# On 3-Layer Crossings and Pseudo Arrangements

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**Abstract.** Let $G = (V_0, V_1, V_2, E)$ be a 3-layer graph. The 3-layer drawings of $G$ in which $V_0$, $V_1$, and $V_2$ are placed on 3 parallel lines and each edge in $E$ is drawn using one straight line segment, are studied. A generalization of the linear arrangement problem which we call the 3-layer pseudo linear arrangement problem is introduced, and it is shown to be closely related to the 3-layer crossing number. In particular, we show that the 3-layer crossing number of $G$ plus the sum of the square of degrees asymptotically has the same order of magnitude as the optimal solution to the 3-layer linear arrangement problem. Consequently, when $G$ satisfies certain (reasonable) assumptions, we derive the first polynomial time approximation algorithm to compute the 3-layer crossing number within a multiplicative factor of $O(\log n)$ from the optimal.

## 1 Introduction

The planar crossing number problem is the problem of placing the vertices of a graph in the plane and drawing the edges with curves, to minimize the number of edge crossings [20]. This problem is known to be NP-hard [9] and has been extensively studied in graph theory [23,16], and theory of VLSI [13]. One of the most important aesthetic objectives in drawing graphs is to have a small number of crossings [17], and therefore the crossing minimization problems have been frequently studied by the graph drawing community e.g. [4,5,11,15].

Let $G = (V, E)$ be an undirected graph with the vertex set $V$ and the edge set $E$. G is called a $k$-layer graph, if a partition of $V$ into $k$ sets $V_0, V_1, ..., V_{k-1}$ exists so that any edge in $E$ has one end point in $V_i$ and the other end point in $V_{i+1}$ for some $i = 0, 1, 2, ..., k-2$. Thus, any bipartite graph is a $2-$layer graph. If $G = (V, E)$ is a $k-$layer graph, then we write $G = (V_0, V_1, ..., V_{k-1}, E)$, where $\{V_0, V_1, ..., V_{k-1}\}$ is the partition of $V$ into $k$ disjoint sets. Let $G =$

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$(V_0, V_2, ..., V_{k-1}, E)$ be a $k$-layer graph, $k \geq 2$, and let $L_0, L_1, ..., L_{k-1}$ be $k$ parallel lines in the plane. A *$k-$layer drawing* [1,4,12,15,21] of $G$ consists of placing the vertices of $V_i$ into distinct points on $L_i$, $i = 0, 1, 2..., k-1$, and then drawing each edge using a straight line segment connecting the points representing the end vertices of the edge. The objective is to minimize the number of crossings between the edge pairs. Note that any $k-$layer drawing of $G$ is identified by a one to one function

$$D : V_0 \cup V_1 ... \cup V_{k-1} \to \Re,$$

where $\Re$ is the set of non-negative real numbers. In particular, note that the restriction of $D$ to $V_i, 0 \leq i \leq k-1$, specifies the order in which the vertices of $V_i$ will appear on $L_i$. When $k = 2$ the corresponding drawing is called a *bipartite drawing* and the problem of minimizing the number of crossings is called the *bipartite crossing number problem* [5,11,15,21,19].

Computing the bipartite crossing number is NP-hard [9][1] and despite a great deal of research which was done on this problem, no polynomial time approximation algorithm had been known for this problem. Very recently a polynomial time approximation algorithm with performance guarantee of $O(\log n)$ from the optimal was discovered for approximating the bipartite crossing number of a large class of graphs on $n$ vertices [19]. Nonetheless, no efficient approximation algorithm for minimizing the number of crossings in a $k$-layer drawing has been known for $k \geq 3$. The main results in [19] were obtained by relating the bipartite crossing number problem to the linear arrangement problem which is another well known problem in theory of VLSI. In this paper we develop a general framework to study the 3-layer crossing number problem by relating it to a very general version of the linear arrangement problem which we call the *pseudo-arrangement problem*. In particular, we derive tight upper and lower bounds for the 3-layer crossing number in terms of the arrangement values. The ratio of the main term of the upper bound to the main term in the lower bound is only 3, and the error term is the sum of the square of degrees in $G$. The result is interesting, since it indicates that the number of edge crossings is closely related to the *length* of the drawing which is defined by the pseudo-arrangement problem. Consequently, we derive the first polynomial time approximation algorithm for the 3-layer crossing problem with the performance guarantee of $O(\log n)$ from the optimal, provided that the graph satisfies certain conditions.

