An Abstract Theoretical Foundation of the Geometry of Digital Spaces

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Abstract. Two approaches to providing an abstract theoretical foundation to the geometry of digital spaces are presented and illustrated. They are critically compared and the possibility of combining them into a single theory is discussed. Such a theory allows us to state and prove results regarding geometrical concepts as they occur in a digital environment independently of the specifics of that environment. In particular, versions of the Jordan Curve Theorem are discussed in this general digital setting.

1 Introduction

One kind of digitization of (three-dimensional) space is obtained by tessellating it into cubes. Then there are natural notions of adjacencies between cubes determined, for example, by the sharing of a single face. However, the adjacencies may also be defined using edges and/or vertices. Tessellations of arbitrary-dimensional spaces into arbitrary polyhedra similarly give rise to various notions of adjacencies of these polyhedra. Alternatively, we may study a grid of points in an N-dimensional space and consider certain points adjacent by whatever criterion seems desirable to us. If we view such models as appropriate for capturing the notion of a digital (as opposed to continuous) space, then we see that the suitable underlying mathematical concept is a (possibly infinite) *graph*, which is a collection of vertices - corresponding to the above-mentioned spatial elements (*spels*, for short) - some pairs of which are considered adjacent. Our aim is to introduce geometrical concepts at this very general level, so that we get away from the specific consequences of the choices of tessellations and adjacencies. If we can prove nontrivial theorems in such a general setting, then these theorems will have nontrivial corollaries in all the specific manifestations of the general theory.

Certain continuous geometrical concepts have an immediate natural equivalent in a digital space. If ρ denotes the adjacency, then a *ρ-path* is a finite sequence of spels each but the last of which is ρ-adjacent to the one following it. A *ρ-connected set* S can then be defined as one in which for any pair of spels in S there is a ρ-path entirely in S from one to the other. A *simple closed ρ-curve* C is a nonempty finite ρ-connected set such that for each element in C there are

A. Braquelaire, J.-O. Lachaud, and A. Vialard (Eds.): DGCI 2002, LNCS 2301, pp. 279–288, 2002.

exactly two other elements in C ρ-adjacent to it. Other notions are more difficult to capture in a natural way. For example, the digital equivalent of the continuous notion of a simply connected set (one in which every simple closed curve can be continuously deformed into a point) is not so obviously definable: we need somehow to capture the digital correspondent of "continuously deformed."

In this article we discuss two (somewhat related) attempts at providing such abstract theoretical foundation to the geometry of digital spaces. We give a concise, but rigorous, description of both, including samples of theorems that have been proved. We follow this up with a discussion of the differences between the two approaches and speculate on a possible synthesis into a single theory. Finally we mention where we see the interesting open problems.

2 Digital Spaces

The material in this section is based on the approach presented in [1]. There a *digital space* is defined as a pair (V, π), with V an arbitrary nonempty set (of spels) and π a symmetric binary relation (called the *proto-adjacency*) on V such that V is π-connected.

A trivial example is when V consists of all the cubes into which space is tessellated and two cubes are in the relation π if, and only if, they share a single face. In mathematical notation, this is the digital space (Z^3, ω_3) defined, more generally for any positive integer N (the number of dimensions), by

$$Z^N = \{(c_1, \cdots, c_N) \mid c_n \in Z, \text{ for } 1 \leq n \leq N\}, \tag{1}$$

with Z being the set of integers, and ω_N the binary relation on Z^N satisfying

$$(c, d) \in \omega_N \;\Leftrightarrow\; \sum_{n=1}^{N} |c_n - d_n| = 1. \tag{2}$$

This trivial example illuminates the thinking underlying [1]: π has the alternative interpretation to just being an adjacency, a (c, d) in π can also be thought to represent the surface element (*surfel*, for short) facing from the spel c to the spel d. Thus any nonempty subset of π is referred to as a *surface* in (V, π). The *boundary between* subsets O and Q of V, defined as

$$\partial(O, Q) = \{(c, d) \mid (c, d) \in \pi,\ c \in O \text{ and } d \in Q\}, \tag{3}$$

is a surface provided that it is not empty.

