

Solvable Map Method for Integrating Nonlinear Hamiltonian Systems

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Abstract. Conventional numerical integration algorithms can not be used for long term stability studies of complicated nonlinear Hamiltonian systems since they do not preserve the symplectic structure of the system. Further, they can be very slow even if supercomputers are used. In this paper, we study the symplectic integration algorithm using solvable maps which is both fast and accurate and extend it to six dimensions. This extension enables single particle studies using all three degrees of freedom.

1 Introduction

Consider a complicated Hamiltonian system that is non-integrable. Suppose we are interested in the long-term stability of this dynamical system. Since the system is assumed to be non-integrable, it is very difficult to give stability criteria in an analytic form. A possible solution is to numerically follow the trajectories of the particles through the system for a large number of iterations, a process which goes by the name tracking. One could then infer the stability of motion in the system by analyzing these tracking results. However, in long term integration of these systems, it is important to preserve the Hamiltonian nature of the system at every integration step. Otherwise, one can get spurious damping or even chaotic behaviour which is not present in the original system. Such problems becomes accentuated when long term integration is performed. This can then obviously lead to wrong predictions regarding the long-term stability of the Hamiltonian system being studied.

The most straightforward method that can be used to perform long term tracking is numerical integration. However, a very short time step has to be used to reduce the spurious behaviour that occurs because of the non-symplectic integration. This makes this method so slow that it is impractical to study the long term behaviour of very complicated systems (like the Large Hadron Collider) even if supercomputers are used. Therefore, we need a method that is both fast and accurate.

Several symplectic integration methods have been discussed in literature [1,2,3,4,5,6,7,8,9,10,11]. Methods using Lie algebraic perturbation theory which give

maps where the final values of variables are explicit functions of initial ones offer several advantages. Further, since the whole complicated system can be represented by a single symplectic map, there can be a substantial saving in computation time as compared to non-map based methods. In this paper, we extend an alternate symplectic integration method using solvable maps [12] to six dimensions. This extension is necessary for single particle studies using all three degrees of freedom. The above method has been shown to give good results in lower dimensions [12].

2 Preliminaries

Let $z = (q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$ denote the $2n$ dimensional phase space variables. Denoting the initial and final locations of the particle by z^i and z^f respectively, the time evolution of a Hamiltonian system $H(z)$ can be described as a symplectic map \mathcal{M} [13]:

$$z^f = \mathcal{M}z^i. \quad (1)$$

The symplectic map is related to the Hamiltonian $H(z)$ by the equation:

$$\frac{d\mathcal{M}}{dt} = \mathcal{M} : -H(z_0), \quad (2)$$

where z_0 is z at time $t = 0$.

Using the Dragt-Finn factorization theorem [13], the symplectic map \mathcal{M} can be factorized as follows.

$$\mathcal{M} = \hat{M} \exp(: f_3(z) :) \cdots \exp(: f_m(z) :) \cdots. \quad (3)$$

Here \hat{M} represents the linear part of the map and $f_m(z)$ is a homogeneous polynomial of degree m in z determined uniquely by the factorization theorem. Further $: f(z) :$ is the Lie operator corresponding to the function $f(z)$ and is defined by

$$: f(z) : g(z) = [f(z), g(z)]. \quad (4)$$

Here $g(z)$ denotes another phase space function and $[f(z), g(z)]$ denotes the usual Poisson bracket of the functions $f(z)$ and $g(z)$. The exponential of a Lie operator is called a Lie transformation and is given as

$$\exp(: f(z) :) = \sum_{n=0}^{\infty} \frac{: f(z) :^n}{n!}. \quad (5)$$

The map \mathcal{M} involves an infinite number of Lie transformations and hence for any practical computation, we have to truncate \mathcal{M} after a finite number of Lie transformations.

$$\mathcal{M} = \hat{M} e^{:f_3:} e^{:f_4:} \dots e^{:f_m:}. \quad (6)$$

Each one of the Lie transformations in the above equation is a symplectic map and hence the map can be truncated at any order without losing symplecticity.

