Domain Decomposition and Multigrid Methods for Obstacle Problems

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Abstract. Uniform linear mesh independent convergence rate estimate is given for some proposed algorithms for variational inequalities in connection with domain decomposition and multigrid methods. The algorithms are proposed for general space decompositions and thus can also be applied to estimate convergence rate for classical block relaxation methods. Numerical examples which support the theoretical predictions are presented.

1 Some subspace correction algorithms

Consider the nonlinear convex minimization problem

$$\min_{v\in K} F(v), \quad K \subset V \ , \tag{1}$$

where F is a convex functional over a reflexive Banach space V and $K \subset V$ is a nonempty closed convex subset. In order to solve the mimization problem efficiently, we shall decompose V and K into a sum of subspaces and subsets of smaller sizes respectively as in [4] [7]. More precisely, we decompose

$$V = \sum_{i=1}^{m} V_i, \quad K = \sum_{i=1}^{m} K_i, \quad K_i \subset V_i \subset V \ , \tag{2}$$

where V_i are subspaces and K_i are convex subsets. We use two constants C_1 and C_2 to measure the quality of the decompositions. First, we assume that there exits a constant $C_1 > 0$ and this constant is fixed once the decomposition (2) is fixed. With such a $C_1 > 0$, it is assumed that any $u, v \in K$ can be decomposed into a sum of $u_i, v_i \in K_i$ and the decompositions satisfy

$$u = \sum_{i=1}^{m} u_i \ , \quad v = \sum_{i=1}^{m} v_i, \quad \text{and} \quad \left(\sum_{i=1}^{m} \|u_i - v_i\|^2\right)^{\frac{1}{2}} \le C_1 \|u - v\| \ . \tag{3}$$

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For given $u, v \in K$, the decompositions u_i, v_i satisfying (3) may not be unique. We also need to assume that there is a $C_2 > 0$ such that for any $w_i \in V, \hat{v}_i \in V_i,\ \tilde{v}_j \in V_j$ it is true that

$$\sum_{i=1}^{m}\sum_{j=1}^{m} |\langle F'(w_{ij}+\hat{v}_i) - F'(w_{ij}), \tilde{v}_j\rangle| \le C_2\Big(\sum_{i=1}^{m}\|\hat{v}_i\|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{m}\|\tilde{v}_j\|^2\Big)^{\frac{1}{2}}, \quad (4)$$

In the above, F' is the Gâteaux differential of F and $\langle\cdot,\cdot\rangle$ is the duality pairing between V and its dual space V', i.e. the value of a linear function at an element of V. We also assume that there exists a constant $\kappa > 0$ such that

$$\langle F'(v_1) - F'(v_2), v_1 - v_2\rangle \ge \kappa\|v_1 - v_2\|_V^2, \quad \forall w, v \in V\ . \quad (5)$$

Under the assumption (5), problem (1) has a unique solution. For some nonlinear problems, the constant κ may depend on v_1 and v_2. For a given approximate solution $u \in K$, we shall find a better solution w using one of the following two algorithms.

Algorithm 1 *Choose a relaxation parameter* $\alpha \in (0, 1/m]$ *and decompose* u *into a sum of* $u_i \in K_i$ *satisfying (3). Find* $\hat{w}_i \in K_i$ *in parallel for* $i = 1, 2, \cdots, m$ *such that*

$$\hat{w}_i = \arg\min_{v_i \in K_i} G(v_i) \quad \textit{with} \quad G(v_i) = F\Big(\sum_{j=1, j\ne i}^{m} u_j + v_i\Big). \quad (6)$$

Set $w_i = (1-\alpha)u_i + \alpha\hat{w}_i$ *and* $w = (1-\alpha)u + \alpha\sum_{i=1}^{m}\hat{w}_i$.

Algorithm 2 *Choose a relaxation parameter* $\alpha \in (0, 1]$ *and decompose* u *into a sum of* $u_i \in K_i$ *satisfying (3). Find* $\hat{w}_i \in K_i$ *sequentially for* $i = 1, 2, \cdots, m$ *such that*

$$\hat{w}_i = \arg\min_{v_i \in K_i} G(v_i) \quad \textit{with} \quad G(v_i) = F\Big(\sum_{j<i} w_j + v_i + \sum_{j>i} u_j\Big) \quad (7)$$

where $w_j = (1-\alpha)u_j + \alpha\hat{w}_j,\ j = 1, 2, \cdots i-1$. *Set* $w = (1-\alpha)u + \alpha\sum_{i=1}^{m}\hat{w}_i$.