The fact that π is a set of ordered pairs allows us to define, for any surface S, its *immediate interior* $II(S)$ and its *immediate exterior* $IE(S)$:

$$II(S) = \{c \mid (c, d) \in S \text{ for some } d \text{ in } V\}, \tag{4}$$

$$IE(S) = \{d \mid (c, d) \in S \text{ for some c in } V\}. \tag{5}$$

We say that a π-path $\langle c^{(0)}, \cdots, c^{(K)} \rangle$ *crosses* S if there is a k, $1 \leq k \leq K$, such that either $(c^{(k-1)}, c^{(k)}) \in S$ or $(c^{(k)}, c^{(k-1)}) \in S$. The surface S is said to be *near-Jordan* if every π-path from an element of $II(S)$ to an element of $IE(S)$ crosses S and it is said to be *Jordan* if it is near-Jordan and no nonempty proper subset of it is near-Jordan. This terminology is justified by the following theorem, which makes use of the notions of *interior* $I(S)$ and *exterior* $E(S)$ of S defined by

$$I(S) = \{c \in V \mid \text{there is a } \pi - \text{path connecting } c \text{ to} \\ \text{an element of } II(S) \text{ which does not cross } S\}, \tag{6}$$

$$E(S) = \{c \in V \mid \text{there is a } \pi - \text{path connecting } c \text{ to} \\ \text{an element of } IE(S) \text{ which does not cross } S\}. \tag{7}$$

Theorem 1. *A Jordan surface S in a digital space (V, π) has the following properties.*

1. $S = \partial(I(S), E(S))$.
2. $I(S) \cup E(S) = V$ *and* $I(S) \cap E(S) = \emptyset$.
3. *Both* $I(S)$ *and* $E(S)$ *are* π*-connected.*
4. *Every* π*-path from an element of* $I(S)$ *to an element of* $E(S)$ *crosses* S.

This theorem, which is Corollary 3.3.6 of [1], says that a Jordan surface S has properties reminiscent of those indicated by the Jordan Curve Theorem for simple closed curves in the plane: S is the boundary between its interior and its exterior, which do not intersect but contain all the spels between them, they are both π-connected, but one cannot get from the interior to the exterior by a π-path without crossing S. It is worthy of note that near-Jordanness is quite powerful by itself: with the possible exception of the third one, a near-Jordan surface has all the properties listed in Theorem 1 (as can be seen from Lemmas 3.2.1 and 3.2.2 of [1]). While it is quite impressive that such powerful-looking results can be stated (and proved) after only just a very few definitions, their practical usefulness is limited by the fact that it maybe very difficult (if not impossible) to check for an arbitrary surface whether or not it is near-Jordan. For this reason, [1] introduces a more desirable "local" property which under some circumstances implies near-Jordanness.

A surfel (c, d) is said to *cross* S if exactly one of $(c, d) \in S$ or $(d, c) \in S$. The surface S is said to be N*-locally-Jordan* (where N is a positive integer) if, for any π-path $P = \langle c^{(0)}, \cdots, c^{(K)} \rangle$ such that $(c^{(0)}, c^{(K)}) \in S$ and $2 \leq K \leq N + 1$, the number of surfels among $(c^{(0)}, c^{(1)}), \ \cdots, \ (c^{(K-1)}, c^{(K)})$ that cross S is odd. N-locally-Jordanness does not by itself imply near-Jordanness; we need to introduce two further conditions: one on the digital spaces (they have to be in some sense simply connected) and one on the surfaces (they have to be certain kinds of boundaries). We now discuss both of these conditions.

If

$$P = \left\langle c^{(1)}, \cdots, c^{(m)}, d^{(0)}, \cdots, d^{(n)}, e^{(1)}, \cdots, e^{(l)} \right\rangle \tag{8}$$

and

$$P' = \left\langle c^{(1)}, \cdots, c^{(m)}, f^{(0)}, \cdots, f^{(k)}, e^{(1)}, \cdots, e^{(l)} \right\rangle \tag{9}$$

are π-paths such that

$$f^{(0)} = d^{(0)},\ f^{(k)} = d^{(n)},\ \text{and } 1 \leq k + n \leq N + 2, \tag{10}$$

then P and P' are said to be *elementarily N-equivalent*. The digital space is said to be *N-simply connected* if, for any π-path $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ such that $c^{(K)} = c^{(0)}$, there is a sequence of π-paths $P_0, \cdots, P_L$ $(L \geq 0)$ such that $P_0 = \left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$, $P_L = \left\langle c^{(0)} \right\rangle$ and, for $1 \leq l \leq L$, P_{l-1} and P_l are elementarily N-equivalent. This is a useful concept, as indicated by Theorem 6.3.5 of [1]:

