Discretization in 2D and 3D Orders

Michel Couprie, Gilles Bertrand, and Yukiko Kenmochi

Laboratoire A[2]SI, ESIEE Cité Descartes, B.P. 99
93162 Noisy-Le-Grand Cedex France, {coupriem,bertrand,kenmochy}@esiee.fr

Abstract. Among the different discretization schemes that have been proposed and studied in the literature, the supercover is a very natural one, and furthermore presents some interesting properties. On the other hand, an important structural property does not hold for the supercover in the classical framework: the supercover of a straight line (resp. a plane) is not a discrete curve (resp. surface) in general.
We follow another approach based on a different, heterogenous discrete space which is an order, or a discrete topological space in the sense of Paul S. Alexandroff. Generalizing the supercover discretization scheme to such a space, we prove that the discretization of a plane in $\mathbb{R}^3$ is a discrete surface, and we prove that the discretization of the boundary of a "regular" set X (in a sense that will be precisely defined) is equal to the boundary of the discretization of X. This property has an immediate corollary for half-spaces and planes, and for convex sets.

Keywords: discretization, topology, orders, supercover, discrete surfaces

1 Introduction

An abundant literature is devoted to the study of discretization schemes. Let $\mathcal{E}$ be an "Euclidean" space, and let $\mathcal{D}$ be a "discrete" space related to $\mathcal{E}$. Typically, one can take $\mathcal{E} = \mathbb{R}^n$ and $\mathcal{D} = \mathbb{Z}^n$ $(n = 2, 3)$, but we do not limit ourselves to this case. A discretization scheme associates, to each subset X of $\mathcal{E}$, a subset $D(X)$ of $\mathcal{D}$ which is called the *discretization of* X. Different discretization schemes have been proposed and compared with respect to some fundamental geometrical, topological and structural properties. We may, for example, ask the following questions: if $X' \subseteq \mathcal{E}$ is the image of X by a symmetry, is $D(X')$ the image of $D(X)$ by the same symmetry ? If X is connected, is $D(X)$ connected (in some sense) ? And if X is a curve, is $D(X)$ a curve (in some sense) ?

In this paper, we consider the discretization scheme called supercover, and we focus on some structural properties. Consider $\mathcal{E} = \mathbb{R}^2$, for simplicity, and let $\mathcal{D}$ be the set of all closed squares in $\mathcal{E}$ with side 1 and the vertices of which have integer coordinates (the elements of $\mathcal{D}$ are often called *pixels*). Let X be a subset of $\mathcal{E}$, the *supercover* of X is the set of all the pixels that have a non-empty intersection with X.

The supercover has many interesting properties, which have been studied by several authors [9,10,2,1,8,21,22]. In particular, Andrès [1] proposed an analytical characterization of the supercover of straight lines, and more generally

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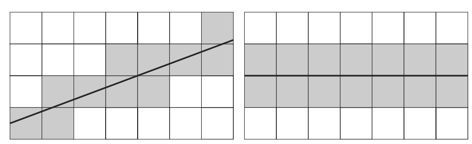
for hyperplanes and for simplices in higher dimensions. Also, Ronse and Tajine showed that the supercover is a particular case of Hausdorff discretization [21, 22].

But the supercover has also a drawback for thin objects such as straight lines. If a straight line δ in $\mathbb{R}^2$ goes through a point with integer coordinates, then the supercover of δ contains the four pixels that cover this point - this configuration is called a "bubble" (Fig. 1(a)). An extreme case is when δ is horizontal or vertical, and hits elements of $\mathbb{Z}^2$ (Fig. 1(b)): the supercover of such a line is 2-pixel thick. Thus, the supercover of a straight line cannot be seen as a discrete curve.



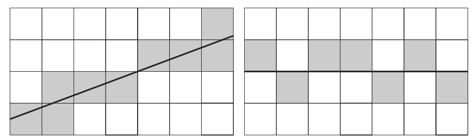
(a) (b)

Fig. 1. (a): A straight line segment and its supercover (shaded), which contains a "bubble" (set of four pixels sharing a common vertex). (b): A horizontal line segment that has a 2-pixel thick supercover.

Another popular discretization scheme for lines, called *grid intersection digitization* [17,23], does guarantee that the discretization of a straight line δ is a digital curve, in the sense of the digital topology [16]. A proof of this property can be found in [15]. The drawback of this discretization scheme is its lack of symmetry: for any intersection of δ with a pixel boundary, the pixel vertex which is closest to this intersection is chosen as an element of the discretization of δ, and if the intersection is at equal distance between two vertices, then an arbitrary choice is made (for example, the rightmost or upmost vertex). This drawback is shared by other discrete models for straight lines and planes, the Bresenham's model [7], the naive model [20] and the standard model [1]. On the other hand, the supercover does not suffer from this lack of symmetry.