Denote u^* the unique solution of (1), the following convergence estimate is correct for Algorithms 1 and 2 (see Tai [6]):

Theorem 1. *Assuming that the space decomposition satisfies (3), (4) and that the functional* F *satisfies (5). Then for Algorithms 1 and 2, we have*

$$\frac{F(w) - F(u^*)}{F(u) - F(u^*)} \le 1 - \frac{\alpha}{(\sqrt{1+C^*} + \sqrt{C^*})^2}, \quad C^* = \Big(C_2 + \frac{[C_1C_2]^2}{2\kappa}\Big)\frac{2}{\kappa}. \quad (8)$$

2 Some Applications

We apply the algorithms for the following obstacle problem:

$$\text{Find } u \in K, \quad \text{such that} \quad a(u, v-u) \geq f(v-u), \quad \forall v \in K, \tag{9}$$

with $a(v,w) = \int_\Omega \nabla v \cdot \nabla w \; dx$, $K = \{v \in H_0^1(\Omega) |\; v(x) \geq \psi(x) \text{ a.e. in } \Omega\}$. It is well known that the above problem is equivalent to the minimization problem (1) assuming that $f(v)$ is a linear functional on $H_0^1(\Omega)$. For the obstacle problem (9), the minimization space $V = H_0^1(\Omega)$. Correspondingly, we have $\kappa = 1$ for assumption (5). The finite element method shall be used to solve (9). It shall be shown that domain decomposition and multigrid methods satisfy the conditions (3) and (4). For simplicity of the presentation, it will be assumed that $\psi = 0$.

2.1 Overlapping domain decomposition methods

For the domain Ω, we first divide it into a quasi-uniform coarse mesh partitions $\mathcal{T}_H = \{\Omega_i\}_{i=1}^M$ with a mesh size H. The coarse mesh elements are also called subdomains later. We further divide each Ω_i into smaller simplices with diameter of order h. We assume that the resulting finite element partition $\mathcal{T}_h$ form a shape regular finite element subdivision of Ω. We call this the fine mesh or the h-level subdivision of Ω with the mesh parameter h. We denote by $S_H \subset W_0^{1,\infty}(\Omega)$ and $S_h \subset W_0^{1,\infty}(\Omega)$ the continuous, piecewise linear finite element spaces over the H-level and h-level subdivisions of Ω respectively. For each Ω_i, we consider an enlarged subdomain Ω_i^δ consisting of elements $\tau \in \mathcal{T}_h$ with $distance(\tau, \Omega_i) \leq \delta$. The union of Ω_i^δ covers $\bar{\Omega}$ with overlaps of size δ. For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. Let Ω_i^c be the union of the subdomains with the i^{th} color, and $V_i = \{v \in S_h |\; v(x) = 0,\; x \notin \Omega_i^c\},\; i = 1, 2, \cdots, m$. By denoting the subspaces $V_0 = S_H$, $V = S_h$, we find that

$$a). \quad V = \sum_{i=1}^m V_i \qquad \text{and} \qquad b). \quad V = V_0 + \sum_{i=1}^m V_i. \tag{10}$$

Note that the summation index is now from 0 to m instead of from 1 to m when the coarse mesh is added. For the constraint set K, we define

$$K_0 = \{v \in V_0 |\; v \geq 0\}, \quad \text{and} \quad K_i = \{v \in V_i |\; v \geq 0\},\; i = 1, 2, \cdots, m. \tag{11}$$

Under the condition that $\psi = 0$, it is easy to see that (2) is correct both with or without the coarse mesh. When the coarse mesh is added, the summation index is from 0 to m. Let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i^c\}_{i=1}^m$, i.e. $\theta_i \in V_i$, $\theta_i \geq 0$ and $\sum_{i=1}^m \theta_i = 1$. It can be chosen so that

$$|\nabla \theta_i| \leq C/\delta, \quad \theta_i(x) = \begin{cases} 1 & \text{if } x \in \tau, \;\; \text{distance } (\tau, \partial\Omega_i^c) \geq \delta \text{ and } \tau \subset \Omega_i^c, \\ 0 & \text{on } \overline{\Omega \backslash \Omega_i^c}. \end{cases} \tag{12}$$

