

$$\emptyset = \rho_1({}^{\diamond}K_1) \cap \rho_2({}^{\diamond}K_2), \tag{4}$$
$$\emptyset = \rho_1({}^{\diamond}K_1) \cap (\rho_2({}^{+}K_2) \cup c^{-1}(\rho_2({}^{-}K_2))), \tag{5}$$
$$\emptyset = \rho_2({}^{\diamond}K_2) \cap (\rho_1({}^{+}K_1) \cup c^{-1}(\rho_1({}^{-}K_1))), \tag{6}$$
$$\emptyset = \rho_1({}^{+}K_1) \cap c^{-1}(\rho_2({}^{-}K_2)), \tag{7}$$
$$\emptyset = \rho_2({}^{+}K_2) \cap c^{-1}(\rho_1({}^{-}K_1)). \tag{8}$$

Then $K = (B_K, E_K, F_K, C_K, \rho_K)$, where $B = B_1 \cup B_2, E = E_1 \cup E_2, F = F_1 \cup F_2, C = C_1 \cup C_2, \rho = \rho_1 \cup \rho_2$, is a process of N.

Definition 16 (Concurrent composition of processes). *With notions of Proposition 3, the process K is called the concurrent composition of the processes K_1 and K_2. It is denoted by $K_1 \parallel K_2$.*

Proof of Proposition 3. No element of F_i $(i = 1, 2)$ contains glued places, i.e. places in $B_1 \cap B_2$. Therefore, (B, E, F, C) is an occurrence net.

Since $b_1 = b_2 \Longrightarrow \rho_1(b_1) = \rho_2(b_2)$ for any $b_1 \in B_1, b_2 \in B_2$, ρ is well defined.

Every slice D of $K_1 \parallel K_2$ can be written in the form

$$D = D_1 \cup D_2$$

with slices D_1 of K_1 and D_2 of K_2. We show that $\rho|D$ is injective. Suppose that this is not true, i.e. that there exists $b_1 \in D_1$ and $b_2 \in D_2$ satisfying $b_1 \neq b_2$ and $\rho(b_1) = \rho(b_2)$. It is enough to consider the following four situations:

a) $\rho(b_1) \notin P, b_1 \in {}^{\diamond}K_1, b_2 \in {}^{\diamond}K_2$: by construction of complement places, there exists $b_1' \in {}^{\diamond}K_1$ and $b_2' \in {}^{\diamond}K_2$ such that $\rho_1(b_1') = \rho_2(b_2') \in P$, contradicting (4),
b) $\rho(b_1) \in P, b_1 \in {}^{\diamond}K_1, b_2 \in {}^{\diamond}K_2$, contradicting (4),
c) $\rho(b_1) \notin P, b_1 \in {}^{\diamond}K_1, b_2 \in {}^{-}K_2$: from properties of complementation there exists $b_1' \in {}^{\diamond}K_1$ such that $\rho_1(b_1') = c^{-1}(\rho_2(b_2))$, contradicting (5),
d) $\rho(b_1) \in P, b_1 \in {}^{\diamond}K_1, b_2 \in {}^{+}K_2$, contradicting (5).

Take a marking m reachable from $m_0 = \rho({}^{\bullet}K_1)$ and let

$$m_0 \xrightarrow{t_1} m_1 \ldots m_{n-1} \xrightarrow{t_n} m_n = m.$$

Since we replaced negative context by positive context, for any marking m' with $m' \cap \bigcup_{0 \leqslant i \leqslant n} m_i = \emptyset$, the marking $m \cup m'$ is reachable from $m_0 \cup m'$, firing the same sequence of transitions. Using the fact that

$$\rho_1({}^{\bullet}K_1) \cap \rho_2({}^{\bullet}K_2) = \rho_1(D_1) \cap \rho_2(D_2),$$

it is easy to see that $\rho(D)$ is reachable from $\rho({}^{\bullet}(K_1 \parallel K_2)) = \rho_1({}^{\bullet}K_1) \cup \rho_2({}^{\bullet}K_2)$.

Since F_i contains no glued places $(i = 1, 2)$, ρ preserves pre- and post-sets of transitions. The preservation of the positive contexts of transitions follows directly from the construction of C.

Thus, $K = K_1 \parallel K_2$ is a process of $\overline{N}$ w.r.t. the initial marking $\rho({}^{\bullet}K)$.

It remains to show that K is also a process of N w.r.t. the initial marking $\rho({}^{\bullet}K) \cap P$, i.e. there is a process $\overline{K} = (\overline{B}, \overline{E}, \overline{F}, \overline{C}, \overline{\rho})$ of $\overline{N}$ w.r.t. the initial marking $\overline{\rho({}^{\bullet}K) \cap P}$ such that $K = (\overline{B} \setminus B^{\overline{K}}_{ICO}, \overline{E}, \overline{F}, \overline{C}, \overline{\rho}|_{\overline{B} \setminus B^{\overline{K}}_{ICO} \cup \overline{E}})$.

Without loss of generality, suppose that $B \cap \overline{P} = \emptyset$. Set

$$\overline{K} = (B \cup (\{c(p) \mid p \notin \rho({}^{\bullet}K) \cap P\} \setminus \rho(B)), E, F, C, \overline{\rho}),$$

where $\overline{\rho} = \rho$ on $B \cup E$ and $\overline{\rho} = id$ on $\{c(p) \mid p \notin \rho({}^{\bullet}K) \cap P\} \setminus \rho(B)$. Clearly $\overline{K}$ is a process of $\overline{N}$ with respect to the initial marking

$$\rho({}^{\bullet}K) \cup (\{c(p) \mid p \notin \rho({}^{\bullet}K) \cap P\} \setminus \rho(B)).$$

Because K_1 and K_2 have no isolated places which are copies of co-places, also $K_1 \parallel K_2$ contains no isolated places which are mapped to co-places, i.e. $B = \overline{B} \setminus B^{\overline{K}}_{ICO}$.

To prove that $\overline{K}$ is a process of $\overline{N}$ w.r.t. the initial marking

$$\overline{\rho({}^\bullet K) \cap P} = (\rho({}^\bullet K) \cap P) \cup \{c(p) \mid p \notin \rho({}^\bullet K) \cap P\}$$

it suffices to show that

$$(\rho({}^\bullet K) \cap P) \cup \{c(p) \mid p \notin \rho({}^\bullet K) \cap P\} = \rho({}^\bullet K) \cup (\{c(p) \mid p \notin \rho({}^\bullet K) \cap P\} \setminus \rho(B)).$$

To see that the first set is a subset of the second set, observe that all co-places removed from the set $\{c(p) \mid p \notin \rho({}^\bullet K) \cap P\}$ belong to the set $\rho({}^\bullet K)$: Because K has no isolated places which are mapped to co-places, for every place $b \in B$ with $\rho(b) \in P'$ and $c^{-1}(\rho(b)) \notin \rho({}^\bullet K) \cap P$, either $b \in {}^- K$ or $b \in {}^\diamond K$. In the first case, $b \in {}^\bullet K$. In the second case, either $b \in {}^\bullet K$ or there exists $e_1 \in E$ such that $b \in e_1^\bullet$. By construction of the complementation, there exists $b_1 \in {}^\bullet e_1$ such that $\rho(b_1) = c^{-1}(\rho(b))$. By the assumption $c^{-1}(\rho(b)) \notin \rho({}^\bullet K) \cap P$ there exists $e_2 \in E$ such that $b_1 \in e_2^\bullet$. By induction, there exists $b_n \in {}^\bullet K$ such that $\rho(b_n) = \rho(b)$.