An attempt to solve the problem of "bubbles", which seems to be the price payed for symmetry, has been made in [8] with the notion of minimal cover. Let X be a subset of $\mathbb{R}^2$. Any set S of pixels, such that X is included in the union of the elements of S, is called a *cover* of X. Let S be a cover of X, we say that S is a *minimal cover of* X if there is no other cover of X which is a proper subset of S. We see in Fig. 2 that the minimal cover of certain straight lines is "thinner" than the supercover, but we see also that the minimal cover is not unique in general.

We follow another approach based on a different, heterogenous discrete space which is an order, or a discrete topological space in the sense of Paul S. Alexandroff [3]. Such spaces have been the subject of intensive research in the recent past, not only from the topology point of view [13,18,11,5], but also in relation with discretization and geometrical models [14,25]. The discrete space $\mathcal{D}$ that



(a) (b)

Fig. 2. (a): A straight line segment and its minimal cover (shaded). (b): A horizontal line segment and one of its possible minimal covers.

we will consider is a partition of the Euclidean space $\mathcal{E}$, composed (in the case of $\mathcal{E} = \mathbb{R}^2$) of open unit squares, unit line segments and singletons. The fact that $\mathcal{D}$ is a partition of $\mathcal{E}$ leads to a fundamental property: for any subset X of $\mathcal{E}$, the supercover of X (relative to $\mathcal{D}$) is the unique minimal cover of X. We will focus on this discretization scheme, and discuss only the 3D case in the sequel (corresponding results in 2D are particular cases).

The two main contributions of this paper are the following results. (i) We prove that the discretization of a plane in $\mathbb{R}^3$ is a discrete surface. (ii) We prove that the discretization of the boundary of a "regular" set X (in a sense that will be precisely defined) is equal to the boundary of the discretization of X. This property has an immediate corollary for half-spaces and planes, and for convex sets.

2 Basic Notions on Orders

In this section, we recall some basic notions relative to orders (see also [13,5,6]).

If X is a set, $\mathcal{P}(X)$ denotes the set composed of all subsets of X, if S is a subset of X, $\overline{S}$ denotes the complement of S in X. If S is a subset of T, we write $S \subseteq T$, the notation $S \subset T$ means that S is a proper subset of T, i.e. $S \subseteq T$ and $S \neq T$. If γ is a map from $\mathcal{P}(X)$ to $\mathcal{P}(X)$, the *dual of* γ is the map $*\gamma$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that, for each $S \subseteq X$, $*\gamma(S) = \overline{\gamma(\overline{S})}$. Let δ be a binary relation on X, *i.e.*, a subset of $X \times X$. We also denote by δ the map from X to $\mathcal{P}(X)$ such that, for each x of X, $\delta(x) = \{y \in X,\ (x, y) \in \delta\}$. We define $\delta^{\square}$ as the binary relation $\delta^{\square} = \delta \setminus \{(x, x); x \in X\}$.

An *order* is a pair (X, α) where X is a set and α is a reflexive, antisymmetric, and transitive binary relation on X. An element of X is also called a *point*. The set $\alpha(x)$ is called the α-*adherence* of x, if $y \in \alpha(x)$ we say that y is α-*adherent to* x.

We illustrate these general notions on orders with the example of Fig. 3, which is composed of the following elements : two triangles t_1, t_2; five edges e_1, e_2, e_3, e_4, e_5; and four vertices v_1, v_2, v_3, v_4. Here, we define the order relation α by: $\alpha(t_1) = \{t_1, e_1, e_2, e_3, v_1, v_2, v_3\}$; $\alpha(t_2) = \{t_2, e_2, e_4, e_5, v_2, v_3, v_4\}$; $\alpha(e_1) = \{e_1, v_1, v_2\}$; $\alpha(e_2) = \{e_2, v_2, v_3\}$; $\alpha(e_3) = \{e_3, v_1, v_3\}$; $\alpha(e_4) = \{e_4, v_3, v_4\}$; $\alpha(e_5) = \{e_5, v_2, v_4\}$; and for $i = 1 \ldots 4$, $\alpha(v_i) = \{v_i\}$.

Let (X, α) be an order. We denote by α the map from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that, for each subset S of X, $\alpha(S) = \cup\{\alpha(x);\ x \in S\}$, $\alpha(S)$ is called the α-

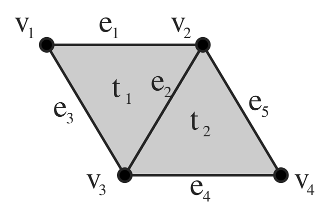


Fig. 3. An example for the basic notions on orders.

closure of S, $*\alpha(S)$ is called the α*-interior of* S. A subset S of X is α*-closed* if $S = \alpha(S)$, S is α*-open* if $S = *\alpha(S)$.

