Reachability on a region bounded by two attached squares

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Abstract

This paper considers a region bounded by two attached squares and a linkage confined within it. By introducing a new movement called *mot*, presents a quadratic time algorithm for reaching a point inside the region by the end of the linkage. It is shown that the algorithm works when a certain condition is satisfied.

keywords: Multi-link arm, reachability, motion planning, concave region, robot arms.

1 Introduction

This paper considers the movement of a linkage in a two-dimensional bounded region and introduces a new algorithm to reach a given point by the end of the linkage. The region considered is the one obtained by two attached squares.

Several papers have been written on reachability problems mainly, on convex region. Hopcroft, Joseph and Whitesides in [1] studied the reconfiguration and reachability problems for a linkage. In [2], they gave a polynomial time algorithm for moving a linkage confined within a circle from one given configuration to another, and proved that the reachability problem for a planar arm constrained by an arbitrary polygon, is NP-hard. Joseph and Plantings [3] proved that the reachability problem for a chain moving within a certain non-convex constraining environment is PSPACE hard.

In [4] and [5], Kantabutra presented a linear time algorithm for reconfiguring certain chains inside squares. He considered an unanchored n-linkage robot arm confined inside a square with side length at least as long as the longest arm link and found a necessary and sufficient condition for reachability in this square. His algorithm requires O(n) time.

This paper extends the previous results by providing a quadratic time algorithm to solve the reachability problem in a special concave region. The

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region is bounded by the union of two squares attached via one edge. In the next section of the paper some preliminaries and useful definitions are given. In section 3 a new movement, by which a linkage moves in a concave corner is formulated and finally in section 4 present the reachability algorithm and the related properties are presented.

2 Preliminaries

An *n-linkage* Γ[0,1,...n] is a collection of n rigid rods or links, $\{A_{i-1}A_i\}_{i=1,...n}$, consecutively joined together at their end points, about which they may rotate freely. Links may cross over one another and none of end points of the linkage are fixed.

We denote the length of links of Γ[0,1,...n] by $l_1, l_2, ...l_n$, where l_i is the length of link with end points A_{i-1} and A_i and $||\Gamma|| = max_{1 \leq i \leq n} l_i$. For $1 \leq i \leq n-1$ the angle obtained by turning clockwise about A_i from A_{i-1} to A_{i+1} is denoted by α_i. We say that a linkage Γ is *bounded by b* if $||\Gamma|| <$ b, i.e no link has a length greater than or equal to b.

For a region P, by *Reaching a given point $p \in P$ by A_n*, the end point of Γ, we mean Γ can move within P from its given initial position to a final position so that A_n reaches p.

For a linkage Γ confined inside a convex region P with boundary denoted by ∂P, we define two special configurations as follows (Figure 1):

We say that Γ is in *Rim Normal Form* (denoted RNF), if all its joints lie on ∂P.

We say that Γ is in *Ordered Normal Form* (denoted ONF), if:

1. Γ is in RNF.
2. Moving from A_0 toward A_n along Γ is always either clockwise or counterclockwise around the boundary polygon.

Algorithms for the reconfiguration of an n-linkage usually break up the motions for the whole reconfiguration into simple motions, in which only a few joints are moved simultaneously (see [2], [6] and [7]). We allow the following type of simple motions:

- No angle at joints changes, but the linkage may translate and rotate as a rigid object.
- At most four angles change simultaneously and the other joints do not change their positions.

3 Movement in a concave environment

In this section we introduce a new movement for a linkage to reach a point inside a certain concave region.

