Problems and Results around the Erdős–Szekeres Convex Polygon Theorem⋆

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1 Introduction

Eszter Klein's theorem claims that among any 5 points in the plane, no three collinear, there is the vertex set of a convex quadrilateral. An application of Ramsey's theorem then yields the classical Erdős–Szekeres theorem [19]: *For every integer $n \geq 3$ there is an N_0 such that, among any set of $N \geq N_0$ points in general position in the plane, there is the vertex set of a convex n-gon.* Let $f(n)$ denote the smallest such number.

Theorem 1 ([20,44]).

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-5}{n-2} + 2 \; .$$

A very old conjecture of Erdős and Szekeres is that the lower bound is tight:

Open Problem 1. *For every $n \geq 3$, $f(n) = 2^{n-2} + 1$.*

Similarly, let $f_d(n)$ denote the smallest number such that, in any set of at least $f_d(n)$ points in general position in Euclidean d-space, there is the vertex set of a convex polytope with n vertices, that is, *n points in convex position.* A simple projective argument [47] shows that $f_d(n) \leq f(n)$. It is conjectured by Füredi [22] that $f_d(n)$ is essentially smaller if $d > 2$, namely that $\log f_d(n) = O(n^{1/(d-1)})$. A lower bound that matches this conjectured upper bound was given recently in [33]. On the other hand, Morris and Soltan [34] contemplate about an exponential lower bound on $f_d(n)$.

In this paper we survey recent results and state some open questions that are related to Theorem 1. In particular, we consider "homogeneous", "partitional", and "modular" versions of the Erdős–Szekeres theorem. We will discuss the question whether empty convex polygons (and then how many of them) can be found among N points in the plane. We will also describe how the convex

⋆ The authors gratefully acknowledge that they have been partially supported by Hungarian Research Grants OTKA T032452 and F030822, respectively.

J. Akiyama, M. Kano, and M. Urabe (Eds.): JCDCG 2000, LNCS 2098, pp. 91–105, 2001.

position condition can be strengthened or relaxed in order to arrive at well-posed questions, and present the results obtained so far.

For further aspects of the Erdős–Szekeres theorem we refer to the very recent and comprehensive survey article [34].

2 Homogeneous Versions

From now on we assume that $X \subset \mathbb{R}^2$ is a finite set of points in general position. We assume further that X has N elements. By the Erdős–Szekeres theorem, any subset of X of size $f(n)$ contains the vertices of a convex n-gon. As a fixed n-set is contained in $\binom{N}{f(n)-n}$ subsets of size $f(n)$, a positive fraction of all the n-tuples from X are in convex position. This is a well-known principle in combinatorics. Maybe one can say more in the given geometric situation, for instance, the many convex position n-tuples come with some structure. The following theorem, due to Bárány and Valtr [7], shows that these n-tuples can be chosen homogeneously:

Theorem 2 ([7]). *Given $n \geq 4$, there is a constant $C(n)$ such that for every $X \subset \mathbb{R}^2$ of N points in general position the following holds. There are subsets $Y_1, \ldots, Y_n$ of X, each of size at least $C(n)N$ such that for every transversal $y_1 \in Y_1, \ldots, y_n \in Y_n$, the points $y_1, \ldots, y_n$ are in convex position.*

We call this result the "homogeneous" Erdős–Szekeres theorem. The proof in [7] is based on another homogeneous statement, the so called same type lemma. We state it in dimension d, but first a definition: Two n-tuples $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are said to be of the *same type* if the orientations of the simplices $x_{i_1}, \ldots, x_{i_{d+1}}$ and $y_{i_1}, \ldots, y_{i_{d+1}}$ are the same for every $1 \leq i_1 < i_2 < \ldots < i_{d+1} \leq n$.

Theorem 3 ([7]). *Given $d \geq 2$ and $k \geq d+1$, there is a constant $C(k,d)$ such that for all finite sets $X_1, \ldots, X_k \subset \mathbb{R}^d$ of points such that $\cup_1^k X_i$ is in general position the following holds. For every $i = 1, \ldots k$, the set X_i contains a subset Y_i of size at least $C(k,d)|X_i|$ such that all transversals $y_1 \in Y_1, \ldots, y_k \in Y_k$ are of the same type.*

The proof is based on the center-point theorem of Rado [7], or on Borsuk's theorem [37]. It uses a reformulation of the definition of same type: all transversals of $Y_1, \ldots, Y_k$ are of the same type if no hyperplane meets the convex hulls of any $d+1$ of these sets. The same type lemma implies the homogeneous version of the Erdős–Szekeres theorem in the following way. Choose $k = f(n)$ and partition $X \subset \mathbb{R}^2$ by vertical lines, say, into sets $X_1, \ldots, X_k$ of almost equal size. Apply the same type lemma to them. All transversals of the resulting subsets $Y_1, \ldots, Y_k$ are of the same type. Fix a transversal $y_1, \ldots, y_k$. As $k = f(n)$, the Erdős–Szekeres theorem implies that some n points of this transversal, $y_{j_1}, \ldots, y_{j_n}$, say, are in convex position. Then by the same type lemma, all transversals of $Y_{j_1}, \ldots, Y_{j_n}$ are in convex position.