In our example of Fig. 3, let S be the set $\{t_1, e_1, e_5, v_2\}$. We see that $\alpha(S) = \{t_1, e_1, e_2, e_3, e_5, v_1, v_2, v_3, v_4\} \neq S$, thus S is not α-closed. We can also see that $*\alpha(S) = \{t_1, e_1\} \neq S$, thus S is not α-open. On the opposite, $\alpha(t_1)$, $\{e_2, e_5, v_2, v_3, v_4\}$, $\{v_1\}$ for example are α-closed, and $\{t_1\}$, $\{t_1, t_2, e_2\}$ for example are α-open.

Let (X, α) be an order. We denote by β the relation $\beta = \{(x, y); (y, x) \in \alpha\}$, β is the *inverse* of the relation α. We denote by θ the relation $\theta = \alpha \cup \beta$. The *dual of the order* (X, α) is the order (X, β).
Notice that $*\alpha(S) = \{x \in S;\ \beta(x) \subseteq S\}$, and $*\beta(S) = \{x \in S;\ \alpha(x) \subseteq S\}$.

In our example of Fig. 3, $\beta(v_2) = \theta(v_2) = \{v_2, e_1, e_2, e_5, t_1, t_2\}$; $\beta(e_2) = \{e_2, t_1, t_2\}$; $\theta(e_2) = \{v_2, v_3, e_2, t_1, t_2\}$; $\beta(t_1) = \theta(t_1) = \{t_1\}$.

The set composed of all α-open subsets of X satisfies the conditions for the family of open subsets of a topology, the same result holds for the set composed of all β-open subsets of X. These topologies are P.S. Alexandroff topologies, *i.e.*, topologies such that every intersection of open sets is open [3].

An order (X, α) is *countable* if X is countable, it is *locally finite* if, for each $x \in X$, $\theta(x)$ is a finite set. A *CF-order* is a countable locally finite order.

Let (X, α) be a CF-order. Let x_0 and x_k be two points of X. A *path* from x_0 to x_k is a sequence $x_0, x_1, ..., x_k$ of elements of X such that $x_i \in \theta(x_{i-1})$, with $i = 1, ..., k$. A CF-order (X, α) is *connected* if for all x, y in X, there is a path from x to y.

If (X, α) is an order and S is a subset of X, the *order relative to* S is the order $|S| = (S, \alpha \cap (S \times S))$.

We will use a general definition for curves and surfaces which has been used in several works (see e.g. [11,6]). This notion is close to the notion of manifold used by Kovalevsky [18]; nevertheless it does not involve the necessity to attach a notion of dimension to each element of X, which allows to have a simpler definition (in particular, no use of isomorphism is made).

Let $|X| = (X, \alpha)$ be a non-empty CF-order.
- The order $|X|$ is a *0-surface* if X is composed exactly of two points x and y such that $y \notin \alpha(x)$ and $x \notin \alpha(y)$.
- The order $|X|$ is an *n-surface*, $n > 0$, if $|X|$ is connected and if, for each x in X, the order $|\theta^\square(x)|$ is an $(n-1)$-surface.
- A *curve* is a 1-surface, a *surface* is a 2-surface.

In our example of Fig. 3, the orders relative to the following sets: $\{v_1, e_1, v_2, e_5, v_4, e_4, v_3, e_3\}$ and $\{v_1, e_1, v_2, t_2, v_3, e_3\}$, are both curves. Conversely, the order depicted in Fig. 3 is not a surface, since for example, $\theta^{\square}(v_1) = \{e_1, t_1, e_3\}$ is not a curve.

3 An Order Associated to $\mathbb{R}^n$

Let $\mathbb{R}$ be the set of real numbers. We consider the families of subsets of $\mathbb{R}$ named $\mathcal{G}_0^1$, $\mathcal{G}_1^1$ and $\mathcal{G}^1$ such that:

$$\mathcal{G}_0^1 = \{\{p + \tfrac{1}{2}\}, p \in \mathbb{Z}\},$$
$$\mathcal{G}_1^1 = \{]p - \tfrac{1}{2}, p + \tfrac{1}{2}[, p \in \mathbb{Z}\},$$
$$\mathcal{G}^1 = \mathcal{G}_0^1 \cup \mathcal{G}_1^1.$$

A subset R of $\mathbb{R}^n$ which is the cartesian product of exactly m elements of $\mathcal{G}_1^1$ and $n - m$ elements of $\mathcal{G}_0^1$ is called an *m-gel* of $\mathbb{R}^n$.