Theorem 2. *For any positive integer N, (Z^N, ω_N) is 2-simply connected.*

The kind of boundaries in which we are particularly interested appear in *binary pictures* over the digital space (V, π). These are defined as triples (V, π, f), with f a function mapping V into $\{0, 1\}$. Those spels which map into 0 are called *0-spels* and those which map into 1 are called *1-spels*. In order to give the intuitively desired interpretation to objects in binary pictures, we are forced to consider simultaneously more than one adjacency. For example, in the two binary pictures shown below (in which the spels are from Z^2), the 1-spels form a letter O and a letter C respectively.

0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	0	1	0	0	1	0	0
0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0

Note that neither the O nor the C forms an ω_2-connected set. On the other hand, if we define the binary relation δ_N on Z^N by

$$(c, d) \in \delta_N \ \Leftrightarrow \ \left(0 < \sum_{n=1}^{N} |c_n - d_n| \leq 2 \text{ and, for } 1 \leq n \leq N,\ |c_n - d_n| \leq 1 \right), \tag{11}$$

then both the O and the C are δ_2-connected, but the inside of the O is also δ_2-connected to its outside. For such reasons, it is customary and useful to consider different adjacencies for the 0-spels and for the 1-spels.

In [1] a symmetric binary relation ρ on V is called a *spel-adjacency* if $\pi \subseteq \rho$. If κ and λ are spel-adjacencies, then a surface S is called a $\kappa\lambda$-boundary in the binary picture (V, π, f) if there is κ-component O of 1-spels and a λ-component

Q of 0-spels, such that $S = \partial(O, Q)$. In the previously-presented binary pictures over the digital space (Z^2, ω_2), the letter O forms a single δ_2-component and its inside and outside each forms a single ω_2-component. Hence there are two $\delta_2\omega_2$-boundaries in the binary picture on the left. On the other hand, there is only one $\delta_2\omega_2$-boundary in the binary picture on the right.

For reasons explained quite in detail in early chapters of [1] (they have to do with being able to prove that certain computer procedures for multidimensional image processing perform as desired), it is useful to choose spel-adjacencies so that every non-empty $\kappa\lambda$-boundary is *$\kappa\lambda$-Jordan*, in the sense that it is near-Jordan, its interior is κ-connected and its exterior is λ-connected. (Note that in view of Theorem 1, a Jordan surface is always a $\pi\pi$-Jordan.) For this we need the concept of a *tight* spel-adjacency ρ, which is defined as having the property that, for all (c, d) in ρ, there exists a π-path $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ from c to d such that, for $0 \le k \le K$, either $(c^{(0)}, c^{(k)}) \in \rho$ or $(c^{(k)}, c^{(K)}) \in \rho$ (or possibly both). Clearly, both ω_N and δ_N are tight spel-adjacencies in (Z^N, ω_N). The following is Theorem 6.2.7 of [1], it holds for all positive integers N.

Theorem 3. *Let κ and λ be tight spel-adjacencies in an N-simply connected digital space (V, π). A $\kappa\lambda$-boundary is in a binary picture over (V, π) is $\kappa\lambda$-Jordan if, and only if, it is N-locally Jordan.*

The advantage of this theorem as compared to Theorem 1 is that the desirable property of being $\kappa\lambda$-Jordan follows from the property of being N-locally Jordan, which appears to be a condition that is easier to check than the condition of being near-Jordan. We now show that under some circumstances this appearance very much corresponds to reality.

We call a π-path $\left\langle c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}, c^{(0)} \right\rangle$ a *unit square* if both $c^{(0)} \neq c^{(2)}$ and $c^{(1)} \neq c^{(3)}$. An unordered pair $\{\kappa, \lambda\}$ of spel-adjacencies in a digital space is said to be a *normal pair* if, for any unit square $\left\langle c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}, c^{(0)} \right\rangle$, we have $(c^{(0)}, c^{(2)}) \in \kappa$ or $(c^{(1)}, c^{(3)}) \in \lambda$ or both. It is easy to prove that, for any positive integer N, $\{\delta_N, \omega_N\}$ is a normal pair in (Z^N, ω_N) (Theorem 6.3.8 of [1]).