To prove that the second set is a subset of the first set, it is enough to show that

$$p \in \rho({}^\bullet K) \Longrightarrow c(p) \notin \rho({}^\bullet K). \tag{9}$$

Assume that this is not true. K_1 and K_2 are processes of N and therefore (9) holds for K_1 and K_2. Without loss of generality, let $b_1 \in {}^\bullet K_1$ and $b_2 \in {}^\bullet K_2$ such that $c(\rho(b_1)) = \rho(b_2)$. We have either $b_1 \in {}^\diamond K_1$ or $b_1 \in {}^+ K_1$. Because K_2 has no isolated placed which are mapped to co-places, there exists $e_2 \in E_2$ such that either $b_2 \in {}^\bullet e_2$ or $b_2 \in {}^+ e_2$. If $b_2 \in {}^\bullet e_2$, by definition of the complementation, there exists $b_2' \in e_2^\bullet$ such that $c(\rho_2(b_2')) = \rho_2(b_2)$ which contradicts $\emptyset = \rho_1({}^\diamond K_1) \cap \rho_2({}^\diamond K_2)$ if $b_1 \in {}^\diamond K_1$, and contradicts $\emptyset = \rho({}^+ K_1) \cap \rho_2({}^\diamond K_2)$ if $b_1 \in {}^+ K_1$. If $b_2 \in {}^+ e_2$, then $b_1 \in {}^\diamond K_1$ contradicts $\emptyset = \rho_1({}^\diamond K_1) \cap c^{-1}(\rho_2({}^- K_2))$, and $b_1 \in {}^+ K_1$ contradicts $\emptyset = \rho_1({}^+ K_1) \cap c^{-1}(\rho_2({}^- K_2))$. $\square$

Given processes K_1, K_2, K_3, K_4 such that $K_1 \parallel K_2, K_3 \parallel K_4$ are defined and $K_1 \simeq K_3, K_2 \simeq K_4$, we have $K_1 \parallel K_2 \simeq K_3 \parallel K_4$, i.e. we have that isomorphism between processes is a congruence w.r.t. the partial operation of concurrent composition defined in the previous proposition.

Proposition 4. *Let $K_1 = (B_1, E_1, F_1, C_1, \rho_1)$ and $K_2 = (B_2, E_2, F_2, C_2, \rho_2)$ be two processes of N with disjoint sets of transitions such that $\forall b_1 \in B_1, b_2 \in B_2$:*

$$b_1 = b_2 \Longleftrightarrow (b_1 \in K_1^\bullet \wedge b_2 \in {}^\bullet K_2 \wedge \rho_1(b_1) = \rho_2(b_2)), \text{ and} \tag{10}$$

$$\rho_1(K_1^\bullet) \cap P = \rho_2({}^\bullet K_2) \cap P. \tag{11}$$

Then $K = (B, E, F, C, \rho)$, where $B = B_1 \cup B_2, E = E_1 \cup E_2, F = F_1 \cup F_2, C = C_1 \cup C_2, \rho = \rho_1 \cup \rho_2$, is a process of N.

Proof. See [19].

Definition 17 (Sequential composition of processes). *With notions of Proposition 4, K is called the sequential composition of the processes K_1 and K_2. It is denoted by $K_1; K_2$.*

Isomorphism between processes is a congruence also w.r.t. the partial operation of sequential composition defined in the previous proposition.

Furthermore, given two isomorphic processes $K_1 \simeq K_2$, we have:

$$\rho_1({}^\bullet K_1) = \rho_2({}^\bullet K_2), \quad \rho_1(K_1^\bullet) = \rho_2(K_2^\bullet),$$

and

$$\rho_1({}^\diamond K_1) = \rho_2({}^\diamond K_2), \quad \rho_1({}^+ K_1) = \rho_2({}^+ K_2), \quad \rho_1({}^- K_1) = \rho_2({}^- K_2).$$

8.3 Relationship between Process Terms and Processes of Elementary Nets with Context

For the most general case of an elementary net with mixed context we prove a one-to-one correspondence between isomorphism classes of its processes and equivalence classes of process terms of the corresponding $(\mathcal{M}, \mathcal{I})$-net from Section 7 with respect to $\sim$. As a consequence, the partial order constructed in a canonical way from an equivalence class of process terms by considering the ordering of transitions of all process terms in the equivalence class coincides with the partial order derived from the corresponding process net.

In the sequel, let $\mathcal{A}_N$ together with inf be the $(\mathcal{M}, \mathcal{I})$-net corresponding to an elementary net with context $N = (P, T, F, C_+, C_-)$, as defined in Section 7. With the help of the above definitions and propositions we will inductively construct isomorphism classes A_α of processes of N associated to process terms $\alpha : a \to b \in \mathcal{P}(\mathcal{A}_N)$ with information $\mathit{Inf}(\alpha)$ according to the four construction rules of process terms. We will also show that processes $K_\alpha \in A_\alpha$ enjoy the following properties:

$$\rho_\alpha({}^\bullet K_\alpha) \cap P = a \text{ and } \rho_\alpha(K_\alpha^\bullet) \cap P = b, \tag{12}$$

$$(\rho_\alpha({}^\diamond K_\alpha),\, \rho_\alpha({}^+ K_\alpha),\, c^{-1}(\rho_\alpha({}^- K_\alpha))) = \mathit{Inf}(\alpha). \tag{13}$$

Proposition 5. *Let $m : m \to m$ be the reflexive process term of a marking m of N with associated information $\mathit{Inf}(m) = (\emptyset, m, \emptyset)$. According to Definition 14, $K(m)$ is a process of N. Clearly the properties (12) and (13) hold for $K(m)$.*

Definition 18 (Isomorphism class of processes associated to markings). *With notions of Proposition 5 define $A_m = [K(m)]_\simeq$ to be the isomorphism class of processes associated with the elementary term m.*

Proposition 6. *Let $t : \mathit{pre}(t) \to \mathit{post}(t)$ be the process term generated by a transition t with associated information $\mathit{Inf}(t) = ({}^\bullet t \cup t^\bullet,\, {}^+t,\, {}^-t)$. The process $K(t)$ of N satisfies properties (12) and (13).*

Definition 19 (Isomorphism class of processes associated to transitions). *With notions of Proposition 6 define $A_t = [K(t)]_{\simeq}$ to be the isomorphism class associated with the elementary term t.*

Proof of Proposition 6. According to Definition 15, $K(t)$ is a process of N. Property (12) follows from ${}^\bullet t \cup {}^+t = pre(t)$ and $t^\bullet \cup {}^+t = post(t)$, where ${}^\bullet t$, $t^\bullet$ and ${}^+t$ are taken w.r.t. N, and ${}^\bullet K(t) = {}^\bullet t \cup {}^+t \cup \{c(p) \mid p \in {}^-t\}$, $K(t)^\bullet = t^\bullet \cup {}^+t \cup \{c(p) \mid p \in {}^-t\}$. Property (13) follows from:

$$ {}^-t = c^{-1}({}^-K(t)). $$

□

Proposition 7. *Let α_1, α_2 be process terms of $\mathcal{A}_N$, such that $\alpha = \alpha_1 \parallel \alpha_2$ is a defined process term. Then there exist processes $K_1 = (B_1, E_1, F_1, C_1, \rho_1) \in A_{\alpha_1}$ and $K_2 = (B_2, E_2, F_2, C_2, \rho_2) \in A_{\alpha_2}$, such that the preconditions for concurrent composition of K_1 and K_2 are fulfilled. Moreover, the process*

$$ K_\alpha = K_1 \parallel K_2 = (B_\alpha, E_\alpha, F_\alpha, C_\alpha, \rho_\alpha), $$

satisfies the properties (12) and (13).