For a given integer m, we denote by $\mathcal{G}_m^n$ the set of all m-gels of $\mathbb{R}^n$, and we denote by $\mathcal{G}^n$ the union of all the sets $\mathcal{G}_m^n$, for all $m = 0 \ldots n$. An element of $\mathcal{G}^n$ is called a *gel*.

For example, with $n = 2$, a 0-gel is a singleton (a set containing a single point), a 1-gel is a line segment which does not contain its extremities (either of the form $\{p + \frac{1}{2}\} \times]q - \frac{1}{2}, q + \frac{1}{2}[$ or $]p - \frac{1}{2}, p + \frac{1}{2}[\times \{q + \frac{1}{2}\}$), and a 2-gel is an open square.

We remark that, according to the "standard" topology of $\mathbb{R}^n$, only the n-gels are open subsets of $\mathbb{R}^n$ (they are open hypercubes), and that only the 0-gels are closed subsets of $\mathbb{R}^n$ (they are singletons). For $0 < m < n$, an m-gel is neither open nor closed.

On the opposite, all pixels (see section 1) are closed subsets of $\mathbb{R}^2$. Notice also that $\mathcal{G}^n$ is a partition of $\mathbb{R}^n$, this is not the case with the covering of $\mathbb{R}^2$ with pixels.

Let x be a gel, we denote by $\mathrm{cl}(x)$ the closure of x (according to the "standard" topology of $\mathbb{R}^n$).

We consider the order $(\mathcal{G}^n, \alpha)$ defined by: $\forall x, y \in \mathcal{G}^n$, $y \in \alpha(x)$ if $y \subseteq \mathrm{cl}(x)$. For example, with $n = 2$, let x be an open square (a 2-gel). Then, $\alpha(x)$ is composed of x itself, of the four line segments that border x (without the vertices), and of the four singletons containing each a vertex of $\mathrm{cl}(x)$.

Notice that these orders are equivalent to those obtained in the framework of connected ordered topological spaces introduced by E.D. Khalimsky [12]. As far as we know, the first mention of $(\mathcal{G}^n, \alpha)$ as a discrete topological space can be found in the classical topology textbook by P.S. Alexandroff and H. Hopf [4], as one of the first examples used to illustrate the notion of a topological space.

4 Generalized Covers and Supercovers

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^n$ ($n \geq 1$). We say that the family $\mathcal{F}$ *covers* $\mathbb{R}^n$ if $\mathbb{R}^n$ is equal to the union of all the elements of $\mathcal{F}$. In the following, we will consider the families $\mathcal{G}^n$ and $\mathcal{G}^n_n$. Notice that $\mathcal{G}^n$ does cover $\mathbb{R}^n$, but $\mathcal{G}^n_n$ does not. Let R be any subset of $\mathbb{R}^n$, we say that a subset $\mathcal{S}$ of $\mathcal{F}$ is an $\mathcal{F}-$*cover* of R, if R is included in the union of all the elements of $\mathcal{S}$ (this definition generalizes the notion of cover in [8], but is different from the notion of cover discretization in [22]).

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^n$, and let R be any subset of $\mathbb{R}^n$. We consider the hit and miss transforms as defined in [24]. The *hit of R in $\mathcal{F}$*, denoted by $\mathcal{F}(R)$, is the set of all the elements of $\mathcal{F}$ which intersect R: $\mathcal{F}(R) = \{x \in \mathcal{F}, x \cap R \neq \emptyset\}$. In a dual way, we may consider the set $*\mathcal{F}(R)$ composed of all elements of $\mathcal{F}$ which are included in R. If $\mathcal{F}$ is a family that covers $\mathbb{R}^n$, then $\mathcal{F}(R)$ is called the $\mathcal{F}-$*supercover* of R. The $\mathcal{F}-$supercover is obviously a particular case of $\mathcal{F}-$cover, and is uniquely defined for any given R.

If we choose $n = 2$ and $\mathcal{F}$ equals the set of all pixels, we retrieve the notion of supercover presented in the introduction.

In this paper, we focus on supercovers based on the family $\mathcal{G}^n$. The fact that $\mathcal{G}^n$ is a partition of $\mathbb{R}^n$ leads to several interesting properties. Furthermore:

Property 1 *Let $\mathcal{F}$ be a family of sets covering $\mathbb{R}^n$. Then, the two following propositions are equivalent:*

(i) for any subset R of $\mathbb{R}^n$, the $\mathcal{F}$-supercover of R is the unique minimal $\mathcal{F}$-cover of R.

(ii) $\mathcal{F}$ is a partition of $\mathbb{R}^n$.

This property is a direct consequence of Prop. 2, which is stated in the more general framework of binary relations.