This proof gives a doubly exponential lower bound for $C(n)$. An alternative proof, with a better bound for $C(n)$ was found by Solymosi [39]. A sketch of Solymosi's neat argument goes as follows. As we have seen above, a positive fraction of the $2n$ element subsets of X are in convex position. Write such a $2n$ element subset as $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ with the points coming in this order on the boundary of their convex hull. Choose $a_1 = x_1, a_2 = x_2, \ldots, a_n = x_n$ so that the number of possible convex extensions $a_1, y_1, a_2, y_2, \ldots, a_n, y_n$ is maximal. Averaging shows that the number of such extensions is at least const$\binom{N}{n}$ where $|X| = N$. A simple geometric argument explains that the possible y_is all lie in the triangle T_i formed by the lines through the pairs of points (a_{i-1}, a_i), (a_i, a_{i+1}) and (a_{i+1}, a_{i+2}). It is not hard to check then that $Y_i = X \cap T_i$ $(i = 1, \ldots, n)$ satisfies the requirements.

This proof gives $C(n) \approx 2^{-16n^2}$, while the lower bound for $f(n)$ shows that $C(n)$ is at least 2^{n-2}. Better bounds are available for $n = 4, 5$ [7]: $C(4) \geq 1/22$ and $C(5) \geq 1/352$. The reader is invited to prove or improve these bounds.

Pach [37] uses the same type lemma to prove a homogeneous version of Caratheodory's theorem that was conjectured in [6]: Given $X_i \subset \mathbb{R}^d$ $(i = 1, \ldots, d+1)$, there is a point $z \in \mathbb{R}^d$ and there are subsets Z_i, each of size $c_d|X_i|$ at least $(i = 1, \ldots, d+1)$, such that the convex hull of each transversal $z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}$ contains the point z. (Here c_d is a constant depending only on d.) Pach's nice argument uses, besides the same type lemma, a quantitative version of Szemerédi's regularity theorem.

We expect that the same type lemma will have many more applications. Also, several theorems from combinatorial convexity extend to positive fraction or homogeneous versions. For instance, a positive fraction Tverberg theorem is proved in [7]. One question of this type concerns Kirchberger's theorem [14]. The latter says that finite sets $A, B \subset \mathbb{R}^d$ can be separated by a hyperplane if and only if for every $S \subset A \cup B$ of size $d + 2$ there is a hyperplane separating $A \cap S$ and $B \cap S$. This suggests the following question:

Open Problem 2. *Let $A, B \subset \mathbb{R}^d$ be finite sets, each of size N, with $A \cup B$ in general position. Assume that for $(1 - \varepsilon)$ fraction of the $\binom{2N}{d+2}$ $(d+2)$-tuples $S \subset A \cup B$ there is a hyperplane separating $A \cap S$ from $B \cap S$. Does it follow then that there are subsets $A' \subset A$ and $B' \subset B$ that are separated by a hyperplane and $|A'|, |B'| \geq (1 - g(\varepsilon))N$ with $g(\varepsilon)$ tending to zero as $\varepsilon \to 0$?*

Partial results in this direction are due to Attila Pór [40]. One word of caution is in place here: the condition $g(\varepsilon) \to 0$ is important since by the ham-sandwich theorem (or Borsuk's theorem, if you like) any two finite sets A, B in $\mathbb{R}^d$ can be simultaneously halved by a hyperplane H. Then half of A is on one side of H while half of B is on the other side.

3 Partitional Variants

Let P be any set of points in general position in the plane. Let C_1, C_2 be subsets of P, each in convex position. We say that the *convex polygons* C_1 and C_2 are *vertex disjoint* if $C_1 \cap C_2 = \emptyset$. If, moreover, their convex hulls are also disjoint, we simply say that the two polygons are *disjoint*. A polygon is called *empty* if its convex hull does not contain any point of P in its interior.