Definition 20 (Isomorphism class of processes associated to concurrent composed process terms). *With notions of Proposition 7 define $A_\alpha = [K_\alpha]_{\simeq}$ to be the isomorphism class associated with the term α.*

Proof of Proposition 7. Take processes $K_1 \in A_{\alpha_1}, K_2 \in A_{\alpha_2}$, such that the sets $B_1 \setminus ({}^+K_1 \cup {}^-K_1), B_2 \setminus ({}^+K_2 \cup {}^-K_2), \overline{P}$ are disjoint, and ${}^+K_i \cup {}^-K_i \subseteq \overline{P} \wedge \rho_i|_{{}^+K_i \cup {}^-K_i} = id$ for $i = 1, 2$ (what can be achieved by an appropriate renaming). Then the precondition (3) formulated in Proposition 3 is fulfilled.

Denoting $\mathit{Inf}(\alpha_1) = (w_1, p_1, n_1)$ and $\mathit{Inf}(\alpha_2) = (w_2, p_2, n_2)$ we have by the definition of $dom_{\parallel}$:

$$ w_1 \cap w_2 = w_1 \cap (p_2 \cup n_2) = w_2 \cap (p_1 \cup n_1) = p_1 \cap n_2 = p_2 \cap n_1 = \emptyset. $$

From property (13) of K_1 and K_2 we have for $i = 1, 2$:

$$ w_i = \rho_i({}^\diamond K_i), \quad p_i = \rho_i({}^+K_i), \quad n_i = c^{-1}(\rho_i({}^-K_i)). $$

Therefore the remaining preconditions for concurrent composition of K_1 and K_2 formulated in Proposition 3 are fulfilled.

We have that

$$ {}^\bullet(K_1 \parallel K_2) = {}^\bullet K_1 \cup {}^\bullet K_2, (K_1 \parallel K_2)^\bullet = K_1^\bullet \cup K_2^\bullet, $$

and, because joined places are neither in ${}^\diamond K_1$ nor in ${}^\diamond K_2$,

$$ \begin{aligned} {}^\diamond(K_1 \parallel K_2) &= {}^\diamond K_1 \cup {}^\diamond K_2, \\ {}^+(K_1 \parallel K_2) &= {}^+K_1 \cup {}^+K_2, \\ {}^-(K_1 \parallel K_2) &= {}^-K_1 \cup {}^-K_2, \end{aligned} $$

which easily implies properties (12) and (13). □

Proposition 8. *Let α_1 and α_2 be two process terms such that $\alpha = \alpha_1;\alpha_2$ is a defined process term. Then there exist $K_1 = (B_1, E_1, F_1, C_1, \rho_1) \in A_{\alpha_1}$ and $K_2 = (B_2, E_2, F_2, C_2, \rho_2) \in A_{\alpha_2}$ such that $K_\alpha = K_1; K_2$ is a defined process, which fulfills properties (12) and (13).*

Definition 21 (Isomorphism class of processes associated to sequential composed process terms). *With notions of Proposition 8 define $A_\alpha = [K_\alpha]_\simeq$ to be the isomorphism class associated with the term α.*

Proof of Proposition 8. Take processes $K_1 \in A_{\alpha_1}, K_2 \in A_{\alpha_2}$ such that the sets $B_1 \setminus K_1^\bullet, B_2 \setminus {}^\bullet K_2, \overline{P}$ are disjoint, $K_1^\bullet \subseteq \overline{P} \wedge \rho_1|_{K_1^\bullet} = id$ and ${}^\bullet K_2 \subseteq \overline{P} \wedge \rho_2|_{{}^\bullet K_2} = id$ (what can be achieved by an appropriate renaming). Then the precondition (10) formulated in Proposition 4 is fulfilled.

From property (12) of processes K_1, K_2 and from $post(\alpha_1) = pre(\alpha_2)$ we have $\rho_1(K_1^\bullet) \cap P = \rho_2({}^\bullet K_2) \cap P$ and therefore precondition (11) formulated in Proposition 4 is fulfilled.

The new process $K_\alpha = K = (B, E, F, C, \rho)$ obviously satisfies property (12).

We have ${}^\diamond K = {}^\diamond K_1 \cup {}^\diamond K_2$ and therefore

$$\rho({}^\diamond K) = \rho_1({}^\diamond K_1) \cup \rho_2({}^\diamond K_2)$$

Moreover, ${}^+K = ({}^+K_1 \cup {}^+K_2) \setminus {}^\diamond K$. Since ρ is injective on ${}^+K_1 \cup {}^+K_2$, we have

$$\rho({}^+K) = (\rho_1({}^+K_1) \cup \rho_2({}^+K_2)) \setminus \rho({}^\diamond K).$$

Let ${}^\diamond K' = \{b \in B \mid \rho(b) \notin P \wedge (\exists e \in E : \, b \in {}^\bullet e \cup e^\bullet)\}$.
Then ${}^-K = ({}^-K_1 \cup {}^-K_2) \setminus {}^\diamond K'$.
By injectivity of ρ on ${}^-K_1 \cup {}^-K_2$ we have $\rho({}^-K) = (\rho_1({}^-K_1) \cup \rho_2({}^-K_2)) \setminus \rho({}^\diamond K')$. By construction of complementation we have $c^{-1}(\rho({}^\diamond K') \subseteq \rho({}^\diamond K)$. Since $p \in \rho({}^\bullet K) \Rightarrow c(p) \notin \rho({}^\bullet K)$, by induction we have

$$p \in \rho(D) \Rightarrow c(p) \notin \rho(D) \tag{14}$$

for each slice D of K. Since each slice of K contains ${}^-K = ({}^-K_1 \cup {}^-K_2) \setminus {}^\diamond K'$, we have $(\rho({}^\diamond K) \setminus c^{-1}(\rho({}^\diamond K'))) \cap \rho({}^-K) = \emptyset$. Thus, we have

$$c^{-1}(\rho({}^-K)) = (c^{-1}(\rho_1({}^-K_1)) \cup c^{-1}(\rho_1({}^-K_1))) \setminus \rho({}^\diamond K).$$

Since each slice of K contains ${}^+K \cup {}^-K$, by (14) we have

$$\rho({}^+K) \cap c^{-1}(\rho({}^-K)) = \emptyset.$$

Thus, process K_α enjoys property (13). □

Definition 22. *Given an elementary net with mixed context N, let $\tau : \mathcal{P}(\mathcal{A}_N) \to (\mathcal{P}(N))/_\simeq$ be the mapping defined by $\tau(\alpha) = A_\alpha$.*

Lemma 9. *Let $K = (B_K, E_K, F_K, C_K, \rho_K)$ be a process of N and $e_1, e_2 \in E_K$ with $e_1 \not\leqslant e_2 \wedge e_2 \not\leqslant e_1$. Then $\rho_K(e_1) \parallel \rho_K(e_2)$ is a defined process term.*

Proof. It suffices to show that:

(a) $(\rho_K({}^\bullet e_1 \cup e_1^\bullet) \cap P) \cap (\rho_K({}^\bullet e_2 \cup e_2^\bullet) \cap P) = \emptyset$,
(b) $((\rho_K({}^+e_1) \cap P) \cup c^{-1}(\rho({}^+e_1) \cap P')) \cap (\rho_K({}^\bullet e_2 \cup e_2^\bullet) \cap P) = \emptyset$, and
(c) $(\rho_K({}^+e_1) \cap P) \cap c^{-1}(\rho_K({}^+e_2) \cap P') = \emptyset$.