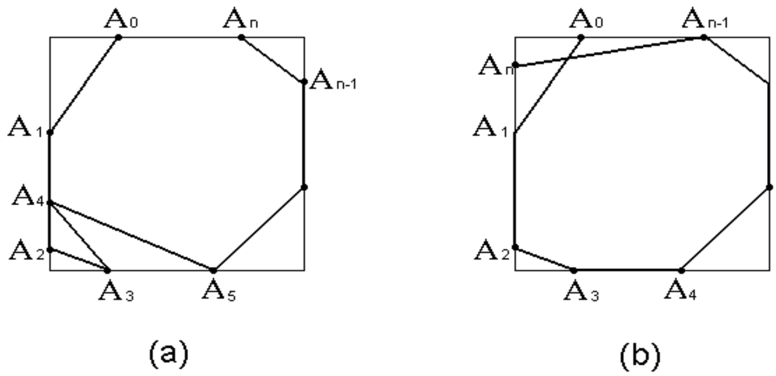

Figure 1: *An n-linkage in (a): Rim Normal Form, (b): Ordered Normal Form.*

Theorem 1. Suppose that S is a region where its boundary polygon ∂S, is a square with side length s, $\Gamma[0, 1, ...n]$ is an n-linkage confined within S and $\|\Gamma\| <$ s. Then Γ can be brought to ONF using O(n) simple motions.

Proof: See [5].

Lemma 2. If ∂S, the boundary polygon of the region S, is a square with side length s and $\Gamma[0, 1, ...n]$ is an n-linkage with $\|\Gamma\| <$ s confined within S, initially in ONF. Then any joint of Γ can be moved along ∂S in either direction, in such a manner that the linkage always remain in ONF. This can be done with O(n) simple motions.

Proof: See [5].

To understand our new movement, it helps to first consider a special case of 2-linkage $\Gamma[1, 2, 3]$ consisting of joints A_1, A_2 and A_3. We define a movement for $\Gamma[1, 2, 3]$ from its initial configuration to a specified final configuration in which, A_1 gets the position of A_2, and A_3 moves forward in a given path (Figure 2).

Unless otherwise specified, by $\angle A_1A_2A_3$ ($\angle\gamma_1\gamma_2$, which γ_1 and γ_2 are two crossing line segments), we mean the angle obtained by turning clockwise from A_1 to A_3 about A_2 (from γ_1 to γ_2).

Circumstances: Consider two line segments γ_1 and γ_2 which intersect at q and $\angle\gamma_1\gamma_2$ is in $[\pi, 2\pi]$. Let ρ be the line segment which starts at q and divides the angle $\angle\gamma_1\gamma_2$ into two angles $\angle\gamma_1\rho$ and $\angle\rho\gamma_2$ in such a way that $\angle\gamma_1\rho$ is in $[\pi/2, \pi]$. Initial configuration of Γ[1,2,3] is defined as follows: Let A_1 be at point p on line segment γ_1, A_2 at q and A_3 at point r on line segment γ_2 (Figure 2-a). By this assumption we can define our movement in a concave region.

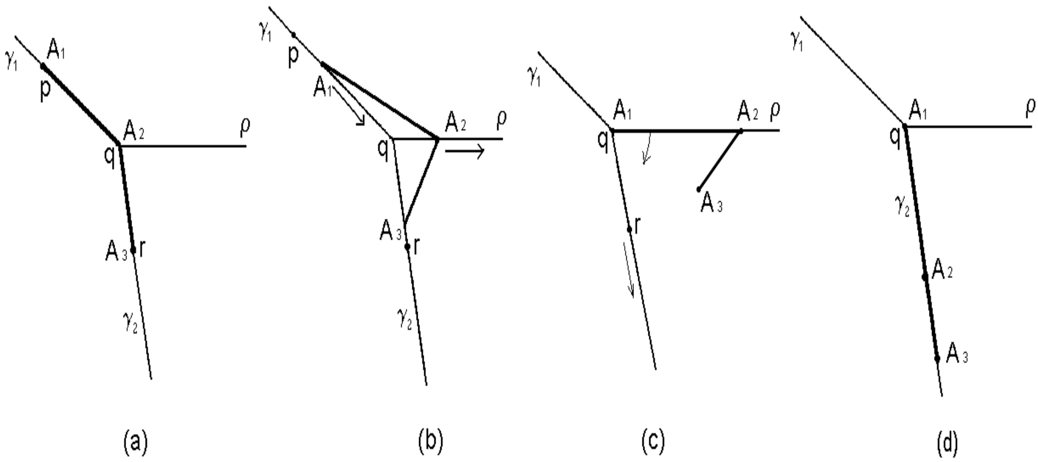

Figure 2: *(a): Initial configuration of* $\Gamma[1,2,3]$, *(b): middle-joint-up(A_1, A_2, A_3, ρ) motion, (c): front-link-forward(A_1, A_2, A_3, ρ) motion, (d): final configuration of* $\Gamma[1,2,3]$.