## 2 Basic Concepts and Notations

Let $G = (V, E)$ be a graph, we denote by $d_v$ the degree of $v \in V$. Throughout this paper $G = (A, B, E)$ denotes a bipartite graph on the partite sets $A$ and $B$, and the edge set $E$. $G = (V_0, V_1, V_2, E)$ denotes a 3-layer graph, with vertex set $V = V_0 \cup V_1 \cup V_2$, $|V| = n$, and the edge set $E$.

[1] Technically speaking, the NP-hardness of the problem was proved for multigraphs, but it is widely assumed that it is also NP-hard for simple graphs.

For $G = (V_0, V_1, V_2, E)$, let $G_0$ and $G_2$ denote, respectively, the induced subgraphs of $G$ on the vertex sets $V_0 \cup V_1$, and $V_2 \cup V_1$. Note that $G_0$ and $G_2$ are bipartite graphs. Note that there is no edge in $G$ with one end vertex in $V_0$ and the other in $V_2$. For any $v \in V_1$, let $d_{v,0}$ and $d_{v,2}$ denote the degrees of $v$, in $G_0$ and $G_2$, respectively, and note that $d_{v,0} + d_{v,2} = d_v$.

A 3-layer drawing of $G = (V_0, V_1, V_2)$ is a one-to-one function

$$D : V_0 \cup V_1 \cup V_2 \to \Re.$$

Let $D_i$ denote the restriction of $D$ to $V_i$, $0 \leq i \leq 2$. Note that $D_i$, $0 \leq i \leq 2$ specifies the order in which the vertices in $V_i$ are placed on the line $L_i$. Observe that $(D_0, D_1)$ is a bipartite drawing for $G_0$ and $(D_1, D_2)$ is a bipartite drawing for $G_2$. For any $e \in E$, let $cr_0(e)$ denote the number of crossings of $e$ with other edges in the drawing $(D_0, D_1)$, and $cr_2(e)$ denote the number of crossings of $e$ with other edges in the drawing $(D_1, D_2)$ . We define $cr_0$ and $cr_2$ to be the total number of crossings in the drawings $(D_0, D_1)$ and $(D_1, D_2)$, respectively, and define $cr_D$ to be the total number of edge crossings in $D$. Thus, $cr_D = cr_0 + cr_2$. The *3-layer crossing number* of $G$ is the minimum number of crossings of edges over all 3-layer drawings of $G$.

A *linear arrangement* (LA) of a graph $G = (V, E)$ is a one to one function $f : V \to \{1, 2, ...|V|\}$. The linear arrangement problem is to find a LA so that $\sum_{uv \in E} |f(u) - f(v)|$ is minimized [2,3,7,10,18]. This problem is known to be NP-hard but can be approximated in polynomial time using a variety of algorithms[7,10,18]. Crucial to our work are generalizations of this problem defined for bipartite and 3-layer graphs.

For $x, y \in \Re$, let $(x, y)$ denote the open interval between $x$ and $y$. Let $G = (A, B, E)$. A pseudo linear arrangement (PLA) for $G$ is a one to one function $f : A \cup B \to \Re$ so that $f(B) = \{1, 2..., |B|\}$. Hence, any vertex in $B$ is assigned a unique integer which is at most equal to $|B|$. Let $ab \in E$, with $f(a) < f(b)$. We define the *length* of $e$, denoted by $L^e_f$, to be

$$\sum_{x \in B, f(x) \in (f(a), f(b))} d_x.$$

We define the length of $f$, denoted by $L_f$, to be $\sum_{ab \in E} L^e_f$. The pseudo linear arrangement problem is to find a PLA of minimum length. We denote this minimum value by $\bar{L}_G$. It follows from the recent work on spreading matrices [7,18] that for any graph on $n$ vertices $\bar{L}_G$ can be approximated to within a factor of $O(\log n)$ from the optimal in polynomial time.