Let A, B be two sets. A *relation Γ from A to B* is a subset of the cartesian product $A \times B$. If $(a, b) \in \Gamma$, we also write that $(b, a) \in \Gamma^{-1}$, that $b \in \Gamma(a)$ and that $a \in \Gamma^{-1}(b)$, and we say that b is a *successor of a* and that a is a *predecessor of b*. We say that the relation Γ is *surjective* if each b in B has at least one predecessor, and Γ is a *map from A to B* if each a in A has a unique successor. Let R be a subset of A, we write $\Gamma(R) = \cup_{a \in R} \Gamma(a)$.

Let A, B be two sets, let Γ be a relation from A to B. We say that Γ *defines a covering of A by B* if both Γ and Γ^{-1} are surjective, i.e. if each element of A has at least one successor and each element of B has at least one predecessor.

Let A, B be two sets, let Γ be a relation defining a covering of A by B. Let R be a subset of A. We say that $S \subseteq B$ is a *Γ-cover of R* if $R \subseteq \Gamma^{-1}(S)$. Furthermore, we say that the Γ-cover S is *minimal* if there is no other Γ-cover of R strictly included in S. The set $\Gamma(R)$ is called the *Γ-supercover* of R. It is obviously a Γ-cover of R, which is uniquely defined for any given R, but in general it is not a minimal Γ-cover of R.

For example, if we take $A = \mathbb{R}^n$ and choose for B a family of sets covering $\mathbb{R}^n$, and define $\Gamma(x)$ as the set of elements of B which hit x, then we retrieve the notions and the results of the beginning of this section. In particular, the following property generalizes Prop. 1.

Property 2 *Let A, B be two sets, let Γ be a relation defining a covering of A by B. Then, the two following propositions are equivalent:*
(i) for any subset R of A, the Γ-supercover of R is the unique minimal Γ-cover of R.
(ii) Γ is a map from A to B.

Proof: (ii) $\Rightarrow$ (i). Let $S = \Gamma(R)$. Is S minimal ? Suppose that there exists another Γ-cover S' strictly included in S, and let s be an element of $S \setminus S'$. Since $S = \Gamma(R)$, there exists an x in R such that $s \in \Gamma(x)$, and since Γ is a map, we have $\Gamma(x) = \{s\}$. Thus there is an element x of R which has no successor in S', a contradiction. Is S the unique minimal Γ-cover ? Suppose that there exists another minimal Γ-cover $S' \neq S$. Since both S and S' are minimal, there must exist at least an element s in $S \setminus S'$ and an element s' in $S' \setminus S$. This leads to the same contradiction.
(i) $\Rightarrow$ (ii). Suppose that (i) and that Γ is not a map. Then, there exists an element x of A that has either 0 or more than two successors. As Γ^{-1} is a surjection, x has at least one successor. Let y, z be two distinct successors of x, and let us consider the set $R = \{x\}$. The set $\{y\}$ is a strict subset of $\Gamma(R)$ which is also a Γ-cover of R, a contradiction. □

The following properties can also be easily proved. They are mentioned in [9] for the particular case of $\mathbb{R}^2$ and a covering with pixels. Notice that the condition that the relation Γ is a map is not required.

Property 3 *Let A, B be two sets, let Γ be a relation defining a covering of A by B. Then, $\forall R, S \subseteq A$ we have:*
(i) $\Gamma(R \cup S) = \Gamma(R) \cup \Gamma(S)$
(ii) $\Gamma(R \cap S) \subseteq \Gamma(R) \cap \Gamma(S)$
(iii) $R \subseteq S \Rightarrow \Gamma(R) \subseteq \Gamma(S)$

Furthermore, if Γ is a map defining a covering of A by B, then the cardinality of $\Gamma(R)$ is less or equal to the cardinality of R, for any R subset of A.

5 Properties

This section contains the main results of the paper. We first show that an analytic characterization of $\mathcal{G}^3$-supercovers of planes can be given, by adapting the result of Andrès for the "classical" supercover. Such an analytical characterization is essential to design fast algorithms that generate discrete planar objects.

We consider the plane π defined by:
$\pi = \{(x, y, z) \in \mathbb{R}^3 / ax + by + cz + d = 0\}$, where a, b, c, d belong to $\mathbb{R}$.

The following result, adapted from Andrès ([1], Th. 18) gives us an analytical characterization of the elements of $\mathcal{G}_3^3(\pi)$.

Remind that $\mathcal{G}_3^3(\pi)$ is the set of the elements (open cubes) of $\mathcal{G}_3^3$ that have a non-empty intersection with π.