Eszter Klein's theorem implies that P can be partitioned into vertex disjoint convex quadrilaterals plus a remainder set of size at most 4. The following result answers a question posed by Mitchell.

Theorem 4 ([30]). *Let P be any set of $4N$ points in general position in the plane, N sufficiently large. Then there is a partition of P into N vertex disjoint convex quadrilaterals if and only if there is no subset A of P such that the size of A is odd but the the size of $A \cap C$ is even for every convex quadrilateral C.*

There is also an $N \log N$-time algorithm [30] which decides if such a partition exists. The following problem seems to be more difficult.

Open Problem 3. *Is there a fast algorithm which decides if a given set of $4N$ points in general position in the plane admits a partition into disjoint convex quadrilaterals?*

For $k \geq 3$ the *Ramsey-remainder* $rr(k)$ was defined by Erdős et al. [21] as the smallest integer such that any sufficiently large set of points in general position in the plane can be partitioned into vertex disjoint polygons, each of size $\geq k$, and a remaining set of size $\leq rr(k)$. Thus, $rr(k) < f(k)$ for every k. In particular, $rr(3) = 0$ and $rr(4) = 1$. Partial results on $rr(k)$ in general were proved in [21]. It is known, for example, that $rr(k) \geq 2^{k-2} - k + 1$. The solution of the following problem could make an essential step to settle Problem 1, see [21].

Open Problem 4. *Is it true that $rr(k) = 2^{k-2} - k + 1$?*

There is no Ramsey-remainder in higher dimensions. The following result is due to Károlyi [30].

Theorem 5. *Let $k > d \geq 3$. If N is sufficiently large, then every set of N points in general position in $\mathbb{R}^d$ can be partitioned into subsets of size at least k each of which is in convex position.*

The main observation here is that, for large enough N, every point of P belongs to some k-element subset which is in convex position.

A problem in close relation to Problem 3 is the following. Given natural numbers k and n, let $F_k(n)$ denote the maximum number of pairwise disjoint empty convex k-gons that can be found in every n-element point set in general position in the plane. The study of this function was initiated in [27]. Horton's result mentioned in Section 5 implies $F_7(n) = 0$ for every n. Thus, the interesting functions are F_4, F_5 and F_6. Nothing is known about F_6, in fact Problem 6 is equivalent to asking whether $F_6(n) > 0$ for some n. Since every 5-point set determines an empty convex quadrilateral, obviously $F_4(n) \geq \lfloor n/5 \rfloor$. Similarly, it follows from a result of Harborth [23] that $F_5(n) \geq \lfloor n/10 \rfloor$ for every n.

The non-trivial lower bound $F_4(n) \geq \lfloor 5n/22 \rfloor$ is presented in [27], based on the following observation. Suppose P is any set of $2m+4$ points in general position in the plane. Then there is a partition of the plane into 3 convex regions such that one region contains 4 points of P in convex position, and the other regions contain m points of P each. There is no counterpart of this lemma for pentagons, and in fact no lower bound is known about F_5 beyond what is said above. As for F_4, an even stronger lower bound $F_4(n) \geq (3n-1)/13$ has been proved for an infinite sequence of integers n.

Concerning upper bounds, a construction in [27] shows that $F_5(n) \leq 1$ if $n \leq 15$. It is not too difficult to prove that $F_5(n) < n/6$, but no nontrivial upper bound is known for $F_4(n)$ in general.

For any positive integer n let $F(n)$ denote the smallest integer such that every set of n points in general position in the plane can be partitioned into $F(n)$ empty convex polygons, with the convention that point sets consisting of at most two points are always considered as empty convex polygons. Urabe [45] proved $\lceil (n-1)/4 \rceil \leq F(n) \leq \lceil 2n/7 \rceil$. The upper bound follows from the fact that every 7-point set can be partitioned into an empty triangle and an empty convex quadrilateral.

An improved upper bound $F(n) \leq \lceil 5n/18 \rceil$ is presented in [27] along with an infinite sequence of integers n for which also $F(n) \leq (3n+1)/11$.

An other function $H(n)$ was also introduced in [45] as the smallest number of vertex disjoint convex polygons into which any n-element point set can be partitioned in the plane. An application of Theorem 1 gives that the order of magnitude of this function is $n/\log n$.

Finally we mention that the functions F and H can be naturally defined in any dimension; denote the corresponding functions in d-space by F_d and H_d. Urabe [46] proves that $\Omega(n/\log n) \leq F_3(n) \leq \lceil 2n/9 \rceil$ and that $H_3(n) = o(n)$. The proof technics of [45] coupled with the bounds given in Section 1 on f_d in fact yield $\Omega(n/(\log^{d-1} n) \leq F_d(n) \leq O(n/\log n)$.