(a) follows from: $e_1 \not\leqslant e_2 \wedge e_2 \not\leqslant e_1$ implies that the sets ${}^\bullet e_1 \cup {}^\bullet e_2 \cup {}^+K$ and $e_1^\bullet \cup e_2^\bullet \cup {}^+K$ are subsets of slices of K. Since ρ_K is injective on slices, $\rho_K({}^\bullet e_1) \cap \rho_K({}^\bullet e_2) = \rho_K(e_1^\bullet) \cap \rho_K(e_2^\bullet) = \emptyset$. Assume there is a place $p \in \rho_K({}^\bullet e_1) \cap \rho_K(e_2^\bullet)$ or $p \in \rho_K({}^\bullet e_2) \cap \rho_K(e_1^\bullet)$. Without loss of generality let $p \in \rho_K({}^\bullet e_1) \cap \rho_K(e_2^\bullet)$. There are places $b_1 \in {}^\bullet e_1$ and $b_2 \in e_2^\bullet$ such that $\rho_K(b_1) = \rho_K(b_2)$. Then either $b_1 \not\leqslant b_2 \wedge b_2 \not\leqslant b_1$ (which would be a contradiction to the injectivity of ρ_K on slices) or $b_2 \leqslant b_1$ (which would be a contradiction to $e_2 \not\leqslant e_1 \wedge e_1 \not\leqslant e_2$ by the transitivity of $\leqslant$) or finally $b_1 \leqslant b_2$, which would imply $e_1 \leqslant e_2$ because places are unbranching, what is again a contradiction.

To show (b), assume there is a place $p \in (\rho_K({}^+e_1) \cap P) \cup c^{-1}(\rho_K({}^+e_1) \cap P')) \cap (\rho({}^\bullet e_2 \cup e_2^\bullet) \cap P)$. Then there are places $b_1 \in {}^+e_1$ and $b_2 \in ({}^\bullet e_2 \cup^\bullet e_2) \cap P$ such that $c(\rho_K(b_1)) = \rho_K(b_2)$ or $\rho_K(b_1) = \rho_K(b_2)$. In the first case we observe:

- Assume $b_2 \in {}^\bullet e_2$. By construction of the complementation $\overline{N}$ of N there is a place $b_2' \in e_2^\bullet$ such that $\rho_K(b_1) = \rho_K(b_2')$. We can distinguish 4 situations:
  $b_1 \not\leqslant b_2' \wedge b_2' \not\leqslant b_1$ leads to a contradiction similar as in case (a).
  $b_1 = b_2'$ implies $e_2 \leqslant e_1$.
  $b_1 < b_2'$ implies the existence of a transition $e' \in E$ such that $b_1 \in {}^\bullet e' \wedge e_1 < e'$. Because places are unbranched, this implies $e' < e_2$ and therefore $e_1 < e_2$.
  $b_2' < b_1$ implies the existence of a transition e' such that $b_2 < e'$ and $b_1 \in (e')^\bullet$. It follows $e' < e_1$ and therefore $e_2 < e_1$.
- The proof for $b_2 \in e_2^\bullet$ is similar.

The second case obviously reduces to the situations considered in the first case.

Finally we obtain (c) by assuming that there is a place $p \in (\rho_K({}^+e_1) \cap P) \cap c^{-1}(\rho_K({}^+e_2) \cap P')$. Then there are places $b_1 \in {}^+e_1$ and $b_2 \in {}^+e_2$ with $c(\rho_K(b_1)) = \rho_K(b_2)$. Since $p \in \rho_K({}^\bullet K) \Rightarrow c(p) \notin \rho_K({}^\bullet K)$, by induction we have $p \in \rho_K(D) \Rightarrow c(p) \notin \rho_K(D)$ for each slice D of K. This implies either $b_1 < b_2$ or $b_2 < b_1$ which again gives a contradiction to $e_1 \not\leqslant e_2 \wedge e_2 \not\leqslant e_1$. □

Remark 5. (a) Given process terms α_i, $i = 1, \ldots, 4$ of $\mathcal{A}_N$ such that the terms $\alpha = ((\alpha_1 \parallel \alpha_2); (\alpha_3 \parallel \alpha_4))$ and $\beta = ((\alpha_1; \alpha_3) \parallel (\alpha_2; \alpha_4))$ are defined. Then $\mathit{Inf}(\alpha) = \mathit{Inf}(\beta)$.
(b) For any two process terms α_1 and α_2 such that $\alpha_1 \parallel \alpha_2$ is defined, we have $\alpha_1 \parallel \alpha_2 \sim (\alpha_1; \mathit{post}(\alpha_1)) \parallel (\mathit{pre}(\alpha_2); \alpha_2) \sim (\alpha_1 \parallel \mathit{pre}(\alpha_2)); (\alpha_2 \parallel \mathit{post}(\alpha_1))$ and analogously $\alpha_1 \parallel \alpha_2 \sim (\alpha_2 \parallel \mathit{pre}(\alpha_1)); (\alpha_1 \parallel \mathit{post}(\alpha_2))$

(c) If $(\alpha_1;\alpha_2) \parallel m$, m being a marking, is defined, then we have $(\alpha_1;\alpha_2) \parallel m \sim (\alpha_1 \parallel m);(\alpha_2 \parallel m)$.

Theorem 5. *The mapping* $\tau : \mathcal{P}(\mathcal{A}_N) \to (\mathcal{P}(N))/_{\simeq}$ *is surjective.*

Proof. Let $K = (B, E, F, C, \rho)$ be a process of N. We inductively construct a process term α with $K_\alpha = K$ by the method of maximal steps analogously to the proof of the similar theorem in [6, Theorem 1]: Beginning with the slice $D = {}^\bullet K$, we take all transitions $\{e_1, \ldots, e_m\} \in E$ with ${}^\bullet e_i \subset D$ such that there is no transition $e \in E$ with $e < e_i$ $(1 \leqslant i \leqslant m)$. Then the transitions $\rho(e_1), \ldots, \rho(e_m)$ can be composed by $\parallel$ as process terms. The resulting process term then is sequentially composed with the next one, which we derive by the same procedure now starting with the follower slice of D after firing $e_1, \ldots, e_m$. This is repeated until the follower slice equals $K^\bullet$. $\square$

Theorem 6. *For two process terms* $\alpha, \beta \in \mathcal{P}(\mathcal{A}_N)$, $\alpha \sim \beta$ *implies* $\tau(\alpha) = \tau(\beta)$.

Proof. It is sufficient to show the proposition for every (of the seven) construction rules of $\sim$ (Definition 2).

(1) The proof for the rule (1) is obvious.