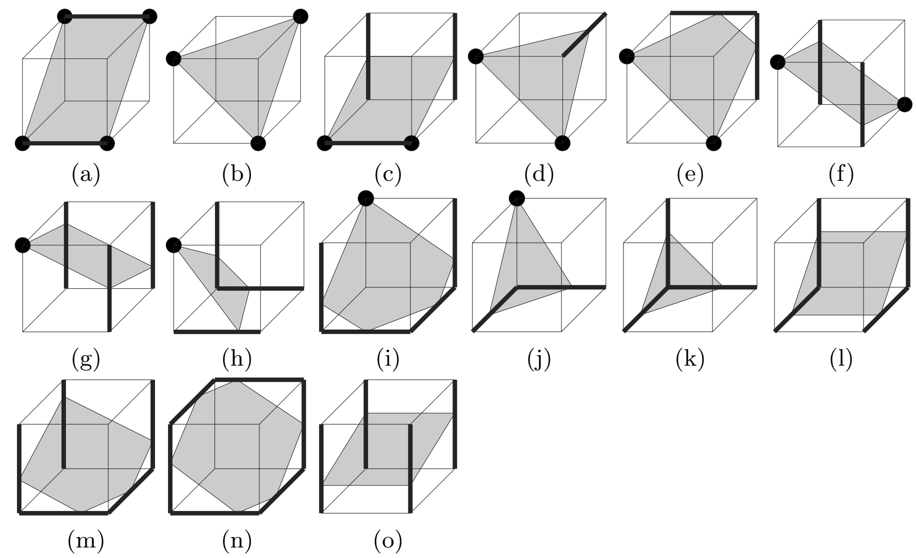


Fig. 4. The fifteen ways for a plane to hit a 3-gel and its θ-neighborhood. The 1-gels and 0-gels that are hit by the plane are highlighted. The 2-gels (squares) that are hit by the plane have not been highlighted, in order to preserve the readability of the figure.

Property 4 *Let $a, b, c, d \in \mathbb{R}$, $ab \neq 0$ or $bc \neq 0$ or $ac \neq 0$, let $\pi = \{(x, y, z) \in \mathbb{R}^3 / ax + by + cz + d = 0\}$. Let $(p, q, r) \in \mathbb{Z}^3$, we denote by G_{pqr} the 3-gel $]p - \frac{1}{2}, p + \frac{1}{2}[\times]q - \frac{1}{2}, q + \frac{1}{2}[\times]r - \frac{1}{2}, r + \frac{1}{2}[$. Then,*

$\mathcal{G}_3^3(\pi) = \{G_{pqr} \in \mathcal{G}_3^3,\ -\frac{|a|+|b|+|c|}{2} < ap + bq + cr + d < \frac{|a|+|b|+|c|}{2}\}$.

Let $x_0 \in \mathbb{R}$, let $\pi = \{(x, y, z) \in \mathbb{R}^3 / x = x_0\}$. If $x_0 - \frac{1}{2} \in \mathbb{Z}$, then $\mathcal{G}_3^3(\pi) = \emptyset$, else $\mathcal{G}_3^3(\pi) = \{G_{pqr} \in \mathcal{G}_3^3,\ |p - x_0| \leq \frac{1}{2}\}$. For planes defined by $y = y_0$ or $z = z_0$, a similar statement holds.

In order to have a complete characterization of the elements of $\mathcal{G}^3(\pi)$, we must characterize also the elements of $\mathcal{G}_2^3(\pi)$, of $\mathcal{G}_1^3(\pi)$ and those of $\mathcal{G}_0^3(\pi)$.

Let $s = \{(p, q, r)\}$ be an element of $\mathcal{G}_0^3$. We denote by $\pi(s)$ the index which characterizes the *position of s relative to π*:

$$\pi(s) = \begin{cases} -1 \text{ if } ap + bq + cr + d < 0, \\ \ \ 0 \text{ if } ap + bq + cr + d = 0, \\ +1 \text{ if } ap + bq + cr + d > 0 \end{cases}$$

Property 5 *Let $a, b, c, d \in \mathbb{R}$, let $\pi = \{(x, y, z) \in \mathbb{R}^3 / ax + by + cz + d = 0\}$.*
a) If $b = c = 0$ and $\frac{d}{a} - \frac{1}{2} \in \mathbb{Z}$, then:

$\mathcal{G}_0^3(\pi) = \{\{(\frac{-d}{a}, q + \frac{1}{2}, r + \frac{1}{2})\},\ q, r \in \mathbb{Z}\}$, *and*

$\mathcal{G}_1^3(\pi) = \{\{\frac{-d}{a}\} \times]q - \frac{1}{2}, q + \frac{1}{2}[\times \{r + \frac{1}{2}\},\ q, r \in \mathbb{Z}\} \cup \{\{\frac{-d}{a}\} \times \{q + \frac{1}{2}\} \times]r - \frac{1}{2}, r + \frac{1}{2}[,\ q, r \in \mathbb{Z}\}$ *and*