Theorem 4. *If $\{\kappa, \lambda\}$ is a normal pair of spel-adjacencies in a digital space and S is a $\kappa\lambda$-boundary in a binary picture over the digital space, then S is 2-locally Jordan.*

This result (which is Lemma 6.3.3 of [1]) together with Theorems 2 and 3 implies that, for any positive integer N, every $\delta_N\omega_N$-boundary in any binary picture over (Z^N, ω_N) is $\delta_N\omega_N$-Jordan. To emphasize what this means in the special case of tessellating space into cubes, we spell out in full its consequences in that space (Corollary 6.3.9 of [1]).

Theorem 5. *Let A be a nonempty proper subset of Z^3. Let O be a δ_3-component of A and Q be an ω_3-component of $Z^3 \backslash A$, such that $\partial(O, Q)$ is not empty. Then there exists two uniquely defined subsets I and E of Z^3 with the following properties.*

1. *$O \subseteq I$ and $Q \subseteq E$.*
2. *$\partial(O, Q) = \partial(I, E)$.*
3. *$I \cup E = Z^3$ and $I \cap E = \emptyset$.*
4. *I is a δ_3-connected subset of Z^3 and E is an ω_3-connected subset of Z^3.*
5. *Every ω_3-path connecting an element of I to an element of E crosses $\partial(O, Q)$.*

From the point of view of our discussion here, the important aspect of this theorem is not what it says but rather the fact that the general results discussed earlier immediately yield similar theorems for other tessellations of three and other dimensional spaces and even for digital spaces which are obtained by means other than tessellating a Euclidean space. Many specific examples of this are presented in [1].

3 Generic Axiomatized Digital Surface-Structures

The material in this section is based on the approach presented in [2]. That paper aims at providing an axiomatic foundation of digital topology; it succeeds in this only partially inasmuch that it deals only with discrete structures which model subsets of the Euclidean plane and of other surfaces. Its main advantage over the approach of the previous section is its treatment of the digital version of "continuously deformed": instead of having the awkward hierarchy of elementary N-equivalences, the allowable deformations are embedded into the very definitions of the structures, in the form of some "loops."

A basic difference between the conventions of [1] and [2] is that in the latter adjacencies are represented by unordered pairs. In order to accommodate the terminology of [2], we define a *proto-edge* in a digital space to be any two-element set $\{c, d\}$ such that (c, d) is a surfel.

A *2D digital complex* is defined as a triple $(V, \pi, \mathcal{L})$, where (V, π) is a digital space and $\mathcal{L}$ is a set of simple closed π-curves (its elements are called *loops*) such that the following conditions hold:

1. V contains more than one spel and if $(c, d) \in \pi$, then $c \neq d$.
2. For any two distinct loops L_1 and L_2, $L_1 \cap L_2$ is either empty, or consists of a single spel, or is a proto-edge.
3. No proto-edge is included in more than two loops.
4. Each spel belongs to only a finite number of proto-edges.

If $\mathcal{L}_{2\times 2}$ is the set of all $\left\{c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}\right\}$ such that $\left\langle c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}, c^{(0)}\right\rangle$ is a unit square in (Z^2, ω_2), then $(Z^2, \omega_2, \mathcal{L}_{2\times 2})$ is a 2D digital complex. Now we see why we use the term 2D digital complex: if we tried to do the same for the space (Z^3, ω_3) we would violate Condition 3 of the definition, since each proto-edge would be included in four loops.

For an arbitrary spel-adjacency ρ in (V, π), let the P in (8) and the P' in (9) be two ρ-paths. They are said to be *elementarily loop-equivalent* in $(V, \pi, \mathcal{L})$ if

1. either there is a proto-edge $\{c, d\}$ such that one of $\left\langle d^{(0)}, \cdots, d^{(n)}\right\rangle$ and $\left\langle f^{(0)}, \cdots, f^{(k)}\right\rangle$ is $\langle c\rangle$ and the other is $\langle c, d, c\rangle$,

2. or $f^{(0)} = d^{(0)}$, $f^{(k)} = d^{(n)}$, and there is a loop which contains $d^{(0)}, \cdots, d^{(n)}$ and $f^{(0)}, \cdots, f^{(k)}$.