(2) Given $\alpha_1, \alpha_2, \alpha_3$ such that terms $(\alpha_1 \parallel \alpha_2) \parallel \alpha_3$ and $\alpha_1 \parallel (\alpha_2 \parallel \alpha_3)$ are defined, take processes

$$K_1 \in A_{\alpha_1},\ K_2 \in A_{\alpha_2},\ K_3 \in A_{\alpha_3},$$

such that sets

$$B_1 \setminus ({}^+K_1 \cup {}^-K_1),\ B_2 \setminus ({}^+K_2 \cup {}^-K_2),\ B_3 \setminus ({}^+K_3 \cup {}^-K_3),\ \overline{P}$$

are disjoint and

$${}^+K_i \cup {}^-K_i \subseteq \overline{P} \wedge \rho_i|_{{}^+K_i \cup {}^-K_i} = id, \quad i \in \{1, 2, 3\}$$

(what can be achieved by an appropriate renaming). Then processes $(K_1 \parallel K_2) \parallel K_3, K_1 \parallel (K_2 \parallel K_3)$ are defined and equal.

(3) Given $\alpha_1, \alpha_2, \alpha_3$ such that terms $(\alpha_1;\alpha_2);\alpha_3$ and $\alpha_1;(\alpha_2;\alpha_3)$ are defined, let G be a set satisfying $|G| = |\overline{P}|$ and $G \cap \overline{P} = \emptyset$, and let $g : G \to \overline{P}$ be a bijection. Take processes

$$K_1 \in A_{\alpha_1},\ K_2 \in A_{\alpha_2},\ K_3 \in A_{\alpha_3},$$

such that sets

$$B_1 \setminus K_1^\bullet,\ B_2 \setminus ({}^\bullet K_2 \cup K_2^\bullet),\ B_3 \setminus {}^\bullet K_3,\ G,\ \overline{P}$$

are disjoint and

$$K_1^\bullet \subseteq \overline{P} \wedge \rho_1|_{K_1^\bullet} = id, \ \ {}^\bullet K_2 \subseteq \overline{P} \wedge \rho_2|_{{}^\bullet K_2} = id,$$

$$K_2^\bullet \setminus ({}^+K_2 \cup {}^-K_2) \subseteq G \wedge \rho_2|_{K_2^\bullet \setminus ({}^+K_2 \cup {}^-K_2)} = g|_{K_2^\bullet \setminus ({}^+K_2 \cup {}^-K_2)},$$

$${}^\bullet K_3 \subseteq G \wedge \rho_3|_{{}^\bullet K_3} = g|_{{}^\bullet K_3}$$

(what can be achieved by an appropriate renaming).
Set $Q = {}^\bullet K_3 \cap g^{-1}({}^+K_2 \cup {}^-K_2)$. Now, take the process $K_3' \in A_{\alpha_3}$ obtained from the process K_3 by renaming every place $b \in Q$ by the place $g(b) \in {}^+K_2 \cup {}^-K_2$. Then processes $(K_1; K_2); K_3'$, $K_1; (K_2; K_3)$ are defined and equal.

(4) Given $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that terms $(\alpha_1 \parallel \alpha_2); (\alpha_3 \parallel \alpha_4)$ and $(\alpha_1; \alpha_3) \parallel (\alpha_2; \alpha_4)$ are defined, take processes

$$K_1 \in A_{\alpha_1}, \ K_2 \in A_{\alpha_2}, \ K_3 \in A_{\alpha_3}, \ K_4 \in A_{\alpha_4},$$

such that sets

$$B_1 \setminus K_1^\bullet, \ B_2 \setminus K_2^\bullet, \ B_3 \setminus {}^\bullet K_3, \ B_4 \setminus {}^\bullet K_4, \ \overline{P}$$

are disjoint and

$$K_1^\bullet \subseteq \overline{P} \wedge \rho_1|_{K_1^\bullet} = id, \ K_2^\bullet \subseteq \overline{P} \wedge \rho_2|_{K_2^\bullet} = id,$$

$${}^\bullet K_3 \subseteq \overline{P} \wedge \rho_3|_{{}^\bullet K_3} = id, \ \ {}^\bullet K_4 \subseteq \overline{P} \wedge \rho_4|_{{}^\bullet K_4} = id$$

(what can be achieved by an appropriate renaming).
Then processes $(K_1 \parallel K_2); (K_3 \parallel K_4), (K_1; K_3) \parallel (K_2; K_4)$ are defined and equal.

(5-7) The proof for rules (5-7) is similar.

□

Theorem 7. *For two process terms $\alpha, \beta \in \mathcal{P}(\mathcal{A}_N)$, $\tau(\alpha) = \tau(\beta)$ implies $\alpha \sim \beta$.*

Proof. Without loss of generality let α and β be process terms with $K(\alpha) = K(\beta) = K = (B, E, F, C, \rho)$ and $\gamma = \gamma_1; \ldots; \gamma_m$ be the process term constructed from the process K in the proof of Theorem 5 by considering maximal steps. Then γ_i is of the form

$$\gamma_i = \rho(e_1^i) \parallel \ldots \parallel \rho(e_{n_i}^i) \parallel \rho(a^i),$$

$e_1^i, \ldots, e_{n_i}^i \in E$ and $a^i \subseteq B$, $i = 1, \ldots, m$. We show that α is equivalent to γ. By symmetry, the same holds for β, and we are done.

According to Remark 5, we assume without loss of generality that α is of the form

$$\alpha = \rho(e_1) \parallel (\rho(a_1) \cap P); \ldots; \rho(e_k) \parallel (\rho(a_k) \cap P)$$

with transitions $e_1, \dots, e_k \in E$ and subsets $a_1, \dots, a_k \subseteq B$. We will use shorthands $\alpha = e_1; \dots; e_k$, and ignore the sets $a_1, \dots, a_k$, because they are determined by the definition of the sequential composition of process terms. Clearly, α and γ 'contain' the same transitions, i.e.

$$\{e_1, \dots, e_k\} = \{e_1^1, \dots, e_{n_1}^1, \dots, e_1^m, \dots, e_{n_m}^m\}.$$

Assume $e_i = e_1^1$ for an $i \geq 2$. It suffices to prove

$$e_1; \dots; e_i \sim e_1; \dots; e_i; e_{i-1} \sim \dots \sim e_i; e_1; \dots; e_{i-1},$$

because firstly the same procedure applied to $e_2^1, \dots, e_{n_1}^1$ provides $e_1; \dots; e_n \sim \gamma_1; \delta$ (where δ is the rest of the term α after removing transitions of γ_1), and secondly this procedure applied to $\gamma_2, \dots, \gamma_m$ finishes the proof. In fact, it is enough to show that we can exchange e_i and e_{i-1} in α. A sufficient condition is that $\rho(e_i) \parallel \rho(e_{i-1})$ is a defined process term.

We have to distinguish two cases: If $e_{i-1} = e_j^1$ for some $j \in \{2, \dots, n_1\}$, $\rho(e_i) \parallel \rho(e_{i-1})$ is defined according to the process term γ. The other possibility is $e_{i-1} = e_j^l$ for an $l \in \{2, \dots, m\}$ and $j \in \{1, \dots, k_l\}$. By construction of the process K_α from α follows $e_i \not\lessdot e_{i-1}$. On the other hand, by construction of γ follows $e_{i-1} \not\lessdot e_i$. By Lemma 9, $\rho(e_i) \parallel \rho(e_{i-1})$ is defined. □

Remark 6. The set of all processes of an elementary net with mixed context w.r.t. an initial marking m_0 corresponds to the set of all equivalence classes of process terms containing process terms of the form $\alpha = m_0; \beta$ (i.e. process terms starting with m_0).