4 Matrix Partitions

Assume $X_1, \ldots, X_n$ in $\mathbb{R}^d$, are pairwise disjoint sets, each of size N, with $\cup X_i$ in general position. A matrix partition, or μ-partition for short, of the X_is with m columns is the partition $X_i = \cup_{k=1}^m M_{ik}$ for $i = 1, \ldots, n$ if $|M_{ik}| = |M_{jk}|$ for every $i, j = 1, \ldots, n$ and every $k = 1, \ldots, m$. In other words, a μ-partition of $X_1, \ldots, X_n$ with m columns is an $n \times m$ matrix M whose (i, k) entry is a subset M_{ik} of X_i such that row i forms a partition of X_i and the sets in column k are of the same size. Gil Kalai asked [28] whether the homogeneous Erdős–Szekeres theorem admits a partitioned extension:

Open Problem 5. *Show that for every $n \geq 4$ there is an integer $m = g(n)$ such that for every finite set $X \subset \mathbb{R}^2$ of N points in general position there is a subset $X_0 \subset X$, of size less than $f(n)$, (this is the Erdős–Szekeres function from*

Section 1), and there exists a partition of $X \setminus X_0$ *into sets* $X_1, \ldots, X_n$ *of equal size such that the following holds. The sets* $X_1, \ldots, X_n$ *admit a* μ*-partition* M *with* m *columns so that every transversal* $x_1 \in M_{1k}, x_2 \in M_{2k}, \ldots, x_n \in M_{nk}$ *is in convex position, for all* $k = 1, \ldots, m$*.*

By the homogeneous version one can choose the sets for the first column of a μ-partition, each of size $C(n)N/n$, then for the second, third, etc columns from the remaining part of X, but this would result in a suitable μ-partition with too many, namely $\log N$, columns. The remainder set X_0 is needed for two simple reasons: when N is smaller than $f(n)$ there may not be a convex n-gon at all, and when N is not divisible by n.

Partial solution to Problem 2 is due to Attila Pór [41]. He first proved a partitioned extension of the same type lemma. To state this result we define the sets $Y_1, \ldots, Y_n \subset \mathbb{R}^d$ with $n \geq d+1$ *separated* if every hyperplane intersects at most d sets of the convex hulls of $Y_1, \ldots, Y_n$. As we mentioned in Section 2, the sets $Y_1, \ldots, Y_n$ are separated if and only if every transversal $y_1 \in Y_1, \ldots, y_n \in Y_n$ is of the same type.

Theorem 6 ([41]). *For all natural numbers* n, d *with* $n \geq d+1$ *there is a natural number* $m = m(n, d)$ *such that if finite sets* $X_1, \ldots, X_n \subset \mathbb{R}^d$ *have the same size and* $\cup_1^n X_i$ *is in general position, then there exists a* μ*-partition with* m *columns such that the sets* $M_{1k}, \ldots, M_{nk}$ *in every column are separated.*

This is exactly the partitioned version of the same type lemma. The proof is based on a clever induction argument and a third characterization for sets $Y_1, \ldots, Y_n$ being separated. The result is used by A. Pór [41] to solve the first interesting case, $n = 4$ of Problem 2.

Theorem 7 ([41]). *Assume* $X \subset \mathbb{R}^2$ *is a finite set of* N *points in general position. Then there is an* $X_0 \subset X$ *of size at most 4, and a partition of* $X \setminus X_0$ *into sets* X_1, X_2, X_3, X_4 *of equal size such that they admit a* μ*-partition* M *with 30 columns so that every transversal* $x_1 \in M_{1k}, \ldots, x_4 \in M_{4k}$ *is in convex position.*

The proof starts by cutting up X into four sets of almost equal size by vertical lines, say. Then the same type lemma (matrix partition version) is applied to these four sets giving a matrix partition with few columns. The columns are of two types: either every transversal is a convex quadrangle and there is nothing to do, or every transversal is a triangle with the fourth point inside it. In the latter case one has to partition the column further. This can be done with a topological argument: the interested reader should consult the paper [41]. The method does not seem to work for $n \geq 5$, apparently new ideas are needed.