The product $e^{f_3} e^{f_4} \dots e^{f_m}$ in Eq. (6) give the nonlinear part of \mathcal{M} . Each of these Lie transformations gives an infinite number of terms when acting on phase space variables. Since we can not evaluate an infinite number of terms in any algorithm, we need to overcome this problem in some manner. The most straightforward method is to truncate the Taylor series expansion of each Lie transformation, after a finite number terms. This however violates the symplectic condition. Though this method is justifiable in short term tracking, it does not work well in long term tracking as the non-symplecticity can lead to spurious damping or even chaotic behaviour which is not present in the original system [10]. Therefore, we refactorize \mathcal{M} in terms of simpler symplectic maps that can be evaluated both exactly and quickly. We achieve this refactorization through the so-called “solvable maps” [11,12].

3 Solvable Map Method

Solvable maps are generalizations of Cremona maps. The class of Cremona maps includes only those symplectic maps for which the Taylor series expansion terminates when acting on phase space coordinates. The class of solvable maps also includes those symplectic maps for which the Taylor series expansion can be summed up explicitly. One simple example of such a map is $\exp(: a q_1^{l+2} + b q_1^{l+1} p_1 :)$ [12].

The basic idea behind the solvable map method is to represent each nonlinear factor $\exp(: f_m :)$ in Eq. (6) as a product of solvable maps, i.e.,

$$\exp(: f_m :) = \exp(: h_1 :) \exp(: h_2 :) \cdots \exp(: h_n :), \text{ for } m \geq 3. \quad (7)$$

We also ensure that the number of solvable maps is a minimum. For simplicity, we restrict ourselves to a general fourth order symplectic map in six dimensions.

We first set up a notation that will facilitate the indexing of monomials in six phase space variables. Let P_j denote the following basis monomial of degree m in six phase-space variables

$$P_j^{(m)} = q_1^{r_1} p_1^{r_2} q_2^{r_3} p_2^{r_4} q_3^{r_5} p_3^{r_6}, \quad (8)$$

where $1 \leq r_i \leq m$, $r_1 + r_2 + \dots + r_6 = m$. For this six dimensional case, the number of independent monomials of degree 3 is 56 and the number of independent monomials of degree 4 is 126.

Consider a nonlinear Hamiltonian system in six phase space dimensions. Performing a Taylor expansion around the origin in the spirit of perturbation theory, the Hamiltonian $H(z)$ can be written as (up to fourth order in the phase space variables z)

$$H(z) = H_2(z) + H_3(z) + H_4(z). \quad (9)$$

Here $H_m(z)$ is a homogeneous polynomial of degree m in z . Concentrating on the nonlinear part for the moment, H_3 and H_4 can be written as a linear combination

of the basis monomials introduced above:

$$H_3(z) = \gamma_1 P_1(z) + \gamma_2 P_2(z) + \cdots + \gamma_{56} P_{56}(z), \quad (10)$$

$$H_4(z) = \gamma_{57} P_{57}(z) + \gamma_{58} P_{58}(z) + \cdots + \gamma_{182} P_{182}(z). \quad (11)$$

Now we express the time evolution of the Hamiltonian system $H(z)$ as a symplectic map \mathcal{M}_4 (truncated at order 4) using the Dragt-Finn factorization:

$$\mathcal{M}_4 = \hat{M} \exp(\cdot f_3 \cdot) \exp(\cdot f_4 \cdot), \quad (12)$$

where \hat{M} is obtained from the linear part $H_2(z)$ of the Hamiltonian $H(z)$. Using the basis monomials, f_3 and f_4 can be represented as:

$$f_3(z) = \alpha_1 P_1(z) + \alpha_2 P_2(z) + \cdots + \alpha_{56} P_{56}(z), \quad (13)$$

$$f_4(z) = \alpha_{57} P_{57}(z) + \alpha_{58} P_{58}(z) + \cdots + \alpha_{182} P_{182}(z). \quad (14)$$

It should be noted that there is a specific algorithm for determining the coefficients α_i in terms of the coefficients γ_i and the linear part $H_2(z)$. For a specific Hamiltonian, many of the coefficients α_i could be zero.