Finally, looking at the definition of τ, we can state the main result for elementary nets with mixed context, which now follows easily from the previous theorems.

Theorem 8. *Given any elementary net N, there exists a one-to-one correspondence between the isomorphism classes of processes $\mathcal{P}(N)$ of N and the $\sim$-congruence classes of the process terms $\mathcal{P}(\mathcal{A}_N)$ of the corresponding algebraic $(\mathcal{M}, \mathcal{I})$-net defined in Section 7. This correspondence preserves the initial marking, final marking and the information about write places, positive context places and negative context places of processes and process terms, as well as concurrent composition and sequential composition of processes (resp. congruence classes of process terms).*

Remark 7. Clearly, according to Remarks 2 and 4 the previous theorem holds also for elementary nets without context and elementary nets with negative context, although in these examples we considered a slightly different process term semantics.

Using terminology from partial algebra [4] we can rephrase Theorem 8 as follows: Given an elementary net with context N and a process term $\alpha \in$

$\mathcal{P}(\mathcal{A}_N)$ of the corresponding net over partial algebra, the congruence class $[\alpha]_\sim \in (\mathcal{P}(\mathcal{A}_N))/_\sim$ corresponds to the isomorphism class $\tau(\alpha) = [K]_\simeq$ of a process $K \in \mathcal{P}(N)$ such that the initial and final marking are preserved, i.e. $\rho({}^\bullet K) \cap P = \mathit{pre}(\alpha)$, $\rho(K^\bullet) \cap P = \mathit{post}(\alpha)$, and information for concurrent composition is preserved, i.e. $(\rho({}^\diamond K), \rho({}^+K), c^{-1}(\rho({}^-K))) = \mathit{Inf}(\alpha)$. The factor algebra $(\mathcal{P}(\mathcal{A}_N))/_\sim$ is isomorphic to the factor algebra $(\mathcal{P}(N))/_\simeq$, (i.e. τ is a surjective closed homomorphism between $\mathcal{P}(\mathcal{A}_N)$ and $(\mathcal{P}(N))/_\simeq$).

9 Place/Transition Nets

In this section we give algebraic definitions of place/transition Petri nets with inhibitor arcs and place/transition Petri nets with capacities.

Here we provide semantics corresponding to collective token philosophy [5]. In this case an equivalence class of process terms corresponds to an equivalence class of partial orders, according to collective token semantics of place/transition nets without capacity restriction (see [1] and [5]). In the case of individual token philosophy, where the single partial orders are of interest, one can use more sophisticated algebras, such as for example concatenated processes [24].

Let us brielfy mention another possibility how to deal with individual token semantics without using different algebra from those used for the collective token semantics. As it was discussed in Introduction, any process term defines naturally a partial order of events labeled by transitions. Thus, an equivalence class of process terms defines a set of partial orders. As we have illustrated in the example from Introduction, one can modify these partial orders comparing each other and removing causalities which are not defined by the net itself. The idea for further research is to generalize this modification procedure in order to obtain the set of partial orders containing only those causalities which are given by the net itself. Such set of partial orders would correspond to collective token semantics, while obtained single partial orders would correspond to individual process semantics.

Clearly, one can also combine restrictions given by inhibitor arcs and capacities and extend them further, or combine them with other approaches such as positive context to get a more complicated enabling rule. In such cases one could use more complicated algebras, see e.g. [11,3].

Definition 23 (Place/transition nets). *A place/transition Petri net (shortly a p/t net) is a quadruple $N = (P, T, F, W)$, where P, T and F are defined as for elementary nets, and $W : F \to \mathbb{N}^+$ is the weight function. Given a transition t, define ${}^\bullet t, t^\bullet \in \mathbb{N}^P$ as follows:*

$$
{}^\bullet t(p) = \begin{cases} W((p,t)) & \text{if } (p,t) \in F, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
t^\bullet(p) = \begin{cases} W((t,p)) & \text{if } (t,p) \in F, \\ 0 & \text{otherwise}. \end{cases}
$$

9.1 Place/Transition Nets with Inhibitor Arcs

Definition 24 (Place/transition nets with inhibitor arcs). *A p/t net with inhibitor arcs is a five-tuple $N = (P, T, F, W, C_-)$, where (P, T, F, W) is a p/t net, and $C_- \subseteq P \times T$ is an inhibitor relation (set of inhibitor arcs) satisfying $(F \cup F^{-1}) \cap C_- = \emptyset$. As usual, ${}^-t = \{p \mid (p,t) \in C_-\}$ for each $t \in T$. A marking of N is a multi-set $m \in \mathbb{N}^P$. A transition t is enabled to occur at m iff $\forall p \in P : m(p) \geqslant {}^\bullet t(p) \wedge ((p,t) \in C_- \Rightarrow m(p) = 0)$. Its occurrence leads to the marking $m' = m - {}^\bullet t + t^\bullet$.*

For p/t nets with inhibitor arcs the cardinality of the information set I is smaller than the cardinality of the marking set of the net:

$\mathcal{M} = (M, +) = (\mathbb{N}^P, +)$, where $+$ is multi-set addition. For concurrent composition it is obviously enough to check that one process does not use negative context places of the other process as write places. Therefore, the necessary information for concurrent composition consists of the set of those places which appear in a marking of the process term and the set of negative context places. For a marking m over the set P of places we denote $m_s = \{p \mid m(p) \neq 0\}$. It follows $\mathcal{I} = (I, \dot{+}, dom_{\dot{+}})$ with $I = 2^P \times 2^P$, $dom_{\dot{+}} = \{((w,n),(w',n')) | w \cap n' = w' \cap n = \emptyset\}$ and $\forall ((w,n)(w',n')) \in dom_{\dot{+}} : (w,n) \dot{+} (w',n') = (w \cup w', n \cup n')$.

The partial groupoid $\mathcal{I}$ satisfies the requirements given in Section 2.

For a transition t and a marking m define

$$
\begin{aligned}
&pre(t) = {}^\bullet t, post(t) = t^\bullet, \\
&\mathit{inf}(m) = (m_s, \emptyset), \mathit{inf}(t) = ((pre(t))_s \cup post(t))_s, {}^-t).
\end{aligned}
$$

The function *inf* preserves property (a) from Definition 1. One can also easily prove that the independence relation of $\mathcal{I}$ encodes the restriction of the occurrence rule by restriction of concurrent occurrences of a transition and a marking.

Lemma 10. *Let $supp : 2^I \to I$ be defined by*

$$
supp(A) = \left(\bigcup_{(w,n) \in A} w, \bigcup_{(w,n) \in A} n \right).
$$

Then relation $\cong$ defined by $x \cong y \Leftrightarrow supp(x) = supp(y)$ is the greatest closed congruence on the partial algebra $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$.

Proof. It is a straightforward observation that *supp* is a surjective closed homomorphism from $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$ to $(I, dom_{\dot{+}}, \dot{+}, \circ)$, where $\forall (w,n),(w',n') \in I : (w,n) \circ (w',n') = (w \cup w', n \cup n')$. Hence $\cong$ is a closed congruence.