5 Empty Convex Polygons

For a long time it had been conjectured that every sufficiently large point set, in general position in the plane contains the vertex set of an *empty* convex n-gon, that is, n points which form the vertex set of a convex polygon with no other point of the set in its interior. Harborth [23] showed that every 10-element point set determines an empty convex *pentagon*, and that here 10 cannot be replaced by any smaller number. Finally, in 1983 a simple recursive construction of arbitrarily large finite point sets determining no empty convex *heptagons* was found by Horton [24]. The corresponding problem for *hexagons* is still open:

Open Problem 6. *Is it true that every sufficiently large set of points in general position in the plane contains the vertex set of an empty convex hexagon?*

We strongly believe that the answer is yes, but there is no proof in sight.

Several algorithms had been designed [4,15,36] to determine if a given set of points contains an empty 6-gon, and to construct large point sets without any empty hexagon. The current world record, a set of 26 points that does not contain an empty convex 6-gon was discovered by Overmars et al. [36] in 1989.

A surprising number of questions can be related to this seemingly particular problem. The first one, due to Solymosi [43], relates it to a Ramsey type problem for geometric graphs. A *geometric graph* is a graph drawn in the plane such that the vertices are represented by points in general position while the edges are straight line segments that connect the corresponding vertices.

Open Problem 7. *Let G be a complete geometric graph on n vertices whose edges are colored with two different colors. Assume that n is sufficiently large. Does it follow then that G contains an empty monochromatic triangle?*

Were the answer to this question negative, it would imply that there are arbitrarily large point sets without an empty convex 6-gon. For assume, on the contrary, that every sufficiently large point set contains such an empty polygon. Color the edges of the corresponding complete geometric graph with two colors, it induces a coloring of the edges that connect the vertices of the empty 6-gon. It follows from Ramsey's theorem that this two-colored graph on 6 vertices contains a monochromatic triangle (which is also empty), a contradiction.

An other related problem has been studied recently by Hosono et al. [26]. Let P denote a simple closed polygon together with its interior. A *convex subdivision* of P is a 2-dimensional cell complex in the plane whose vertex set coincides with the vertex set of P, whose body is P, and whose faces are all convex polygons. Denote by $F'(n)$ the smallest integer for which any set of n points in general position in the plane can be connected with a closed simple polygon that admits a convex subdivision with at most $F'(n)$ faces. Since each face in a convex subdivision is an empty convex polygon, it follows from Horton's construction that $F'(n) \geq n/4$ for an infinite sequence of n. It is proved for every n in [26] where an upper bound $F'(n) \leq \lceil 3n/5 \rceil$ is also presented.

Open Problem 8. *Is it true that $F'(n) \geq (n-2)/3$?*

A negative answer would give an affirmative solution to the empty hexagon problem.

Essential combinatorial properties of Horton's construction were studied and extended into higher dimensions by Valtr [47], resulting in constructions that yield the following general result. Denote by $h(d)$ the largest integer h with the following property: every sufficiently large point set in general position in $\mathbb{R}^d$ contains an *h-hole*, that is, h points which are vertices of an empty convex d-polytope. Thus, $5 \leq h(2) \leq 6$.

Theorem 8 ([47]). *The integer $h(d)$ exist for any $d \geq 2$ and satisfies*

$$2d + 1 \leq h(d) \leq 2^{d-1}(P_{d-1} + 1) \ ,$$

where P_i denotes the product of the first i positive prime numbers.

It is also known that $h(3) \leq 22$.

We close this section by turning back to the plane: there are certain nontrivial classes of point sets where large empty convex polygons can be found. For example, if every triple in the point set determines a triangle with *at most one* point in its interior, then it is said to be *almost convex*.

Theorem 9 ([32]). *For any $n \geq 3$, there exists an integer $K(n)$ such that every almost convex set of at least $K(n)$ points in general position in the plane determines an empty convex n-gon. Moreover, we have $K(n) = \Omega(2^{n/2})$.*

This result has been extended recently by Valtr [50] to point sets where every triple determines a triangle with at most a fixed number of points in its interior. It also must be noted that Bisztriczky and Fejes Tóth [10] proved the following related result.

Theorem 10. *Let l, n denote natural numbers such that $n \geq 3$. Any set of at least $(n-3)(l+1)+3$ points in general position in the plane, with the property that every triple determines a triangle with at most l of the points in its interior, contains n points in convex position. Namely, its convex hull has at least n vertices, and in this respect this bound cannot be improved upon.*

6 The Number of Empty Polygons

Let $X \subset \mathbb{R}^2$ be a set of N points in general position, and write $g_n(X)$ for the number of empty convex n-gons with vertices from X. Of course, $n \geq 3$. Define $g_n(N)$ as the minimum of $g_n(X)$ over all planar sets X with N points in general position. Horton's example shows that $g_n(N) = 0$ when $n \geq 7$. Problem 6 is, in fact, to decide whether $g_6(N) = 0$ or not.