Definition 3. The *mot(A_1, A_2, A_3, ρ)* movement changes the initial configuration of $\Gamma[1,2,3]$ to a final configuration by which Γ lies on γ_2. This is done by two consecutive motions:

- *Middle-joint-up(A_1, A_2, A_3, ρ)*: moves A_2 along ρ away from q until A_1 reaches q. During the movement A_1 remains on γ_1, and A_3 remains on γ_2 as much as possible.
- *Front-link-forward(A_1, A_2, A_3, ρ)*: fixes A_1 at q and brings down A_3 on γ_2 (if not already there). To straighten Γ, it moves A_3 along γ_2 away from q.

We show the mot(A_1, A_2, A_3, ρ) movement can be done in finite number of simple motions.

Assume Γ is in the initial configuration. We show how each of the middle-joint-up motion and front-link-forward motion is done in finite number of simple motions.

Middle-joint-up(A_1, A_2, A_3, ρ):
Move A_2 along ρ away from q (Figure 2-b). If $\angle\rho\gamma_2 \geq \pi/2$, during the movement, A_1 and A_3 approach q, while staying on lines γ_1 and γ_2 respectively.

If $\angle\rho\gamma_2 < \pi/2$, during the movement, A_3 moves away from q and it is possible that A_2A_3 becomes perpendicular to γ_2. If this happens, first turn A_2A_3 about A_2 until qA_2A_3 folds, then if needed, move A_2A_3 along ρ away from q in a way that α_2 increases until $A_1A_2A_3$ folds and A_1 reaches q. This requires a finite number of simple motions.

Front-link-forward(A_1, A_2, A_3, ρ):
If during middle-joint-up motion A_1 reaches q first, for applying front-link-forward motion, it is enough to keep A_1 at q fixed, and move A_3 along γ_2 until Γ straightens.

If A_3 reaches q first and A_1 arrives later, for applying front-link-forward motion, turn A_2A_3 about A_2 in a way that α_2 decreases, until A_3 hits γ_2 or $\alpha_2 = 3\pi/2$. If $\alpha_2 = 3\pi/2$ before A_3 hits γ_2, rotate Γ about A_1 in a way that $\angle A_2A_1r$ decreases until A_3 reaches γ_2, then keep A_1 fixed at q and move A_3 along γ_2 away from q so that Γ straightens. This requires a finite number of simple motions (Figure 2-c).

If A_3 hits γ_2 first, keep A_1 fixed at q and move A_3 along γ_2 away from q so that Γ straightens.

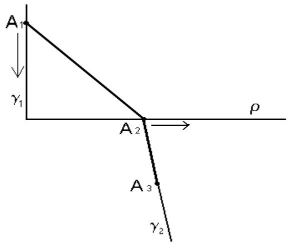

Figure 3: γ_1 *can be a convex path instead of a line segment.*

In the definition 3, during mot(A_1, A_2, A_3, ρ) movement, A_1 moves along the line segment γ_1. The line segment γ_1 can be replaced by a composition of two line segments in such a way that the path where A_1 belongs to is convex. See figure 3.

In our algorithm, to reach p we have to apply mot$(A_{i-1}, A_i, A_{i+1}, \rho)$ movement several times. At the end, possibly p can be reached by A_n somewhere during one of the middle-joint-up or the front-link-forward. It means that algorithm stops before the last mot$(A_{i-1}, A_i, A_{i+1}, \rho)$ movement is terminated. Such a movement is called *partial-mot($A_{i-1}, A_i, A_{i+1}, \rho$)* movement. This is a movement in according with the mot$(A_{i-1}, A_i, A_{i+1}, \rho)$ movement, the movement stops somewhere during one of the middle-joint-up or the front-link-forward motion in such a way that A_3 remains on γ_2.