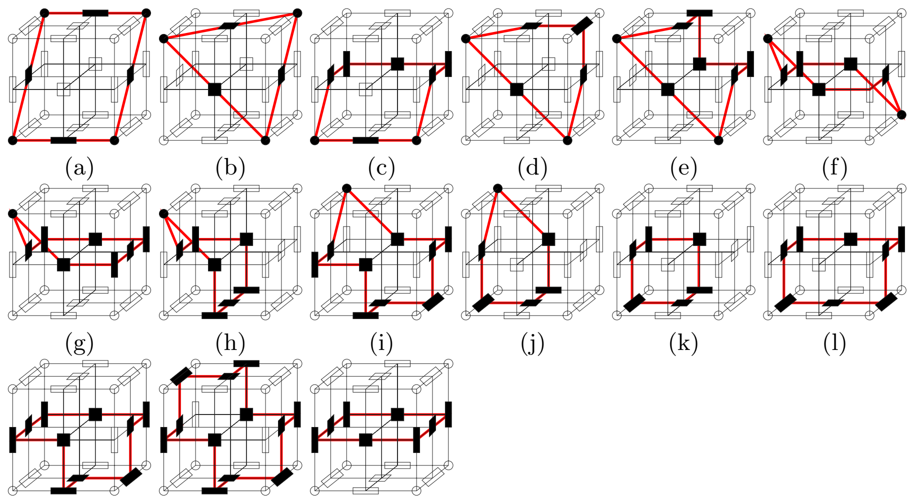


Fig. 5. Neighborhoods of a 3-gel (open cube)

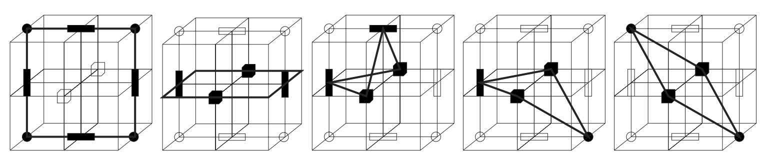


Fig. 6. Neighborhoods of a 2-gel (square)

$\mathcal{G}_2^3(\pi) = \{\{\frac{-d}{a}\} \times]q - \frac{1}{2}, q + \frac{1}{2}[\times]r - \frac{1}{2}, r + \frac{1}{2}[,\ q, r \in \mathbb{Z}\}$.
Similar characterizations are obtained for the cases $a = b = 0$ *and* $a = c = 0$.
b) Other cases: let $v \in \mathcal{G}_0^3$*, let* $i \in \mathcal{G}_1^3$*, with* $\alpha(i) = \{v_1, v_2\}$*, and let* $s \in \mathcal{G}_2^3$*. Then:*
$v \in \mathcal{G}_0^3(\pi)$ *iff* $\pi(v) = 0$,
$i \in \mathcal{G}_1^3(\pi)$ *iff* $(\pi(v_1) = \pi(v_2) = 0)$ *or* $(\pi(v_1).\pi(v_2) < 0)$,
$s \in \mathcal{G}_2^3(\pi)$ *iff at least two vertices of* s *have different non-null indices.*

Now we are ready to state the first main result of this paper. It says that the discretization of a plane is a surface, in the sense defined in section 2. This result can be easily transposed in 2D, where it states that the discretization of a straight line is a curve.

Property 6 *Let* $a, b, c, d \in \mathbb{R}$*, let* $\pi = \{(x, y, z) \in \mathbb{R}^3 / ax + by + cz + d = 0\}$. *The order* $|\mathcal{G}^3(\pi)|$ *is a surface.*

The proof of Prop. 6 involves the examination of the different configurations of a plane π hitting an open cube, a square, a line segment and a single point, and

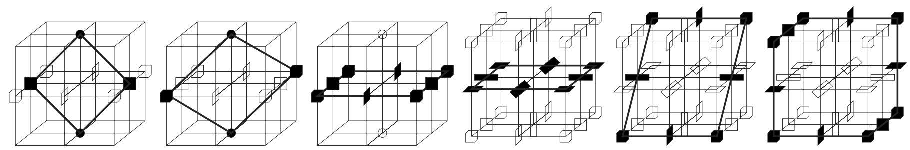


Fig. 7. (a,b,c): neighborhoods of a 1-gel (segment), (d,e,f): neighborhoods of a 0-gel (singleton)

their respective θ-neighborhoods. For the open cube, these configurations (up to rotations and symmetries) are only 15, they are depicted in Fig. 4. We can easily check (see Fig. 5) that for each of these configurations, the $\theta^{\square}$-neighborhood of any 3-cube in $\mathcal{G}^3(\pi)$ is a curve. For the cases of a single point, a line segment and a square, the numbers of possible configurations (up to rotations and symmetries) are respectively 3, 3 and 5. Figs. 6 and 7 shows the $\theta^{\square}$-neighborhood of such an element in each possible configuration, again we can verify that it forms a curve.