A ρ-path $\langle c^{(0)}, \cdots, c^{(K)} \rangle$ such that $c^{(K)} = c^{(0)}$, is said to be ρ-reducible in $(V, \pi, \mathcal{L})$ if there is a sequence of ρ-paths $P_0, \cdots, P_L$ such that $P_0 = \langle c^{(0)}, \cdots, c^{(K)} \rangle$, $P_L = \langle c^{(0)} \rangle$ and, for $1 \leq l \leq L$, P_{l-1} and P_l are elementarily loop-equivalent in $(V, \pi, \mathcal{L})$. The 2D digital complex $(V, \pi, \mathcal{L})$ is said to be *simply connected* if every π-path $\langle c^{(0)}, \cdots, c^{(K)} \rangle$ such that $c^{(K)} = c^{(0)}$ is π-reducible in it.

A spel c of $(V, \pi, \mathcal{L})$ is called *interior* if every proto-edge that contains c is included in two loops. A 2D digital complex is called a *pseudomanifold* if all its spels are interior. (Trivially, every proto-edge of a pseudomanifold is included in exactly two loops.) It is easy to see that $(Z^2, \omega_2, \mathcal{L}_{2\times 2})$ is a pseudomanifold.

Two loops L and L' of a 2D digital complex are said to be *adjacent* if $L \cap L'$ is a proto-edge. A subset $\mathcal{L}'$ of $\mathcal{L}$ is said to be *strongly connected* if, for any two loops L and L' of $\mathcal{L}'$, there exists a sequence $L_0, \cdots, L_K$ of loops in $\mathcal{L}'$ such that $L_0 = L$, $L_K = L'$ and, for $1 \leq k \leq K$, L_{k-1} and L_k are adjacent. The 2D digital complex $(V, \pi, \mathcal{L})$ is said to be *strongly connected* if $\mathcal{L}$ is strongly connected. A spel c is said to be a singularity of a 2D digital complex if the set of all loops that contain c is not strongly connected. The following is Proposition 3.7 of [2].

Theorem 6. *A 2D digital complex that is both simply connected and strongly connected has no singularities.*

For an arbitrary spel-adjacency ρ in (V, π), let C be a simple closed ρ-curve. A ρ-path $\langle c^{(0)}, \cdots, c^{(|C|)} \rangle$, where $|C|$ denotes the number of elements in C, which is such that $c^{(|C|)} = c^{(0)}$ and $C = \{c^{(1)}, \cdots, c^{(|C|)}\}$ is called a ρ-parameterization of C. (Note that this exists.) We say that in this parameterization $c^{(k-1)}$ *precedes* $c^{(k)}$ and $c^{(k)}$ *follows* $c^{(k-1)}$, for $1 \leq k \leq |C|$.

Let L and L' be two adjacent loops of a 2D digital complex and let $\{c, d\} \in L \cap L'$. We say that a π-parameterization of L is *coherent* with a π-parameterization of L' if c precedes d in one of the π-parameterizations and c follows d in the other. A 2D digital complex $(V, \pi, \mathcal{L})$ is said to be *orientable* if there is a function Ω with domain $\mathcal{L}$ such that:

1. For each loop L in $\mathcal{L}$, $\Omega(L)$ is a π-parameterization of L.
2. For all pairs of adjacent loops L and L', $\Omega(L)$ is coherent with $\Omega(L')$.

It is easily seen that $(Z^2, \omega_2, \mathcal{L}_{2\times 2})$ is orientable.

A *generic axiomatized digital surface-structure* (or *GADS*, for short) is a pair $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$, where $(V, \pi, \mathcal{L})$ is a 2D digital complex (called the *complex* of $\mathcal{G}$, whose spels, proto-edges and loops are also referred to as the spels, proto-edges and loops of $\mathcal{G}$) and κ and λ are spel-adjacencies in (V, π) that satisfy:

Axiom 1. If $(c, d) \in \kappa \cup \lambda$, then $c \neq d$.
Axiom 2. If $(c, d) \in (\kappa \cup \lambda) \setminus \pi$, then some loop contains both c and d.
Axiom 3. If $\{c, d\}$ is a subset of a loop L, but it is not a proto-edge, then
(a) $(c, d) \in \lambda$ if, and only if, $L \setminus \{c, d\}$ is not κ-connected and
(b) $(c, d) \in \kappa$ if, and only if, $L \setminus \{c, d\}$ is not λ-connected.

(Note that if in the underlying complex it is the case that, for every unit square $\langle c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}, c^{(0)} \rangle$, $\{c^{(0)}, c^{(1)}, c^{(2)}, c^{(3)}\}$ is a loop, then Axiom 3 implies that $\{\kappa, \lambda\}$ is a normal pair of spel-adjacencies.) The following is Theorem 6.1 of [2].