4 The reachability algorithm

In this section, we study reachability in a region bounded by two squares in which the whole or a part of a side of one square coincides with a part of a side of the other.

Assume S_1 and S_2 are two regions bounded by squares ∂S_1 and ∂S_2 with side lengths s_1 and s_2 respectively. Let squares ∂S_1 and ∂S_2 be attached via one side (the whole or a part of a side) and S = $S_1 \cup S_2$. Let $\Gamma = [0, 1, ...n]$ be an n-linkage confined within S_1 (Figure 4-a). In the following theorem we explain how A_n, the end of Γ, can reach a point p $\in S_2$.

Let ρ be the line segment shared by S_1 and S_2 and let v_1 and v_2 be two end points of ρ, where v_1 is the farthest point of ρ from p (Figure 4-b).

The following theorem presents sufficient condition for reachability of a given point in S by the end of a linkage confined within S.

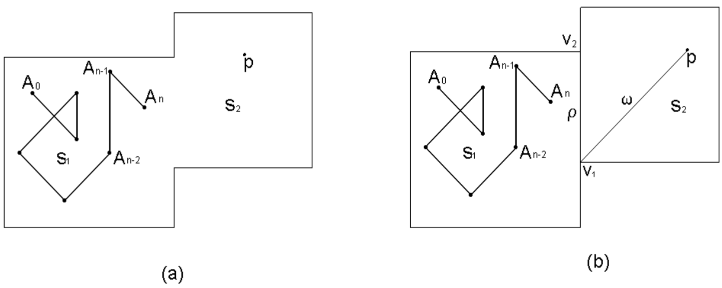

Figure 4: Γ *confined within* S_1 *and* $p \in S_2$.

Theorem 4. Suppose $p \in S_2$, Γ confined within S_1, $\|\Gamma\| < \mathrm{Min}\{\frac{\sqrt{2}}{2}s_1, \|\rho\|\}$, then with O($n^2$) simple motions – in the worst case – p can be reached by A_n.

Proof: We introduce an algorithm to bring A_n to p using O(n^2) simple motions, in the worst case.

Assume that ω is the line including v_1p, and moving from v_2 to v_1 on the side of ∂S_1 which includes v_2 and v_1 is clockwise. At the beginning we bring Γ to ONF in S_1. By theorem 1, this is done in O(n) simple motions. Without loss of generality we assume that Γ is placed on ∂S in counterclockwise order of indices of links'joints. Then Γ is moved along ∂S_1 counterclockwise until A_n reaches v_1. This can be done while no joint of Γ leaves ∂S_1.

We consider two cases: $d(p, v_1) \geq ||A_{n-1}A_n||$ and $d(p, v_1) < ||A_{n-1}A_n||$.

Case 1: $d(p, v_1) \geq ||A_{n-1}A_n||$. The algorithm consists of three steps. In the first step A_n is brought into S_2. In the second step Γ is moved so that $\Gamma[0, k_0]$ takes ONF in S_1 (k_0 will be defined in step 2), A_{k_0} coincides with v_1, and $\Gamma[k_0, n] \subset \omega$, and finally, in the last step A_n reaches p.

Step 1: Move Γ along ∂S_1 counterclockwise until A_{n-1} reaches v_1, because $\|\Gamma\| < \|\rho\|$, A_n doesn't pass v_2, this takes O(n) (Figure 5-a). Then rotate A_n clockwise about $A_{n-1} = v_1$ toward ω until A_n lies on ω. If $d(p, v_1) = ||A_{n-1}A_n||$, A_n reaches p and we are done. If not, we pass to the second step. This step takes O(n).