We now refactorize the symplectic map \mathcal{M}_4 representing the original Hamiltonian system in terms of solvable maps as follows (for further details see [14]):

$$\mathcal{M} = \hat{M} e^{i h_1} e^{i h_2} \cdots e^{i h_{20}}, \quad (15)$$

where

$$\begin{aligned} h_1 = & q_1^4 [\beta_{57}] + q_1^3 [\beta_1 + \beta_{59} q_2 + \beta_{61} q_3] + \\ & q_1^2 [\beta_3 q_2 + \beta_5 q_3 + \beta_{68} q_2^2 + \beta_{70} q_2 q_3 + \beta_{75} q_3^2] + \\ & q_1 [\beta_{12} q_2^2 + \beta_{14} q_2 q_3 + \beta_{19} q_3^2 + \beta_{93} q_2^3 + \beta_{95} q_2^2 q_3 + \beta_{100} q_2 q_3^2 + \\ & \quad \beta_{109} q_3^3] + \\ & [\beta_{39} q_2^2 q_3 + \beta_{44} q_2 q_3^2 + \beta_{150} q_2^3 q_3 + \beta_{155} q_2^2 q_3^2 + \beta_{164} q_2 q_3^3] + \\ & p_1 [\beta_{27} q_2^2 + \beta_{29} q_2 q_3 + \beta_{34} q_3^2 + \beta_{128} q_2^3 + \beta_{130} q_2^2 q_3 + \\ & \quad \beta_{135} q_2 q_3^2 + \beta_{144} q_3^3] + \\ & q_1 p_1 [\beta_8 q_2 + \beta_{10} q_3 + \beta_{83} q_2^2 + \beta_{85} q_2 q_3 + \beta_{90} q_3^2], \end{aligned}$$

$$\begin{aligned} h_2 = & p_1^4 [\beta_{113}] + p_1^3 [\beta_{22} + \beta_{115} p_2 + \beta_{117} p_3] + \\ & p_1^2 [\beta_{24} p_2 + \beta_{26} p_3 + \beta_{122} p_2^2 + \beta_{124} p_2 p_3 + \beta_{127} p_3^2] + \\ & p_1 [\beta_{31} p_2^2 + \beta_{33} p_2 p_3 + \beta_{36} p_3^2 + \beta_{138} p_2^3 + \beta_{140} p_2^2 p_3 + \\ & \quad \beta_{143} p_2 p_3^2 + \beta_{147} p_3^3] + \\ & [\beta_{49} p_2^2 p_3 + \beta_{52} p_2 p_3^2 + \beta_{170} p_2^3 p_3 + \beta_{173} p_2^2 p_3^2 + \beta_{177} p_2 p_3^3] + \\ & q_1 [\beta_{16} p_2^2 + \beta_{18} p_2 p_3 + \beta_{21} p_3^2 + \beta_{103} p_2^3 + \beta_{105} p_2^2 p_3 + \\ & \quad \beta_{108} p_2 p_3^2 + \beta_{112} p_3^3] + \\ & q_1 p_1 [\beta_9 p_2 + \beta_{11} p_3 + \beta_{87} p_2^2 + \beta_{89} p_2 p_3 + \beta_{92} p_3^2], \end{aligned}$$

$$\begin{aligned} h_3 = & q_2^4 [\beta_{148}] + q_2^3 [\beta_{37} + \beta_{151} p_3] + \\ & q_2^2 [\beta_{40} p_3 + \beta_{118} p_1^2 + \beta_{131} p_1 p_3 + \beta_{157} p_3^2] + \end{aligned}$$

$$q_2 \left[\beta_{23} p_1^2 + \beta_{30} p_1 p_3 + \beta_{46} p_3^2 + \beta_{114} p_1^3 + \beta_{121} p_1^2 p_3 + \right. \\ \left. \beta_{137} p_1 p_3^2 + \beta_{167} p_3^3 \right] + \\ q_2 p_2 \left[\beta_{28} p_1 + \beta_{43} p_3 + \beta_{119} p_1^2 + \beta_{134} p_1 p_3 + \beta_{163} p_3^2 \right] ,$$

$$h_4 = p_2^4 \left[\beta_{168} \right] + p_2^3 \left[\beta_{47} + \beta_{169} q_3 \right] + \\ p_2^2 \left[\beta_{48} q_3 + \beta_{72} q_1^2 + \beta_{104} q_1 q_3 + \beta_{171} q_3^2 \right] + \\ p_2 \left[\beta_4 q_1^2 + \beta_{17} q_1 q_3 + \beta_{50} q_3^2 + \beta_{60} q_1^3 + \beta_{73} q_1^2 q_3 + \beta_{106} q_1 q_3^2 + \beta_{174} q_3^3 \right] + \\ q_2 p_2 \left[\beta_{13} q_1 + \beta_{42} q_3 + \beta_{69} q_1^2 + \beta_{98} q_1 q_3 + \beta_{161} q_3^2 \right] ,$$