The first result on $g_n(N)$ is due to Katchalski and Meir [29] who showed $g_3(N) \leq 200N^2$. In Bárány and Füredi [5] lower and upper bounds for $g_n(N)$ are given. The lower bounds are:

Theorem 11 ([5]).

$$g_3(N) \geq N^2 - O(N \log N) \ ,$$
$$g_4(N) \geq \frac{1}{2}N^2 - O(N \log N) \ ,$$
$$g_5(N) \geq \lfloor \frac{N}{10} \rfloor \ .$$

The last estimate can be easily improved to $g_5(N) \geq \lfloor \frac{N-4}{6} \rfloor$.

Of these inequalities, the most interesting is the one about g_3. Its proof gives actually more than just $g_3(N) \geq N^2(1 + o(1))$. Namely, take any line ℓ and project the points of X onto ℓ. Let $z_1, \dots, z_N$ be the projected points on ℓ in this order, and assume z_i is the projection of $x_i \in X$. We say that pair z_i, z_j supports the empty triangle x_i, x_k, x_j if this triangle is empty and $i < k < j$. Now the proof of the lower bound on $g_3(N)$ follows from the observation that all but at most $O(N \log N)$ pairs z_i, z_j support at least two empty triangles. (This fact implies, further, the lower bound on g_4 as well.) It is very likely that a small but positive fraction of the pairs supports three or more empty triangles but there is no proof in sight. If true, this would solve the next open problem in the affirmative:

Open Problem 9. *Assume X is a finite set of N points in general position in $\mathbb{R}^2$. Show that $g_3(N) \geq (1 + \varepsilon)N^2$ for some positive constant ε.*

The upper bounds from [5] have been improved upon several times, [48], [16], and [8]. The constructions use Horton sets with small random shifts. We only give the best upper bounds known to date [8].

Theorem 12.

$$g_3(N) \leq (1 + o(1))1.6195...N^2 \ ,$$
$$g_4(N) \leq (1 + o(1))1.9396...N^2 \ ,$$
$$g_5(N) \leq (1 + o(1))1.0205...N^2 \ ,$$
$$g_6(N) \leq (1 + o(1))0.2005...N^2 \ .$$

It is worth mentioning here that the function $g_n(X)$ satisfies two linear equations. This is a recent discovery of Ahrens et al. [1] and Edelman-Reiner [17]. Since the example giving the upper bounds in the last theorem is the same point set X and $g_7(X) = 0$, only two of the numbers $g_n(X)$ $(n = 3, 4, 5, 6)$ have to be determined.

There is a further open problem due to the first author, that appeared in a paper by Erdős [18]. Call the degree of a pair $e = \{x, y\}$ (both x and y coming from X) the number of triples x, y, z with $z \in X$ that are the vertices of an empty triangle, and denote it by $\deg(e)$.

Open Problem 10. *Show that the maximal degree of the pairs from X goes to infinity as the size of X, $N \to \infty$.*

The lower bound on g_3 implies that the average degree is at least $6 + o(1)$ in the following way. Write T for the set of triples from X that are the vertices of an empty triangle. We count the number, M, of pairs (e, t) where $t \in T$, $e \subset T$ and e consists of two elements of X in two ways. First $M = \sum \deg(e)$ the sum taken over all two-element subsets of X. Secondly, as every triangle has three sides, $M = 3|T| = 3g_3(X) \geq (3 + o(1))N^2$ from the lower bound on $g_3(N)$, showing indeed that the average degree is at least $6 + o(1)$.

We show next that the maximal degree is at least 10 when N is large enough, a small improvement that is still very far from the target. Choose first a vertical line ℓ_1 having half of the points of X on its left, the other half on its right. (Throw away the leftmost or rightmost point if N is odd.) Then choose a line ℓ_2, by the ham-sandwich theorem, halving the points on the left and right of ℓ_1 simultaneously (throwing away, again, one or two points if necessary). We have now four sectors, S_1, S_2, S_3, S_4 each containing m points from X with $m = \lfloor N/4 \rfloor$. (S_1, S_4 are on the left of ℓ_1 and S_1, S_2 are below ℓ_2, say.) Let $e = \{x, y\}$ with $x, y \in X$ and define $\deg(e; S_i)$ as the number of points $z \in X \cap S_i$ such that $\{x, y, z\} \in T$. The observation following the lower bounds for g_n gives that, when $e = \{x, y\}$ with $x \in X \cap S_1$ and $y \in X \cap S_2$, then for all but at most $O(m \log m)$ of the possible pairs $\deg(e; S_1 \cup S_2) \geq 2$, so

$$\sum_{x \in S_1} \sum_{y \in S_2} \deg(\{x, y\}; S_1 \cup S_2) \geq (2 + o(1))m^2.$$