Step 2: We define $k_0 = \min\ \{\mathrm{k}\ | d(p, v_1) \geq \sum_{i=k+1}^{n} l_i\}$. Since $d(p, v_1) \geq l_n$, then $k_0 \leq n-1$. Suppose that, for $j > k_0$, $\Gamma[j, n] \subset \omega$ is straight, A_j coincides with v_1, and $\Gamma[1, j]$ gets ONF in S_1, by using mot$(A_{j-1}, A_j, A_{j+1}, \rho)$,

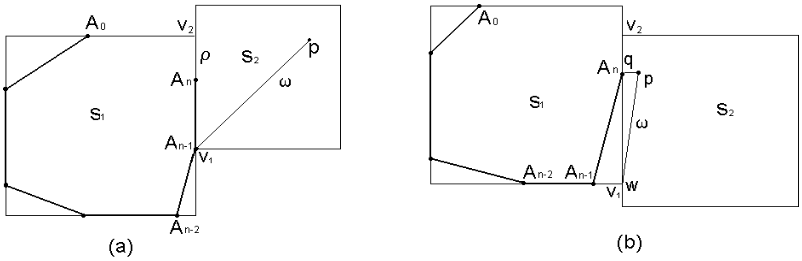

Figure 5: *(a):* $d(p, v_1) > \|A_{n-1}A_n\|$, *(b):* $d(p, v_1) < \|A_{n-1}A_n\|$ *and* $v_1 = \mathrm{w}$

Γ is moved to a configuration in which $\Gamma[j-1, n] \subset \omega$ straightens, A_{j-1} coincides with v_1, and $\Gamma[1, j-1]$ is in ONF in S_1.

By repeating this process, Γ can move to a configuration in which, $\Gamma[1, k_0]$ gets ONF, A_{k_0} coincides with v_1, and $\Gamma[k_0, n] \subset \omega$.

If $k_0 > 0$, since $\sum_{i=k_0}^{n} l_i > d(p, v_1) > \sum_{i=k_0+1}^{n} l_i$, A_n reaches p during mot$(A_{k_0-1}, A_{k_0}, A_{k_0+1}, \rho)$. Therefore we move Γ according to partial-mot$(A_{k_0-1}A_{k_0}, A_{k_0+1}, \rho)$, depending on values of $\angle v_2v_1p$, l_{k_0} and $d(p, v_1)$, A_n reaches p during one of the middle-joint-up motion or the front-link-forward motion. This step takes O(k_0n) and is O(n^2) in the worst case.

If $k_0 = 0$, A_n doesn't reach p during this step and we pass to step 3.

Step 3: In the case of $k_0 = 0$, i.e. $\sum_{i=1}^{n} l_i < d(p, v_1)$, by step2, Γ may move to a configuration in which, A_0 coincides with v_1 and $\Gamma \subset \omega$ straightens. It is enough to move Γ along ω toward p until A_n reaches p. This step takes O(1).

Case 2: $d(p, v_1) < \|A_{n-1}A_n\|$. Assume that ω intersects ∂S_1 at w (it is possible that w may coincides with v_1 (Figure 5-b)). Let the circle $C(v_1, \|pv_1\|)$ intersect v_1v_2 at q. To reach p, move Γ counterclockwise along ∂S_1 until A_n reaches q. Depending on the position of A_{n-1} on ∂S_1 one of the three following subcases occurs.

Subcase 2.1: A_{n-1} resides on the side of ∂S_1 containing v_1v_2. In this situation v_1 belongs to the link $A_{n-1}A_n$ and $C(p, l_n)$ intersects the line segment ω at point g . Rotate $A_{n-1}A_n$ clockwise about v_1 toward p. Because $\|\Gamma\| < \frac{\sqrt{2}}{2}s_1$, $C(g, l_{n-1})$ cannot contain S_1 i.e. A_{n-2} does not need to exit S_1. Continue rotation until A_{n-1} reaches g and A_n reaches p. During rotation, A_{n-1} exits ∂S_1 and if $C(g, l_{n-1})$ intersects ∂S_1, A_{n-2} can be stayed on ∂S_1 and $\Gamma[0...n-2]$ remains in ONF (Figure 6-a).

Otherwise if $C(g, l_{n-1})$ does not intersect ∂S_1, consider the largest $0 < k_0$ in such a way $C(g, l_{n-1}... + l_{k_0})$ intersects ∂S_1, otherwise let $k_0 = 1$. During rotation we let A_{n-1},...,A_{k_0} exit ∂S_1 while making $\alpha_{n-1} = ... = \alpha_{k_0+1} = \pi$, keeping $\Gamma[k_0...n-1]$ straight and remaining $\Gamma[0...k_0]$ in ONF.