Later in this paper, we use I_h as the linear Lagrangian interpolation operator which uses the function values at the h-level nodes. In addition, we aslo need a nonlinear interpolation operator $I_H^{\ominus} : S_h \mapsto S_H$. Assume that $\{x_0^i\}_{i=1}^{n_0}$ are all the interior nodes for $\mathcal{T}_H$ and let ω_i be the support for the nodal basis function of the coarse mesh at x_0^i. The nodal values for $I_H^{\ominus} v$ for any $v \in S_h$ is defined as $(I_H^{\ominus} v)(x_0^i) = \min_{x\in\omega_i} v(x)$, c.f [6]. This operator satisfies

$$I_H^{\ominus} v \leq v,\ \forall v \in S_h, \quad \text{and} \quad I_H^{\ominus} v \geq 0,\ \forall v \geq 0, v \in S_h. \tag{13}$$

Moreover, it has the following monotonicity property

$$I_{h_1}^{\ominus} v \leq I_{h_2}^{\ominus} v, \quad \forall h_1 \geq h_2 \geq h, \quad \forall v \in S_h. \tag{14}$$

As $I_H^{\ominus} v$ equals v at least at one point in ω_i, it is thus true that for any $u, v \in S_h$

$$\|I_H^{\ominus} u - I_H^{\ominus} v - (u-v)\|_0 \leq c_d H |u-v|_1, \quad \|I_H^{\ominus} v \ - \ v\|_0 \leq c_d H |v|_1, \tag{15}$$

where d indicates the dimension of the physical domain Ω, i.e. $\Omega \subset R^d$, and

$$c_d = \begin{cases} C & \text{if } d = 1; \\ C\left(1 + \left|\log \frac{H}{h}\right|^{\frac{1}{2}}\right) & \text{if } d = 2, \\ C\left(\frac{H}{h}\right)^{\frac{1}{2}} & \text{if } d = 3, \end{cases}$$

With C being a generic constant independent of the mesh parameters. See Tai [6] for a detailed proof.

2.2 Decompositions with or without the coarse mesh

If we use the overlapping domain decomposition without the coarse mesh, i.e. we use decomposition (10.a), then we will get some domain decomposition algorithms which are essentially the block-relaxation methods. Even in the case $V = R^n$, the analysis of the convergence rate for a general convex functional $F : R^n \mapsto R$ and a general convex set $K \subset R^n$ is not a trivial matter, see [2] [3]. A linear convergence rate has been proved in [1] [5] for the overlapping domain decomposition without the coarse mesh. However, all the proofs require that the computed solutions converge to the true solution monotonically. Numerical evidence shows that linear convergence is true even if the computed solutions are not monotonically increasing or decreasing. In the following, we shall use our theory to prove this fact.

For any given $u, v \in S_h$, we decompose u, v as

$$u = \sum_{i=1}^{m} u_i, \quad v = \sum_{i=1}^{m} v_i, \quad u_i = I_h(\theta_i u), \quad v_i = I_h(\theta_i v). \tag{16}$$

In case that $u, v \geq 0$, it is true that $u_i, v_i \geq 0$. In addition,

$$\sum_{i=1}^{m} \|u_i - v_i\|_1^2 \leq C\left(1 + \frac{1}{\delta^2}\right)\|u - v\|_1^2,$$

which shows that $C_1 \le C(1+\delta^{-1})$. It is known that $C_2 \le \sqrt{m}$ with m being the number of colors. From Theorem 1, the following rate is obtained without requiring that the computed solutions increase or decrease monotonically:

$$\frac{F(w)-F(u^*)}{F(u)-F(u^*)} \le 1 - \frac{\alpha}{1+C(1+\delta^{-2})}.$$

For Algorithm 2, we can take $\alpha = 1$.

Numerical experiments and the convergence analysis for the two-level domain decomposition method, i.e. overlapping domain decomposition with a coarse mesh, seems still missing in the literature. To apply our algorithms and theory, we decompose any $u \in K$ as

$$u = u_0 + \sum_{i=1}^{m} u_i, \quad u_0 = I_H^{\ominus} u,\ u_i = I_h(\theta_i(u-u_0)). \tag{17}$$

From (13) and the fact that $u \ge 0$, it is true that $0 \le u_0 \le u$ and so $u_i \ge 0,\ i = 1, 2, \cdots, m$, which indicates that $u_0 \in K_0$ and $u_i \in K_i,\ i = 1, 2, \cdots, m$. The decomposition for any $v \in K$ shall be done in the same way. It follows from (15) that $\|u_0 - v_0\|_1 \le C\|u-v\|_1$. Note that $u_i - v_i = I_h\big(\theta_i(u - v - I_H^{\ominus}u + I_H^{\ominus}v)\big)$. Using estimate (15) and a proof similar to those for the unconstrained cases, c.f. [7], [8], it can be proven that $\|u_i - v_i\|_1^2 \le c_d\left(1+\frac{H}{\delta}\right)\|u-v\|_1^2$. Thus

$$\left(\|u_0 - v_0\|_1^2 + \sum_{i=1}^{m}\|u_i - v_i\|_1^2\right)^{\frac{1}{2}} \le C(m)c_d\left(1+\left(\frac{H}{\delta}\right)^{\frac{1}{2}}\right)\|u-v\|_1.$$