Our second main result states that, for any "regular" object (in a sense that will be defined below), the boundary operator commutes with the discretization operator.

Let (O, α) be an order, and let P be a subset of O. We define the *$\theta-$boundary of P in O* (or simply the boundary of P) as the set $B(P)$ of elements p of P such that $\theta(p) \cap \overline{P} \neq \emptyset$.

Let X be a subset of $\mathbb{R}^3$. Let C_0 be the unit closed cube centered at the origin: $C_0 = [-\frac{1}{2}, \frac{1}{2}]^3$. Let $C_{\mathbf{u}}$ be the translation of C_0 by the vector $\mathbf{u}$ of $\mathbb{R}^3$. The set X is *morphologically open by the structuring element C_0* if X is equal to the union of all the translations $C_{\mathbf{u}}$ of C_0 which are included in X (see [24]). Notice that this notion is close to the notion of a par$(r,+)$-regular set defined by Latecki in the continuous plane[19]. The (topological) closure of X is denoted by cl(X), and the *boundary* of X is defined by $b(X) = \text{cl}(X) \cap \text{cl}(\overline{X})$.

Property 7 *If X is a closed subset of $\mathbb{R}^3$, then $B(\mathcal{G}^3(X)) \subseteq \mathcal{G}^3(b(X))$. Furthermore, if $cl(\overline{X})$ is morphologically open by C_0, then the boundary of the discretization of X is equal to the discretization of the boundary of X, in other words, $B(\mathcal{G}^3(X)) = \mathcal{G}^3(b(X))$.*

A corollary can be immediately derived from this property, concerning planes and half-spaces. We consider the closed half-space $\gamma = \{(x, y, z) \in \mathbb{R}^3 / ax + by + cz + d \geq 0\}$, where a, b, c, d belong to $\mathbb{R}$. The boundary of γ is the plane π defined by: $\pi = \{(x, y, z) \in \mathbb{R}^3 / ax + by + cz + d = 0\}$. Then we have:

Corollary 8 $\mathcal{G}^3(b(\gamma)) = \mathcal{G}^3(\pi) = B(\mathcal{G}^3(\gamma))$

From this, we can easily deduce a more general corollary which holds for any convex and closed set:

Corollary 9 *If X is a convex closed subset of $\mathbb{R}^3$, then $B(\mathcal{G}^3(X)) = \mathcal{G}^3(b(X))$.*

Fig. 8 illustrates these properties in 2D.

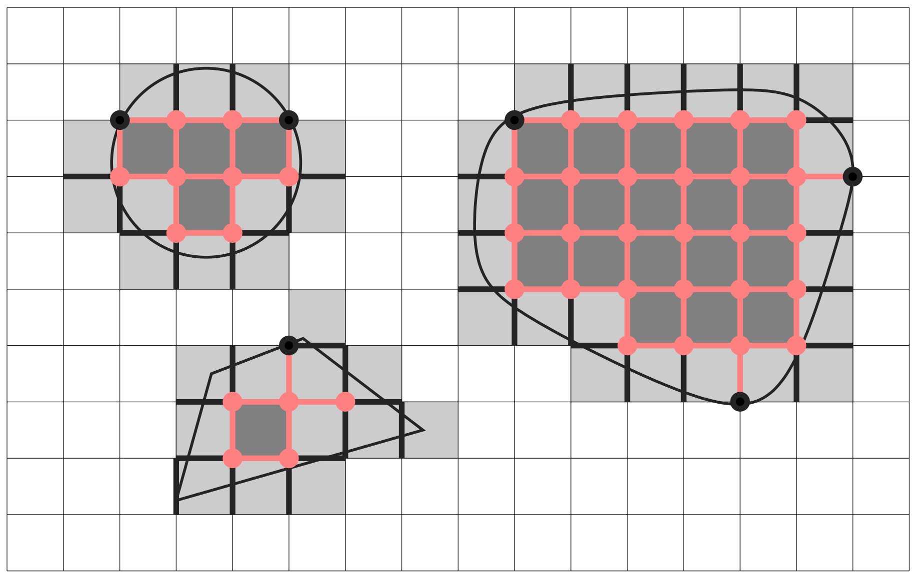


Fig. 8. Illustration of Prop. 7 and Cor. 9 in 2D. We see the discretizations of three objects : a disc, a convex polygon and a third convex set. The boundaries of these objects appear as continuous solid lines. The discretizations of the boundaries, which coincide with the boundaries of the discretizations, are represented by light gray squares, black segments and black dots.

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