A *3-layer pseudo linear arrangement* (3PLA) of $G = (V_0, V_1, V_2, E)$ is a one to one function $f : V_0 \cup V_1 \cup V_2 \to \Re$, so that $f(V_1) = \{1, 2, 3, ..., |V_1|\}$. Note that we may view any 3PLA $f$ of $G$ as a 3-layer drawing of $G$. Let $f_i$ denote the restriction of $f$ to $V_i$, $0 \leq i \leq 2$, and note that that $(f_0, f_1)$ and $(f_1, f_2)$ are PLAs of $G_0$ and $G_2$, respectively. Let $ab \in E$, we define the *length* of $e$, denoted by $L^e_f$ to be $L^e_{(f_0,f_1)}$, provided that $e$ is an edge in $G_0$, otherwise, we define $L^e_f$ to be $L^e_{(f_1,f_2)}$. Note that for any edge $e$ in $G_0$

$$L_f^e = \sum_{x \in V_1, f(x) \in (f(a), f(b))} d_{x,0}$$

whereas, for any $e$ in $G_2$,

$$L_f^e = \sum_{x \in V_1, f(x) \in (f(a), f(b))} d_{x,2}.$$

The length of $f$, denoted by $L_f$, is defined to be $\sum_{e \in E} L_f^e$. The 3-layer pseudo linear arrangement problem is to find a 3PLA for $G$ which has the minimum length. We denote this minimum value by $L_G$. Let $f$ be a 3PLA of $G$. For $v \in V_0 \cup V_2$, let $u_1, u_2, ..., u_{d_v}$ be its neighbors in the set $V_1$ satisfying $f(u_1) < f(u_2) < ... < f(u_{d_v})$. We define the *median vertex* of $v$, denoted by $med(v)$, to be $u_{\lceil \frac{d_v}{2} \rceil}$.

## 3 Arrangements and 3-Layer Drawings

Let $G = (V_0, V_1, V_2, E)$ and $D$ be a 3-layer drawing of $G$. We assume throughout this paper that the vertices of $V_0$ are placed on the line $y = 0$, vertices of $V_1$ are placed on the line, $y = 1$, and vertices of $V_3$ are placed on the line $y = 2$. Moreover, since the number of crossings only depends on the order of vertices, we will assume throughout this paper that the vertices of $V_1$ are placed into the points

$$(1, 1), (2, 1), ..., (|V_1|, 1).$$

Note that for any $v \in V$, $D(v)$ is the $x$-coordinate of $v$, and that $D$ is a 3PLA of $G$. In particular, note that for any $v \in V_0 \cup V_2$, $med(v)$ is the vertex $u_{\lceil \frac{d_v}{2} \rceil}$, where $u_1, u_2, ..., u_{d_v}$ are neighbors of $v$ in the set $V_1$ satisfying $D(u_1) < D(u_2) < ... < D(u_{d_v})$.

**Theorem 1.** *Let $D$ be a 3-layer drawing of $G = (V_0, V_1, V_2, E)$, then*

$$cr_D \geq \frac{1}{2}\left(L_G - \sum_{v \in V_0 \cup V_2} \left\lfloor \frac{d_v}{2} \right\rfloor d_v\right).$$

**Proof.** To show the lower bound on $cr_D$, consider the bipartite drawing $(D_0, D_1)$ of $G_0$. Let $v \in V_0$ with $d_v \geq 2$, and let $u_1, u_2, ..., u_{d_v}$ be its neighbors with $D(u_1) < D(u_2) < ... < D(u_{d_v})$. Let $i$ be an integer, $1 \leq i \leq \lfloor d_v/2 \rfloor$, and let $u$ be a vertex in $V_1$ so that that $D(u_i) < D(u) < D(u_{d_v - i+1})$. Observe that $u$ generates $d_{u,0}$ crossings on the edges $u_i v$ and $u_{d_v-i+1} v$, if it is not adjacent to $v$; similarly, $u$ generates $d_{u,0} - 1$ crossings on the edges $u_i v$ and $u_{d_v-i+1} v$, if it is adjacent to $v$. Thus,

$$cr_0(u_i v) + cr_0(u_{d_v-i+1} v) \geq \sum_{D(u) \in (D(u_i), D(u_{d_v-i+1}))} d_{u,0} - d_v.$$

Hence, for $v \in V_0$ with $d_v \geq 2$,

$$\sum_{i=1}^{d_v} cr_{D_0}(u_i v) \geq \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} \sum_{D(u) \in (D(u_i), D(u_{d_v - i+1}))} d_{u,0} - \left\lfloor \frac{d_v}{2} \right\rfloor d_v.$$

We conclude by taking the sum over all $v \in V_0, d_v \geq 2$ that,

$$2cr_0 \geq \sum_{v \in V_0} \sum_{i=1}^{d_v} cr_0(u_i v) \geq \sum_{v \in V_0} \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} \sum_{D(u) \in (D(u_i), D(u_{d_v - i+1}))} d_{u,0} - \sum_{v \in V_0} \left\lfloor \frac{d_v}{2} \right\rfloor d_v.$$