Theorem 7. *Let C be a simple closed $(\kappa \cap \lambda)$-curve contained in a loop L of a GADS $((V, \pi, \mathcal{L}), (\kappa, \lambda))$. Then C has one of the following properties:*

1. *For all distinct c and d in C, $(c, d) \in \kappa$.*
2. *For all distinct c and d in C, $(c, d) \in \lambda$.*

A GADS $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ is a *subGADS* of a GADS $((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$ if:

1. $V \subseteq V'$, $\pi \subseteq \pi'$ and $\mathcal{L} \subseteq \mathcal{L}'$.
2. For all $L \in \mathcal{L}$, $\kappa \cap L^2 = \kappa' \cap L^2$ and $\lambda \cap L^2 = \lambda' \cap L^2$, where L^2 denotes the set of ordered pairs of elements of L.

(It is a consequence of this definition and Axiom 2 that if $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ is a subGADS of $((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$, then $\kappa \subseteq \kappa'$ and $\lambda \subseteq \lambda'$.)

A GADS assumes the properties of its complex; thus a GADS is said to be simply connected, strongly connected or orientable if its complex is simply connected, strongly connected or orientable, respectively. Thus we can state the following, which is Proposition 5.1 of [2].

Theorem 8. *A subGADS of a simply connected GADS is orientable.*

We are now ready to state Theorem 8.1 of [2], which is a very general version of the Jordan Curve Theorem for GADS. In it we make use of the concept of a pGADS, which is a GADS whose complex is a pseudomanifold.

Theorem 9. *Let $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a GADS that is a subGADS of an orientable pGADS $((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$ whose complex has no singularities. Let P be a κ-parameterization of a simple closed κ-curve C in V such that:*

1. *C is not included in any loop in $\mathcal{L}$.*
2. *Every spel in C is an interior spel of $(V, \pi, \mathcal{L})$.*
3. *P is κ'-reducible in $(V', \pi', \mathcal{L}')$.*

Then $V \setminus C$ has exactly two λ-components and, for each spel c in C, $N_\lambda(c)$ (the set of all spels λ-adjacent to c) intersects both of these λ-components.

To illustrate the applicability of this theorem, we discuss its implication for $((Z^2, \omega_2, \mathcal{L}_{2\times2}), (\delta_2, \omega_2))$, which is easily checked to be a GADS. Based on previously made remarks, we see that it is in fact an orientable pGADS and it is easy to see that its complex has no singularities. In this application of Theorem 9 we can use this same GADS for the two GADS mentioned in that theorem. Furthermore, it is not difficult to prove (similarly how our Theorem 2 is proved in [1]) that every δ_2-path is δ_2-reducible in $(Z^2, \omega_2, \mathcal{L}_{2\times2})$. Recognizing that a simple closed δ_2-curve in Z^2 which contains at least four spels cannot be contained in an element of $\mathcal{L}_{2\times2}$ and that every spel of a pGADS is interior, all this proves the following (which is purposely stated to resemble Theorem 5):

Theorem 10. *Let C be a simple closed δ_2-curve in Z^2 which contains at least four spels. Then there exists two uniquely defined nonempty subsets I and E of Z^2 with the following properties.*

1. *$I \subseteq Z^2 \setminus C$ and $E \subseteq Z^2 \setminus C$.*
2. *For every c in C, both $I \cap N_{\omega_2}(c)$ and $E \cap N_{\omega_2}(c)$ are nonempty.*
3. *$I \cup E \cup C = Z^2$ and $I \cap E = \emptyset$.*
4. *Both I and E are ω_2-connected subsets of Z^2.*
5. *Every ω_2-path connecting an element of I to an element of E contains an element of C.*

Again, the important aspect of such a theorem is that it is just one example of many similar theorems that can be derived from Theorem 9 for a variety of GADS. Some examples of interesting GADS are given in [2].

4 Discussion

Although the approaches of the two previous sections are clearly related, there are many differences between them. An inessential one is expressed by Condition 1 of the definition of a 2D digital complex; such restrictions are not made in the definition of a digital space. However, nothing interesting can be said without this restriction which cannot be said in its presence and so there is no harm in restricting our study of digital spaces to those which satisfy Condition 1.