To prove that $\cong$ is the greatest closed congruence it suffices to show that any congruence $\approx$ satisfying $\cong \subsetneq \approx$ is not closed. The proof is similar to the proof of Lemma 6. Assume there are $A, A' \in 2^I$ such that $A \approx A'$ but $A \not\cong A'$. Then $supp(A) \neq supp(A')$.

We construct a set $C \in 2^I$ such that $(A, C) \in dom_{\{\dot{+}\}}$ but $(A', C) \notin dom_{\{\dot{+}\}}$ or vice versa (which implies that $\approx$ is not closed). If $supp(A) = (\overline{w}, \overline{n})$ and $supp(A') = (\overline{w}\,', \overline{n}\,')$ then $\overline{n} \neq \overline{n}\,' \vee \overline{w} \neq \overline{w}\,'$ (since $supp(A) \neq supp(A')$).

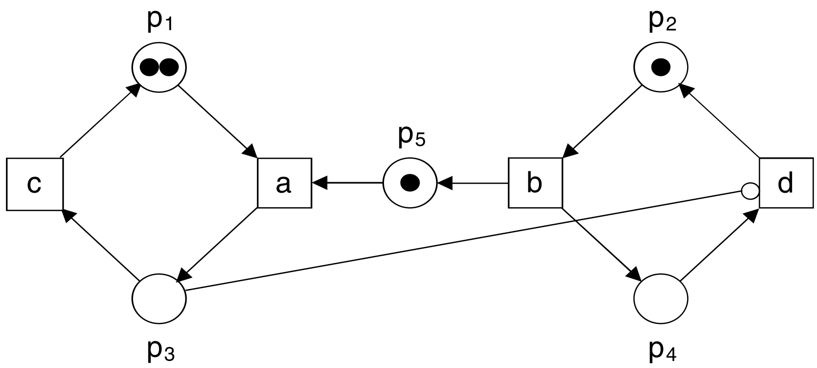

Fig. 13. An example of a p/t net with inhibitor arcs. A possible process term is $(a \parallel b \parallel p_1); (c \parallel (p_1 + p_4 + p_5); d \parallel (2p_1 + p_5)$.

Let $\overline{w} \neq \overline{w}\,'$. Without loss of generality we can assume $\overline{w}\,' \setminus \overline{w} \neq \emptyset$. Set $C = \{(c_w, c_n)\}$ with $c_w = \emptyset$ and $c_n = \overline{w}\,' \setminus \overline{w}$. Therefore $c_w \cap \overline{n} = c_n \cap \overline{w} = \emptyset$, but $c_n \cap \overline{w}\,' \neq \emptyset$, i.e. $(A, C) \in dom_{\{+\}}$, but $(A', C) \notin dom_{\{+\}}$.

Now let $\overline{n} \neq \overline{n}\,'$. Without loss of generality we have $\overline{n}\,' \setminus \overline{n} \neq \emptyset$. Set $C = \{(c_w, c_n)\}$ with $c_w = (\overline{n}\,' \setminus \overline{n})$ and $c_n = \emptyset$. Then $c_w \neq \emptyset$, $c_w \cap \overline{n} = \overline{w} \cap c_n = \emptyset$ and $c_w \cap \overline{n}\,' \neq \emptyset$, and we are finished. $\square$

Because also property (b) from Definition 1 is preserved, we can formulate the following theorem.

Theorem 9. *Given a p/t net with inhibitor arcs $N = (P, T, F, W, C_-)$ with $\mathcal{M}, \mathcal{I}, pre, post, inf$ as defined in this subsection, the quadruple $\mathcal{A}_N = (2^P, T, pre, post)$ together with the mapping inf is an algebraic $(\mathcal{M}, \mathcal{I})$-net. Moreover, it is a corresponding algebraic $(\mathcal{M}, \mathcal{I})$-net to the net N.*

Figure 13 shows an example of a p/t net with inhibitor arcs.

9.2 Nets with Capacities

There are two different interpretations of consuming and producing tokens for Petri nets with capacities (for more details see e.g. [9,10,15]). According to the order of consuming and producing tokens one can distinguish the following situations:

- A transition t first consumes the tokens given by $pre(t)$ yielding an intermediate marking 0 (empty multiset) and then produces tokens $post(t)$. This interpretation corresponds to classical rewriting and such capacities are said to be weak [9].
- A transition t first produces tokens (given by $post(t)$), yielding an intermediate marking $pre(t) + post(t)$ and then consumes tokens (given by $pre(t)$) yielding the marking $post(t)$. Such capacities are said to be strong [9].

Definition 25 (Place/transition nets with capacities). *A place/transition net with capacities is a p/t net together with a partial function $K : P \to \mathbb{N}^+$ with a domain $P_K \subseteq P$.*
A marking of a net with capacities is a multi-set $m \in \mathbb{N}^P$ such that $\forall p \in P_K$: $m(p) \leqslant K(p)$.
A transition t is said to be weakly enabled at a marking m iff $\forall p \in P : m(p) \geqslant {}^\bullet t(p)$ and $\forall p \in P_K : K(p) \geqslant m(p) - {}^\bullet t(p) + t^\bullet(p)$.
A transition tis said to be strongly enabled at a marking m iff $\forall p \in P : m(p) \geqslant {}^\bullet t(p)$ and $\forall p \in P_K : K(p) \geqslant m(p) + t^\bullet(p)$.
The occurrence of an enabled transition t at a marking m leads to the marking $m' = m - {}^\bullet t + t^\bullet$.

The concurrent occurrence of transitions, and more general concurrent composition of processes, have to respect capacities. In the case of strong capacities the information about the intermediate marking $pre(t) + post(t)$ is attached to transition t.

Thus, as the set of markings we set $\mathcal{M} = (\{a \in \mathbb{N}^P \mid \forall p \in P_K : a(p) \leqslant K(p)\}, \tilde{+})$, where the operation $\tilde{+}$ is defined by $a(p)\tilde{+}b(p) = \min(a(p) + b(p), K(p))$ for all $p \in P_K$ and $a(p)\tilde{+}b(p) = a(p) + b(p)$ for all $p \in P \setminus P_K$.

The partial groupoid of information $\mathcal{I} = (I, \dot{+}, dom_{\dot{+}})$ is defined by

$$\begin{aligned} I &= (\{w \in \mathbb{N}^{P_K} | \forall p \in P_K : w(p) \leqslant K(p)\}, \\ dom_{\dot{+}} &= \{(w, w') \in I \times I \mid \forall p \in P_K : w(p) + w'(p) \leqslant K(p)\} \\ \dot{+} &= +|_{dom_{\dot{+}}}. \end{aligned}$$

This partial groupoid satisfies the requirements from Section 2.

Define $pre(t) = {}^\bullet t, post(t) = t^\bullet$ for every transition t. Moreover, for weak capacities define a mapping $\mathit{inf}_w : M \cup T \to I$ by:

- For a marking m, $\mathit{inf}_w(m) = m|_{P_K}$.
- For a transition t and a place $p \in P_K$, $\mathit{inf}_w(t)(p) = max(pre(t)(p), post(t)(p))$.

For strong capacities define a mapping $\mathit{inf}_s : M \cup T \to I$ by:

- For a marking m, $\mathit{inf}_s(m) = m|_{P_K}$.
- For a transition t and a place $p \in P_K$, $\mathit{inf}_s(t)(p) = (pre(t)(p) + post(t)(p))$.[4]

Again, property (a) from Definition 1 is satisfied. The considered independence relation encodes the restriction of the occurrence rule.