$$h_5 = q_3^4 \left[\beta_{178} \right] + q_3^3 \left[\beta_{53} \right] + q_3^2 \left[\beta_{125} p_1^2 + \beta_{141} p_1 p_2 \right] + \\ q_3 \left[\beta_{25} p_1^2 + \beta_{32} p_1 p_2 + \beta_{116} p_1^3 + \beta_{123} p_1^2 p_2 + \beta_{139} p_1 p_2^2 \right] + \\ q_3 p_3 \left[\beta_{35} p_1 + \beta_{51} p_2 + \beta_{126} p_1^2 + \beta_{142} p_1 p_2 + \beta_{172} p_2^2 \right] ,$$

$$h_6 = p_3^4 \left[\beta_{182} \right] + p_3^3 \left[\beta_{56} \right] + p_3^2 \left[\beta_{77} q_1^2 + \beta_{102} q_1 q_2 \right] + \\ p_3 \left[\beta_6 q_1^2 + \beta_{15} q_1 q_2 + \beta_{62} q_1^3 + \beta_{71} q_1^2 q_2 + \beta_{96} q_1 q_2^2 \right] + \\ q_3 p_3 \left[\beta_{20} q_1 + \beta_{45} q_2 + \beta_{76} q_1^2 + \beta_{101} q_1 q_2 + \beta_{156} q_2^2 \right] ,$$

$$h_7 = \beta_{74} q_1^2 p_2 p_3 ,$$

$$h_8 = \beta_{120} p_1^2 q_2 q_3 ,$$

$$h_9 = q_1^2 p_1 \left[\beta_2 + \beta_{64} q_2 + \beta_{65} p_2 + \beta_{66} q_3 + \beta_{67} p_3 \right] ,$$

$$h_{10} = q_1 p_1^2 \left[\beta_7 + \beta_{79} q_2 + \beta_{80} p_2 + \beta_{81} q_3 + \beta_{82} p_3 \right] ,$$

$$h_{11} = q_2^2 p_2 \left[\beta_{38} + \beta_{94} q_1 + \beta_{129} p_1 + \beta_{153} q_3 + \beta_{154} p_3 \right] ,$$

$$h_{12} = q_2 p_2^2 \left[\beta_{41} + \beta_{97} q_1 + \beta_{132} p_1 + \beta_{159} q_3 + \beta_{160} p_3 \right] ,$$

$$h_{13} = q_3^2 p_3 \left[\beta_{54} + \beta_{110} q_1 + \beta_{145} p_1 + \beta_{165} q_2 + \beta_{175} p_2 \right] ,$$

$$h_{14} = q_3 p_3^2 \left[\beta_{55} + \beta_{111} q_1 + \beta_{146} p_1 + \beta_{166} q_2 + \beta_{176} p_2 \right] ,$$

$$h_{15} = \beta_{58} q_1^3 p_1 + \beta_{149} q_2^3 p_2 + \beta_{179} q_3^3 p_3 ,$$

$$h_{16} = \beta_{78} q_1 p_1^3 + \beta_{158} q_2 p_2^3 + \beta_{181} q_3 p_3^3 ,$$

$$h_{17} = \beta_{86} q_1 p_1 q_2 p_3 + \beta_{88} q_1 p_1 p_2 q_3 ,$$

$$h_{18} = \beta_{99} q_2 p_2 q_1 p_3 + \beta_{133} q_2 p_2 p_1 q_3 ,$$

$$h_{19} = \beta_{107} q_3 p_3 q_1 p_2 + \beta_{136} q_3 p_3 q_2 p_1 ,$$

$$h_{20} = \beta_{63} q_1^2 p_1^2 + \beta_{84} q_1 p_1 q_2 p_2 + \beta_{91} q_1 p_1 q_3 p_3 + \beta_{152} q_2^2 p_2^2 + \beta_{162} q_2 p_2 q_3 p_3 + \beta_{180} q_3^2 p_3^2 .$$

Here β_i 's are functions of α_i 's. They can be easily determined by comparing the original factorization for \mathcal{M}_4 in terms of f_i with the solvable map refactorization in terms of h_i order by order and using the Campbell-Baker-Hausdorff theorem [15]. Thus given γ_i 's parameterizing the original Hamiltonian system $H(z)$, β_i 's can be readily determined [14].