Using a similar approach we obtain

$$2cr_2 \geq \sum_{v \in V_2} \sum_{i=1}^{d_v} cr_2(u_i v) \geq \sum_{v \in V_2} \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} \sum_{D(u) \in (D(u_i), D(u_{d_v - i+1}))} d_{2,0} - \sum_{v \in V_2} \left\lfloor \frac{d_v}{2} \right\rfloor d_v.$$

Define a 3PLA by: $f(v) = D(v)$ for all $v \in V_1$, and $f(v) = med(v) + \epsilon_v$ for all $v \in V_0 \cup V_2$, where $\epsilon_v$ is an infinitely small value. Let $e = vx$, where $v \in V_0 \cup V_2$, and $x = med(v)$, then $L_f^e = 0$. It follows that

$$L_f = \sum_{v \in V_0 \cup V_2} \sum_{i=1}^{\lfloor \frac{d_v}{2} \rfloor} \sum_{D(u) \in (D(u_i), D(u_{d_v - i+1}))} d_{u,0}.$$

Hence, $2cr_D \geq L_f - \sum_{v \in V_0 \cup V_2} \left\lfloor \frac{d_v}{2} \right\rfloor d_v$, and the claim follows by observing that $L_G \leq L_f$. □

**Theorem 2.** *Let $f$ be a 3PLA of $G = (V_0, V_1, V_2, E)$, then there is 3-layer drawing $D$ of $G$ so that*

$$cr_D \leq \frac{3}{2} L_f + \sum_{v \in V_1} d_v^2.$$

**Proof.** Define a new 3PLA by: $D(v) = f(med(v))$ for any $v \in V_0 \cup V_2$, and $D(v) = f(v)$, for any $v \in V_1$. If two vertices are placed at the same location, we separate them by placing an arbitrary small distance between them.

$$L_D \leq L_f.$$

To prove the upper bound on $cr_D$, we estimate from above the number of crossings on any edge incident to a vertex $v \in V_0 \cup V_2$. The sum of the number of crossings can be shown to be at most $3/2L_D + \sum_{v \in V_1} d_v^2$ using the method to derive the lower bound. □

By Theorems 1 and 2, in order to construct a 3-layer drawing with small number of crossings, one only needs to find a 3PLA of $G$ with small value. Unfortunately, it is not known how to compute exactly or even approximate the 3-layer arrangement problem, in polynomial time. Nonetheless, we can show that if $G_0$ and $G_2$ do not have vertices of degree zero, then, this problem "essentially" becomes the pseudo linear arrangement problem, provided that $G$ is degree bounded. Let $G = (V_0, V_1, V_2, E)$, we denote the maximum degree among all vertices in $V_1$ by $\Delta_1$.

**Lemma 1.** *Let $G = (V_0, V_1, V_2, E)$, and let $\bar{f}$ be any PLA for the bipartite graph $G = (V_0 \cup V_2, V_1, E)$. Then, $\bar{f}$ induces a 3PLA, denoted by $f$, so that*

$$L_f \leq L_{\bar{f}}.$$

*Moreover, if $d_{v,0} > 0$ and $d_{v,2} > 0$, for any $v \in V_1$, then*

$$\frac{L_{\bar{f}}}{\Delta_1} \leq L_f.$$

We can now present our main result.

**Theorem 3.** *Let $G = (V_0, V_1, V_2, E)$ so that $|E| > (2 + \epsilon)|V|$, for a positive $\epsilon$. Assume that $d_{v,0} > 0$ and $d_{v,2} > 0$, for all $v \in V_1$, and that $\Delta_1$ is bounded by a constant. Then, the 3-layer crossing number can be approximated to within a factor of $O(\log n)$ from the optimal in polynomial time.*

**Proof.** The conditions on $E$ and $\Delta_1$ can be used to show that for any 3-layer drawing $D$,

$$cr_D = \Omega\Big(\sum_{v \in V_0 \cup V_2} d_v^2\Big).$$

Hence, by Theorem 1 $cr_D = \Omega(L_G)$. It remains to construct an $O(\log n)$ times optimal 3PLA, denoted by $f$, since then by Theorem 2 we can construct the desired drawing $D$. To construct $f$, we first construct a $O(\log n)$ times optimal PLA using the algorithm in [18]. By Lemma 1 this gives a $O(\log n)$ times optimal 3PLA, or $f$, since $\Delta_1$ is bounded by a constant. □

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