In the sequel, we define a mapping $supp : 2^I \to I$ and prove that $supp$ is the natural homomorphism of the greatest closed congruence $\cong$ of the partial algebra $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$.

[4] In the case of strong capacities we implicitly suppose for each transition t and each place $p \in P_K$ that $pre(t)(p) + post(t)(p) \leqslant K(p)$. Otherwise transition t is never enabled to occur and therefore according to the Definition 3 it is irrelevant for the corresponding net

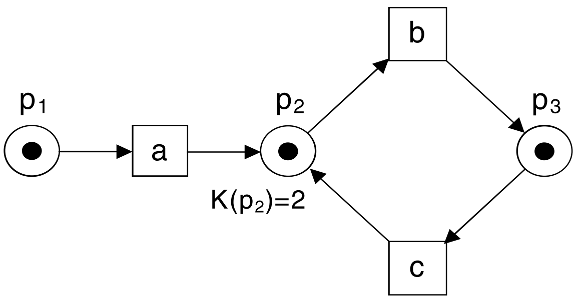

Fig. 14. An example of a p/t net with capacity

Lemma 11. *Given $\mathcal{I}$ as above, let supp* $: 2^I \to I$ *be defined for all* $p \in P_K$ *by*

$$supp(A)(p) = max_{a \in A} a(p).$$

Then the relation $\cong$ *defined by* $A \cong A' \iff supp(A) = supp(A')$ *is the greatest closed congruence on the partial algebra* $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$.

Proof. By the properties of maximum and the definition of the mapping *supp*, *supp* is a surjective closed homomorphism from $(2^I, dom_{\{\dot{+}\}}, \{\dot{+}\}, \cup)$ to $(I, dom_{\dot{+}}, \dot{+}, \circ)$, where $\forall a, a' \in I : a \circ a' = supp(\{a, a'\})$, and therefore $\cong$ is a closed congruence. To prove that $\cong$ is the greatest closed congruence we show that any congruence $\approx$ satisfying $\cong \subsetneq \approx$ is not closed. We construct a set $C \in 2^I$ such that $(A, C) \in dom_{\{\dot{+}\}}$ but $(A', C) \notin dom_{\{\dot{+}\}}$ or vice versa. Assume there are $A, A' \in 2^I$ such that $A \approx A'$ but $A \ncong A'$. Then there is a place $p \in P$ such that $max_{a \in A} a(p) \neq max_{a' \in A'} a'(p)$. Without loss of generality let $max_{a' \in A'} a'(p) > max_{a \in A} a(p)$. It suffices to take, for example, $C = \{a\}$ for the multi-set $a(p) = K(p) - max_{a \in A} a(p)$ and $a(p') = 0$ for all $p' \in P_K$ such that $p' \neq p$. $\square$

The property (b) from Definition 1 is satisfied both for inf_w and inf_s. Thus, we have the following theorem for place/transition nets with capacities.

Theorem 10. *Given a p/t net with capacity* $N = (P, T, F, W, K)$ *with* $\mathcal{M}, \mathcal{I}, pre, post, \mathit{inf}_w, \mathit{inf}_s$ *as defined in this subsection, the quadruple* $\mathcal{A}_N = (M, T, pre, post)$ *together with* inf_w *for weak capacities and* inf_s *for strong capacities is a corresponding* $(\mathcal{M}, \mathcal{I})$*-net to the net* N.

Notice that in the case that there are no self-loops in the net, as it is in Figure 14, weak and strong capacities coincide. Nets with capacities represent a class of $(\mathcal{M}, \mathcal{I})$-nets where information can violate the distributive law (see Definition 2, (4)). For example, we have the following process terms of the net from Figure 14: $\alpha = (b \parallel p_1); (p_3 \parallel a)$ with $\mathit{Inf}(\alpha) = p_2$ and $\beta = (b; p_3) \parallel (p_1; a)$ with $\mathit{Inf}(\beta) = 2p_2$. The information of the term α corresponds to the fact that during the execution of α there is at most one token in place p_2, while the information of β expresses the fact that during the execution of β place p_2 can obtain two tokens. Because terms α and β have different information, they are not equivalent. As a consequence of the difference of information, α can run concurrently with c, but β cannot. If the place p_2 had no capacity restriction, then α and β would be equivalent according to the distributive law and α and β would represent the same run.

10 Conclusion

There are several approaches to unifying Petri nets (see e.g. [22,20,21,16]). They enable to unify different classes of Petri nets which use different underlying algebras and different treatment of data-type part, defining them as formal parameters which can be actualized by choosing an appropriate structure. However, in these approaches enabling condition of the occurrence rule is not a parameter, but it is fixed. Both definitions in [20,16] capture elementary nets but they let open more complicated restrictions of enabling condition in occurrence rule, such as inhibitor arcs or even capacities.

In our paper we have focused on unified description of Petri nets with modified occurrence rule. Namely, we have described a unifying approach to non-sequential semantics of Petri nets with modified occurrence rule. We have demonstrated that methods of partial algebra, such as greatest closed congruence, represent a suitable mathematical tool for such an approach. By restricted domains of operations we were able to generate precisely just those runs of the net which are allowed. In comparison with methods based on partial order where concurrency is defined implicitly if there is no causal connection between runs, we define explicitly when runs can be composed concurrently. Thus, in our approach causality is defined using two partial operations to generate runs, namely concurrent and sequential composition.

On the other hand, we did not discuss unifying of data type part. So, we did not discuss high-level Petri nets in this paper. There are also other restrictions of the occurrence rules in various high-level nets (e.g. transition guards, time intervals, roles etc.) which are of different characters and were not discussed in the paper. It would be interesting to discuss those kinds of restrictions in order to see the implication of the unifying approach for high-level nets. Namely, it would be interesting to combine the approach presented in [22] and the approach presented in this paper.

The presented approach opens many interesting questions. We can further distinguish between synchronous and concurrent occurrences of transitions. In such an extension of our approach one first needs to generate steps from transitions using a partial operation of synchronous composition and then to use this steps to generate process terms using partial operations of concurrent and sequential composition. In terms of causal relationships, such an extension corresponds to the approach described in [13,17], where two kinds of causalities are defined, first saying (as usual) which transitions cannot occur earlier than others, while the second indicating which transitions cannot occur later than others. In [13,17] the principle is illustrated for a variant of nets with inhibitor arcs, where testing for zero precedes the execution of a transition. Thus, if a transition t tests a place for zero, which is in a post-set of another transition

t', this means that t cannot occur later than t' and therefore they cannot occur concurrently - but still can occur synchronously. There are also other net extensions employing steps of transitions (distinguishing between synchronous and concurrent composition), such as nets with asymmetric synchronization [12]. We are currently working on the extension of our approach using a partial operation for synchronous composition to cover such cases.

Another area of further research is to investigate whether the presented framework would lead to a unifying and mathematically elegant way of producing the causal semantics for nets with restricted occurrence rule. Namely, as it was discussed in Introduction, any process term defines naturally a partial order of events labeled by transitions. Thus, an equivalence class of process terms defines a set of partial orders. As we have illustrated in the example from Introduction, one can modify these partial orders comparing each other and removing causalities which are not defined by the net itself. The idea for further research is to generalize this modification procedure in order to obtain the set of partial orders containing only those causalities which are given by the net itself.

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