On the other hand,

$$\sum_{x \in S_1} \sum_{y \in S_2} \deg(\{x, y\}; S_1 \cup S_2) = \sum_{x, z \in S_1} \deg(\{x, z\}; S_2) + \sum_{y, z \in S_2} \deg(\{y, z\}; S_1).$$

The analogous identities and inequalities for pairs in $S_2 \times S_3$, $S_3 \times S_4$, and $S_4 \times S_1$ together yield that

$$\sum_{i=1}^{4} \sum_{x, y \in S_i} \deg(\{x, y\}; S_{i-1} \cup S_{i+1}) \geq (8 + o(1))m^2,$$

where $i + 1$ and $i - 1$ are to be taken modulo 4. This means that, in at least one of the sectors, the average degree of a pair is at least $4 + o(1)$ in the neighboring two sectors. As we have seen, the average degree of a pair is at least $6 + o(1)$ within each sector. This proves the claim.

7 The Modular Version

Bialostocki, Dierker, and Voxman [9] proposed the following elegant "modular" version of the original problem.

Open Problem 11. *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B(n, p)$ such that every set of $B(n, p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is* $0 \bmod p$.

Bialostocki et al. proved this conjecture for every $n \geq p+2$. Their proof goes as follows. Assume, for technical simplicity, that $n = p+2$. Choose an integer m that is very large compared to n. Consider a set P of $f(m)$ points in general position in the plane, by Theorem 1 it contains an m-element set S in convex position. Associate with every triple $\{a,b,c\} \subseteq S$ one of the p colors $0,1,2,\ldots,p-1$; namely color i if triangle abc contains i points of P in its interior modulo p. As a consequence of Ramsey's theorem we can select an n-element subset S' of S all of whose triples are of the same color, given that m is sufficiently large. Consider any triangulation of the convex hull of S', it consists of p triangles. Consequently, the number of points inside this convex n-gon is divisible by p.

This proof implies a triple exponential upper bound on $B(n,p)$, a bound which was later improved essentially by Caro [12], but his proof also relied heavily on the assumption $n \geq p+2$. Recently the conjecture was proved in [32] for every $n \geq 5p/6 + O(1)$. A key factor in this improvement is Theorem 9.

The situation changes remarkably in higher dimensions. For example, a 3-polytope with 5 vertices admits two essentially different triangulations: one into two simplices and an other into three simplices. Based on this observation Valtr [49] proved the following result.

Theorem 13. *For any $n \geq 4$ and $p \geq 2$, there exists an integer $C(n,p)$ such that every set of $C(n,p)$ points in general position in 3-space determines a convex polytope with n vertices such that the number of points in its interior is* $0 \bmod p$.

Indeed, let P be any sufficiently large set of points in general position in 3-space. As in the planar case, we can use the Erdős–Szekeres theorem and then Ramsey's theorem to find at least n and not less than 5 points in convex position such that every tetrahedron determined by these points contains the same number of points, say i, in its interior modulo p. Consider any 5 of these points and triangulate their convex hull in two different ways: first into two tetrahedra, then into three tetrahedra. It follows that $2i \equiv 3i$, and thus $i \equiv 0 \pmod{p}$.

The same argument can be used to extend Theorem 9, and also its generalization by Valtr, to 3-space:

Theorem 14. *Given any natural numbers k and $n \geq 3$, there exists an integer $K_3(k,n)$ such that the following holds. Every set of at least $K_3(k,n)$ points in general position in 3-space, with the property that any tetrahedron determined by these points contains at most k points in its interior, contains an n-hole.*

Similar results are proved also in every odd dimension. First we recall the following strengthening of the Erdős–Szekeres theorem, which seems to be folklore. See [13] or [11, Proposition 9.4.7] for a proof.

Theorem 15. *Let $d \geq 2$. For every $n \geq d+1$ there is an integer $N_d(n)$ such that, among any set of $N \geq N_d(n)$ points in general position in $\mathbb{R}^d$ there is the vertex set of a cyclic d-polytope with n vertices.*

Note that in the above theorem we cannot replace the cyclic polytopes with any class of polytopes of different combinatorial kind: one may select any number of points on the moment curve yet every n-element subset will determine a cyclic polytope.