We will now explicitly obtain the action of some of the above maps e^{h_i} on the phase space variables in a closed form. This will also prove that these maps are solvable maps. A similar analysis can be done for the remaining maps [14]. Consider the action of e^{h_i} on z :

$$z' = e^{h_i} z. \quad (16)$$

From Eq. (2) we see that this action is equivalent to first integrating the equations of motion for the Hamiltonian $h_i(z)$ from time $t = 0$ to time $t = -1$ and then identifying the initial values z_0 (at $t = 0$) with z and the final values (at $t = -1$) for z with z' .

First, we consider the action of e^{h_7} on the phase space variables. Using the above procedure, this action can be easily obtained and is given as follows:

$$\begin{aligned} q'_1 &= q_1 , & p'_1 &= p_1 + 2 \beta_{74} q_1 p_2 p_3 , \\ q'_2 &= q_2 - \beta_{74} q_1^2 p_3 , & p'_2 &= p_2 , \\ q'_3 &= q_3 - \beta_{74} q_1^2 p_2 , & p'_3 &= p_3 . \end{aligned}$$

Thus, e^{h_7} is a solvable map. The action of the solvable map e^{h_8} is similar to e^{h_7} .

The action of e^{h_9} on the phase space variables is obtained as follows. The appropriate Hamiltonian is

$$h_9 = q_1^2 p_1 [\beta_2 + \beta_{64} q_2 + \beta_{65} p_2 + \beta_{66} q_3 + \beta_{67} p_3] = E_9 . \quad (17)$$

Here E_9 denotes the conserved numerical value of h_9 and is obtained by evaluating h_9 at $t = 0$. We note that

$$\frac{d}{dt}(q_1^2 p_1) = 2 q_1 p_1 \dot{q}_1 + q_1^2 \dot{p}_1 = 0 .$$

Therefore, $q_1^2 p_1 = q_{10}^2 p_{10} = \text{constant}$. Consequently,

$$\beta_2 + \beta_{64} q_2 + \beta_{65} p_2 + \beta_{66} q_3 + \beta_{67} p_3 = \frac{E_9}{q_{10}^2 p_{10}} = \text{constant} .$$

Hamilton's equations of motion, give

$$\dot{q}_1 = \frac{\partial h_9}{\partial p_1} = q_1^2 (\beta_2 + \beta_{64} q_2 + \beta_{65} p_2 + \beta_{66} q_3 + \beta_{67} p_3) .$$

Integrating the above equation, we have

$$q_1 = \frac{q_{1_0}^2 p_{1_0}}{q_{1_0} p_{1_0} - E_9 t} ,$$

where we have denoted the values of q_i and p_i at time $t = 0$ by q_{i_0} and p_{i_0} . Also, we will sometimes collectively denote q_1, q_2, q_3 by q and p_1, p_2, p_3 by p . Further, if $q_{1_0}, p_{1_0} \neq 0$, we have $q_1 \neq 0$ and from Eq. (17)

$$\begin{aligned} p_1 &= \frac{E_9}{q_1^2 (\beta_2 + \beta_{64} q_2 + \beta_{65} p_2 + \beta_{66} q_3 + \beta_{67} p_3)} , \\ &= p_{1_0} \left(\frac{q_{1_0}}{q_1} \right)^2 . \end{aligned}$$

If $q_{1_0} = 0$, then $q_1 = q_{1_0}$ and $p_1 = p_{1_0}$.

The equation for q_2 is given by

$$\dot{q}_2 = \frac{\partial h_9}{\partial p_2} = \beta_{65} q_{1_0}^2 p_{1_0} .$$

Once again integrating the above equation, we have

$$q_2 = q_{2_0} + \beta_{65} q_{1_0}^2 p_{1_0} t .$$

Similarly, integrating the equations of motion for the other variables and letting $(q, p) \rightarrow (q', p')$, $(q_0, p_0) \rightarrow (q, p)$, $t \rightarrow -1$, we get the action of e^{h_9} on the phase space variables. Thus,

$$\begin{aligned} q'_1 &= \begin{cases} \frac{q_1^2 p_1}{q_1 p_1 + E_9} & \text{if } q_1, p_1 \neq 0 \\ q_1 & \text{otherwise,} \end{cases} \\ p'_1 &= \begin{cases} p_1 \left(\frac{q_1}{q'_1} \right)^2 & \text{if } q_1, p_1 \neq 0 \\ p_1 & \text{otherwise,} \end{cases} \\ q'_2 &= q_2 - \beta_{65} q_1^2 p_1 , \\ p'_2 &= p_2 + \beta_{64} q_1^2 p_1 , \\ q'_3 &= q_3 - \beta_{67} q_1^2 p_1 , \\ p'_3 &= p_3 + \beta_{66} q_1^2 p_1 . \end{aligned}$$

This shows that e^{h_9} is a solvable map. Similarly, $e^{h_{10}}$, $e^{h_{11}}$, ..., $e^{h_{14}}$ can be shown to be solvable maps and their action on phase space variables can be got by appropriately changing q and p in the above equations.