The estimate for C_2 is known, c.f. [7], [8]. Thus, for the two-level domain decomposition method, we have $C_1 = C(m)c_d\left(1+\frac{\sqrt{H}}{\sqrt{\delta}}\right),\ C_2 = C(m)$, where $C(m)$ is a constant only depending on m, but not on the mesh parameters and the number of subdomains. An application of Theorem 1 will show that the following convergence rate estimate is correct:

$$\frac{F(w)-F(u^*)}{F(u)-F(u^*)} \le 1 - \frac{\alpha}{1+c_d^2(1+H\delta^{-1})}.$$

2.3 Multigrid decomposition

Multigrid methods can be regarded as a repeated use of the two-level method. We assume that the finite element partition $\mathcal{T}_h$ is constructed by a successive refinement process. More precisely, $\mathcal{T}_h = \mathcal{T}_{h_J}$ for some $J > 1$, and $\mathcal{T}_{h_j}$ for $j \le J$ is a nested sequence of quasi-uniform finite element partitions, see [6], [7], [8]. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}. Corresponding to each finite element partition $\mathcal{T}_{h_j}$, let $\{x_j^k\}_{k=1}^{n_j}$ be the set of all the interior nodes. Denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ the nodal basis functions satisfying $\phi_j^i(x_j^k) = \delta_{ik}$. We then define a one dimensional subspace $V_j^i = \mathrm{span}(\phi_j^i)$. Letting $V = \mathcal{M}_J$, we have the following trivial space decomposition:

$$V = \sum_{j=1}^{J} \sum_{i=1}^{n_j} V_j^i. \tag{18}$$

Each subspace V_j^i is a one dimensional subspace. For any $v \geq 0$ and $j \leq J - 1$, define $v_j = I_{h_j}^{\ominus} v - I_{h_{j-1}}^{\ominus} v \in \mathcal{M}_j$. Let $v_J = v - I_{h_{J-1}}^{\ominus} v \in \mathcal{M}_J$. A further decomposition of v_j is given by $v_j = \sum_{i=1}^{n_j} v_j^i$ with $v_j^i = v_j(x_j^i)\phi_j^i$. It is easy to see that

$$v = \sum_{j=1}^{J} v_j = \sum_{j=1}^{J} \sum_{i=1}^{n_j} v_j^i.$$

For any $u \geq 0$, it shall be decomposed in the same way, i.e.

$$u = \sum_{j=1}^{J} \sum_{i=1}^{n_j} u_j^i, \;\; u_j^i = u_j(x_j^i)\phi_j^i, \;\; u_j = I_{h_j}^{\ominus} u - I_{h_{j-1}}^{\ominus} u, \;\; j < J; \;\; u_J = u - I_{h_{J-1}}^{\ominus} u. \tag{19}$$

It follows from (13) and (14) that $u_j^i, v_j^i \geq 0$ for all $u, v \geq 0$, i.e. $u_j^i, v_j^i \in K_j^i = \{v \in V_j^i : v \geq 0\}$ under the condition that $\psi = 0$. Define

$$\tilde{c}_d = \begin{cases} C, & \text{if} \quad d = 1; \\ C(1 + |\log h|^{\frac{1}{2}}), & \text{if} \quad d = 2; \\ Ch^{-\frac{1}{2}}, & \text{if} \quad d = 3. \end{cases}$$

The following estimate can be obtained using approximation properties (15) (see [6]):

$$\sum_{j=1}^{J} \sum_{i=1}^{n_j} \|u_j^i - v_j^i\|_1^2 \leq C \sum_{j=1}^{J} h_j^{-2} \|u_j - v_j\|_0^2 \leq \tilde{c}_d^2 \sum_{j=1}^{J} h_j^{-2} h_{j-1}^2 \, |u - v|_1^2 \leq \tilde{c}_d^2 \gamma^{-2} J \, |u - v|_1^2,$$

which proves that

$$C_1 \cong \tilde{c}_d \gamma^{-1} J^{\frac{1}{2}} \cong \tilde{c}_d \gamma^{-1} |\log h|^{\frac{1}{2}}.$$