Next, suppose that d is odd. In general, any cyclic polytope with $d+2$ vertices admits a triangulation into $(d+1)/2$ simplices, and also a different one into $(d+3)/2$ simplices. Thus, Theorems 13 and 14 have counterparts in every odd dimension [50].

These arguments however cannot be extended to even dimensions: it is known [42] that every triangulation of a cyclic d-polytope, d even, consists of the same number of simplices.

8 Further Problems

Let $h(n,k)$ denote the smallest number such that among at least $h(n,k)$ points in general position in the plane there is always the vertex set of a convex n-gon such that the number of points in its interior is at most k. Horton's result says that $h(n,0)$ does not exist for $n \geq 7$. In general, Nyklová [35], based on Horton's construction, established that $h(n,k)$ does not exist for $k \leq c \cdot 2^{n/4}$. She also determined that $h(6,5) = 19$, yet another step towards the solution of Problem 6.

The following problem was motivated in [30]. For integers $n \geq k \geq 3$, let $g(k,n)$ be the smallest number with the property that among any $g(k,n)$ points in general position in the plane, there exist n points whose convex hull has at least k vertices. Clearly $g(k,n)$ exists and satisfies $f(k) \leq g(k,n) \leq f(n)$. Based on the results of Section 2 one can easily conclude that $g(k,n) < c_1 n + c_2$, where the constants c_1, c_2 (dependent only on k) are exponentially large in k. The true order of magnitude of $g(k,n)$ was found by Károlyi and Tóth [31]. It is not difficult to see that $g(4,n) = \lceil 3n/2 \rceil - 1$. In general the following bounds are known.

Theorem 16 ([31]). *For arbitrary integers $n \geq k \geq 3$,*

$$\frac{(k-1)(n-1)}{2} + 2^{k/2-4} \leq g(k,n) \leq 2kn + 2^{8k} .$$

To obtain the upper bound, peel off convex layers from a set P of at least $2kn + 2^{8k}$ points as follows. Let $P_1 = P$ and Q_1 the vertex set of its convex hull. Having P_i, Q_i already defined, set $P_{i+1} = P_i \setminus Q_i$ and let Q_{i+1} be the set of vertices of the convex hull of P_{i+1}. If there is an integer $i \leq 2n$ such that $|Q_i| \geq k$, then we are ready. Otherwise we have $2n$ convex layers $Q_1, Q_2, \ldots, Q_{2n}$, and at least 4^{4k} further points inside Q_{2n}. Thus, by Theorem 1, P_{2n+1} contains the vertex set of a convex $4k$-gon C, and the desired configuration of n points whose convex hull has at least k vertices can be selected from the nested arrangement of the convex sets $Q_1, Q_2, \ldots, Q_{2n}, C$.

Open Problem 12. *Is it true that* $g(5,n) = 2n-1$*?*

Open Problem 13. *Is it true for any fixed value of* k *that*

$$\lim_{n\to\infty} \frac{g(k,n)}{n} = \frac{k-1}{2} \ ?$$

An *interior* point of a finite point set is any point of the set that is not on the boundary of the convex hull of the set. For any integer $k \geq 1$, let $g(k)$ be the smallest number such that every set of points P in general position in the plane, which contains at least $g(k)$ interior points has a subset whose convex hull contains exactly k points of P in its interior. Avis, Hosono, and Urabe [2] determined that $g(1) = 1$, $g(2) = 4$ and $g(3) \geq 8$. It is not known if $g(k)$ exists for $k \geq 3$. It was pointed out by Pach (see [2]) that if P contains at least k interior points, then it has a subset such that the number of interior points of P inside its convex hull is between k and $\lfloor 3k/2 \rfloor$. A similar problem was studied also in [3].

Open Problem 14. *Prove or disprove that every point set in general position in the plane with sufficiently many interior points contains a subset in convex position with exactly 3 interior points.*

A first step towards the solution may be the following result of Hosono, Károlyi, and Urabe [25]. Let $g_\Delta(k)$ be the smallest number such that every set of points P in general position in the plane whose convex hull is a triangle which contains at least $g(k)$ interior points also has a subset whose convex hull contains exactly k points of P in its interior.

Theorem 17. *If* $g_\Delta(k)$ *is finite then so is* $g(k)$*.*

The proof is based on a result of Valtr [50] which extends Theorem 9.

Note Added in Proof. The answer to Open Problem 2 is yes and the proof is quite simple. Open Problem 5 was solved very recently by Pór and Valtr: the answer is again yes, but the proof is not that simple.

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