The action of the map $e^{h_{15}}$ on the phase space variables can be obtained in the following way. The corresponding Hamiltonian is

$$h_{15} = \beta_{58} q_1^3 p_1 + \beta_{149} q_2^3 p_2 + \beta_{179} q_3^3 p_3. \quad (18)$$

From the equations of motion,

$$\dot{q}_1 = \frac{\partial h_{15}}{\partial p_1} = \beta_{58} q_1^3.$$

Integrating, we have

$$q_1 = \frac{q_{10}}{\sqrt{1 - 2\beta_{58} q_{10}^2 t}}. \quad (19)$$

Substituting (19) in the equation of motion for p_1 and integrating, we see that

$$\dot{p}_1 = -3\beta_{58} q_1^2 p_1, \quad \Rightarrow \quad p_1 = p_{10} (1 - 2\beta_{58} q_{10}^2 t)^{3/2}. \quad (20)$$

Similarly integrating the remaining equations of motion and letting $(q, p) \rightarrow (q', p')$, $(q_0, p_0) \rightarrow (q, p)$, $t \rightarrow -1$, we get the action of $e^{h_{15}}$ on the phase space variables:

$$\begin{aligned} q'_1 &= \frac{q_1}{\sqrt{1 + 2\beta_{58} q_1^2}}, \quad p'_1 = p_1 (1 + 2\beta_{58} q_1^2)^{3/2}, \\ q'_2 &= \frac{q_2}{\sqrt{1 + 2\beta_{149} q_2^2}}, \quad p'_2 = p_2 (1 + 2\beta_{149} q_2^2)^{3/2}, \\ q'_3 &= \frac{q_3}{\sqrt{1 + 2\beta_{179} q_3^2}}, \quad p'_3 = p_3 (1 + 2\beta_{179} q_3^2)^{3/2}. \end{aligned}$$

Thus, $e^{h_{15}}$ is a solvable map. The action of the map $e^{h_{16}}$ is similar to that of $e^{h_{15}}$ and can be got by interchanging q and p and by letting $t \rightarrow -t$.

Finally, the action of the solvable map $e^{h_{20}}$ is as follows:

$$\begin{aligned} q'_1 &= q_1 \exp [-(2\beta_{63} q_1 p_1 + \beta_{84} q_2 p_2 + \beta_{91} q_3 p_3)], \\ p'_1 &= p_1 \exp [(2\beta_{63} q_1 p_1 + \beta_{84} q_2 p_2 + \beta_{91} q_3 p_3)], \\ q'_2 &= q_2 \exp [-(\beta_{84} q_1 p_1 + 2\beta_{152} q_2 p_2 + \beta_{162} q_3 p_3)], \\ p'_2 &= p_2 \exp [(\beta_{84} q_1 p_1 + 2\beta_{152} q_2 p_2 + \beta_{162} q_3 p_3)], \\ q'_3 &= q_3 \exp [-(\beta_{91} q_1 p_1 + \beta_{162} q_2 p_2 + 2\beta_{182} q_3 p_3)], \\ p'_3 &= p_3 \exp [(\beta_{91} q_1 p_1 + \beta_{162} q_2 p_2 + 2\beta_{182} q_3 p_3)]. \end{aligned}$$

Therefore $e^{h_{20}}$ is also a solvable map.

Using the above solvable map method, different nonlinear Hamiltonian systems were studied and good results were obtained. Since the results obtained are very similar to those already demonstrated in Ref. [12] for lower dimensions, the details are omitted here.

4 Conclusions

We have studied a symplectic integration algorithm using solvable maps. The solvable map factorization for three degrees of freedom (up to order 4) was carried out and it involves 20 solvable maps. It has been shown [12] that this algorithm gives good results for various examples. It also provides a fast and accurate symplectic integration method for complicated Hamiltonian systems. Another advantage of this method is that it can be easily extended to higher dimensions since the computations involved are not that difficult when symbolic manipulation programs are used.

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