The estimate for C_2 is known, i.e. $C_2 = C(1 - \gamma^d)^{-1}$, see Tai and Xu [8]. Thus for the multigrid method, the error reduction factor for the algorithms is

$$\frac{F(w) - F(u^*)}{F(u) - F(u^*)} \leq 1 - \frac{\alpha}{1 + \tilde{c}_d^2 \gamma^{-2} J}.$$

2.4 Numerical experiments

We shall test our algorithms for the obstacle problem (9) with $\Omega = [-2, 2] \times [-2, 2]$, $f = 0$ and $\psi(x, y) = \sqrt{x^2 + y^2}$ when $x^2 + y^2 \leq 1$ and $\psi(x, y) = -1$ elsewhere. This problem has an analytical solution [6]. Note that the continuous obstacle function ψ is not even in $H^1(\Omega)$. Even for such a difficult problem,

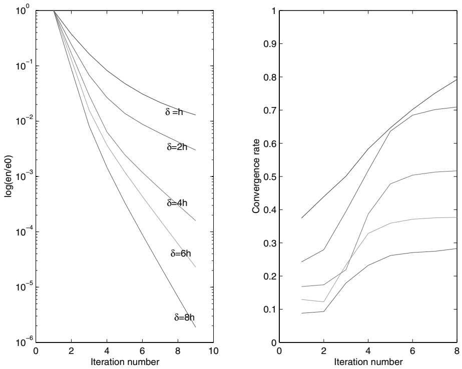

Fig. 1. Convergence for the two-level method for decomposition (17) with different overlaps, $h = 4/128$, and $H = 4/8$.

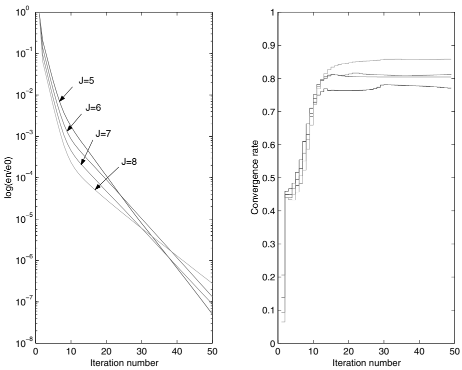

Fig. 2. Convergence for the multigrid method

uniform linear convergence has been observed in our experiments. In the implementations, the non-zero obstacle can be shifted to the right hand side.

Figure 1 shows the convergence rate for Algorithm 2 with different overlapping sizes for decomposition (17). Figure 2 shows the convergence rate for Algorithm 2 with the multigrid method for decomposition (19) and J indicates the number of levels. In the figures en is the H^1-error between the computed solution and the true finite element solution and $e0$ is the initial error. $\log(en/e0)$ is used for one of the subfigures. The convergence rate is faster in the beginning and then approaches a constant after some iterations.

References

1. Lori Badea and Junping Wang. An additive Schwarz method for variational inequalities. *Math. Comp.*, 69(232):1341–1354, 2000.
2. R. W. Cottle, J. S. Pang, and R. E. Stone. *The linear complementary problem.* Academic Press, Boston, 1992.
3. Z.-Q. Luo and P. Tseng. Error bounds and convergence analysis of feasible descent methods: A general approach. *Ann. Oper. Res.*, 46:157–178, 1993.
4. X.-C. Tai. Parallel function and space decomposition methods. In P. Neittaanmäki, editor, *Finite element methods, fifty years of the Courant element, Lecture notes in pure and applied mathematics*, volume 164, pages 421–432. Marcel Dekker inc., 1994. Available online at http://www.mi.uib.no/˜tai.
5. X.-C. Tai. Convergence rate analysis of domain decomposition methods for obstacle problems. *East-West J. Numer. Math.*, 9(3):233–252, 2001. Available online at http://www.mi.uib.no/˜tai.
6. X.-C. Tai. Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities. Technical Report 150, Department of Mathematics, University of Bergen, November, 2000. Available online at http://www.mi.uib.no/˜tai.
7. X.-C. Tai and P. Tseng. Convergence rate analysis of an asynchronous space decomposition method for convex minimization. *Math. Comput.*, 2001.
8. X-C. Tai and J.-C. Xu. Global and uniform convergence of subspace correction methods for some convex optimization problems. *Math. Comput.*, 71:105–124, 2001.