A more interesting difference is due to the use of proto-edges rather than surfels. The concept of surfel is not even mentioned in [2], everything there is developed in terms of proto-edges. That material has been rewritten for Section 3 so as to make it notationally consistent with the previous section. However, there is more than notation at stake here. The use of surfels allowed us to define in a natural way the concept of a surface. The corresponding concept in GADS is a simple closed curve, which is a very different sort of animal: surfaces in digital spaces are sets of surfels, while curves in GADS are sets of spels. The consequences of this difference in approach become evident when comparing Theorems 5 and 10. Since both approaches have been utilized in the literature, it seems desirable to develop a theory capable to deal with them simultaneously.

To investigate this, let us start with a situation in which all the conditions of Theorem 9 are satisfied (as a specific illustration consider Theorem 10 and the letter O on the left of the previously shown figure as the simple closed δ_2-curve C). Creating a binary picture in which the 1-spels are elements of C (which is κ-connected), we see that there are exactly two $\kappa\lambda$-boundaries, namely $\partial(C, Q_1)$ and $\partial(C, Q_2)$ with Q_1 and Q_2 being the two λ-components of $V \setminus C$. Under what additional conditions on the GADS in Theorem 9 are these two boundaries guaranteed to be $\kappa\lambda$-Jordan? (That they can indeed be such is illustrated by the corresponding boundaries of the O in the figure; see also the more general statement after Theorem 4.) If they were, then we would have a way of going from C to surfaces which approximate it and have desirable properties.

On the other hand suppose that we have a $\kappa\lambda$-Jordan surface S in the digital space of the underlying complex of a GADS. Under what conditions is $II(S)$ a simple closed κ-curve? If it is and if the other conditions of Theorem 9 are also satisfied, then one of the λ-components of $V \setminus II(S)$ implied by Theorem 9 has to be $E(S)$ and the other has to be $I(S) \setminus II(S)$. (This implies that $I(S)$ itself is λ-connected, in addition to being κ-connected, giving us one necessary condition.) Symmetrical conditions would be obtained for insuring that $IE(S)$ is a simple closed λ-curve. One might even investigate the circumstances under which $IE(S)$ is a simple closed κ-curve, but this seem less likely to lead to useful results for a $\kappa\lambda$-Jordan surface S.

A major disadvantage of the GADS-based approach is its restriction to 2D digital complexes and consequently (in that approach) to curves to play the role of "surfaces which separate space into two components." An attempt has been made in [1] to introduce more general structures to fulfill such a role. In any digital space (V, π), a nonempty subset of P of V is called a *spel-manifold* if it satisfies the following three conditions:

1. P is π-connected.
2. For each $c \in P$, $N_\pi(c) \setminus P$ has two π-components,
3. For each $c \in P$ and for each $d \in N_\pi(c) \cap P$, $N_\pi(d)$ has a nonempty intersection with both π-components of $N_\pi(c) \setminus P$.

This definition is half satisfactory in the sense that it is the case that [1, Theorem 7.3.1] if P is a spel-manifold in a digital space (V, π), then $V \setminus P$ has at most two π-components, but it is not guaranteed to have more than one π-component, even if we restrict ourselves to rather special digital spaces [1, Theorem 7.3.2]. It appears desirable to find some nice conditions on a set of spels which would guarantee that it is a $\kappa\lambda$-*manifold* in the sense that it is κ-connected and its complement has exactly two λ-components.

Another intriguing approach is to generalize the notion of a 2D digital complex to N dimensions. Even though it is not too difficult to envision how to do this satisfactorily (an inductive definition is a possibility), it is less clear how such a generalization can be combined with spel-adjacencies; in particular, Axiom 3 seems to be very anchored to the two-dimensional environment.

In summary, the previous two sections have illustrated that powerful results can be proven in an abstract framework; these results have immediate consequences in the many specific theories that have been put forward to study geometry in a digital framework. However, as discussed in this final section, not all aspect of such a theoretical foundation have yet been satisfactorily resolved.

References

1. Herman, G.T.: Geometry of Digital Spaces. Birkhäuser, Boston Basel Berlin (1998)
2. Fourey, S., Kong, T.Y., Herman, G.T.: Generic Axiomatized Digital Surface-Structures, submitted for publication in Disc. Appl. Math. (preliminary version appeared in Electronic Notes in Theoretical Computer Science **46** (2001) see http://www.elsevier.nl/gej-ng/31/29/23/86/27/show/Products/notes/index.htt)