Subcase 2.2: A_{n-1} resides on the side of ∂S_1 adjacent to the side containing v_1v_2, and ω intersects link A_nA_{n-1}. To reach p, first fix $\Gamma[0, 1, ...n-1]$ and rotate $A_{n-1}A_n$ about A_{n-1} toward p until link $A_{n-1}A_n$ reaches v_1. Then rotate $A_{n-1}A_n$ about v_1 toward ω until A_n hits ω. During rotation A_n does not hit ∂S_1. Finally slip $A_{n-1}A_n$ on ω until A_n reaches p. During the move-

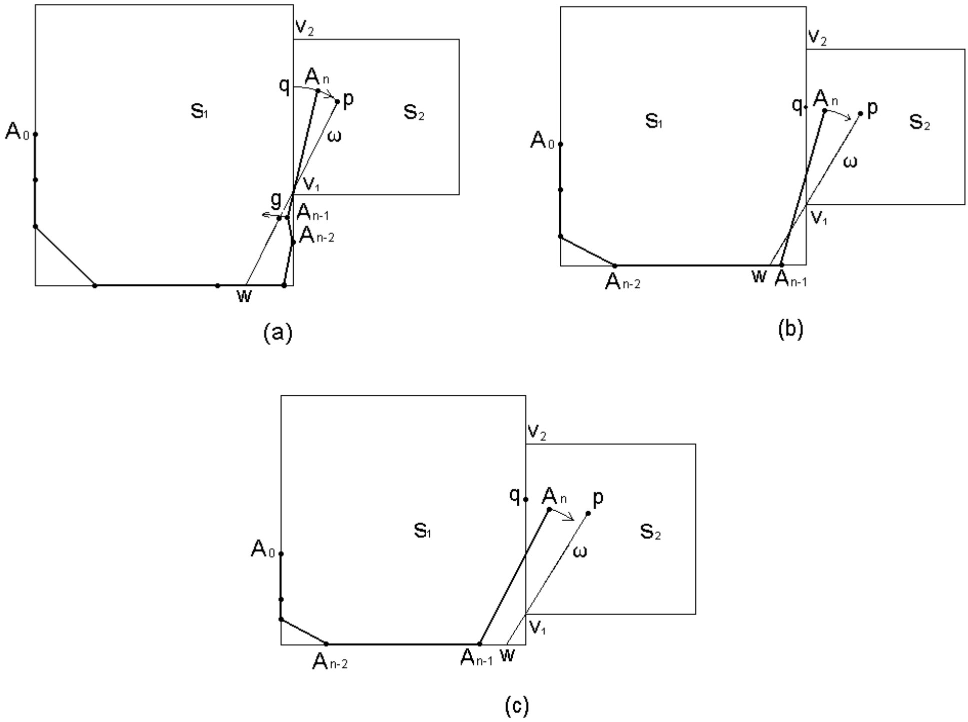

Figure 6: *(a):* A_{n-1} *belongs to the same edge as* v_1*, (b):* A_n *and* A_{n-1} *are in both sides of* ω*, (c):* A_n *and* A_{n-1} *are in the same side of* ω

ment, one of the possibilities similar to the previous situation will happen, which can be treated accordingly (Figure 6-b).

Subcase 2.3: Like case 2.2, but ω does not intersect link A_nA_{n-1}. Suppose that $C(p, l_n)$ intersects ∂S_1at g. i.e. p is visible from g. To reach p, first fix $\Gamma[0, 1, ...n-1]$ and rotate $A_{n-1}A_n$ about A_{n-1} toward ω until A_n reaches ω. Then, move A_n along ω toward p. During movement $\Gamma[0, 1, ...n-1]$ does not exit ∂S_1 and A_n gets to p while A_{n-1} reaches g. Refer to Figure 6-c.

Each of these subcases takes O(n).

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