Mixed-hybrid FEM Discrete Fracture Network Model of the Fracture Flow

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Abstract. A stochastic discrete fracture network model of Darcy's underground water flow in disrupted rock massifs is introduced. Mixed finite element method and hybridization of appropriate lowest order Raviart–Thomas approximation is used for the special conditions of the flow through connected system of 2-D polygons placed in 3-D. Model problem is tested.

1 Introduction

We consider a steady saturated Darcy's law governed flow of an incompressible fluid through a system of 2-D polygons placed in the 3-D space and connected under certain conditions into one network. This may simulate underground water flow through natural geological disruptions of a rock massif, fractures, e.g. for the purposes of finding of suitable nuclear waste repositories. Note that intersection of three or more triangles through one edge in the discretization is possible owing to the special geometrical situation, see Fig. 1. We study the existence and uniqueness of weak and discrete mixed solutions, and finally use the hybridization of the lowest order Raviart-Thomas mixed approximation, see [3], [4] respectively. For technical details of the following, see [5].

2 Mathematical-physical Formulation

We suppose that we have

$$\mathcal{S} = \Big\{ \bigcup_{\ell \in L} \overline{\alpha_\ell} \setminus \partial \mathcal{S} \Big\}, \qquad (1)$$

where α_ℓ is an opened 2-D polygon placed in a 3-D Euclidean space; we call $\overline{\alpha_\ell}$ as a fracture. We denote as L the index set of fractures, $|L|$ is the number (finite) of considered fractures. We suppose that all closures of these polygons are connected into one "fracture network", the connection is possible only through an edge, not a point. Moreover, we require that if $\overline{\alpha_i} \bigcap \overline{\alpha_j} \neq \emptyset$ then $\overline{\alpha_i} \bigcap \overline{\alpha_j} \subset \partial\alpha_i \bigcap \partial\alpha_j$, i.e. the connection is possible only through fracture boundaries, cf.

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Fig. 1 (we state this requirement in order to be able to define correct function spaces).

Let us have a 2-D orthogonal coordinate system in each polygon α_ℓ. We are looking for the fracture flow velocity $\mathbf{u}$ (2-D vector in each α_ℓ), which is the solution of the following problem

$$\mathbf{u} = -\mathbf{K}\Big(\nabla p + \nabla z\Big) \quad \text{in} \quad \mathcal{S}\,, \tag{2}$$

$$\nabla \cdot \mathbf{u} = q \quad \text{in} \quad \mathcal{S}\,, \tag{3}$$

$$p = p_D \quad \text{in} \quad \Lambda_D, \qquad \mathbf{u}\cdot\mathbf{n} = u_N \quad \text{in} \quad \Lambda_N\,, \tag{4}$$

where all variables are expressed in appropriate local coordinates of α_ℓ and also the differentiation is always expressed towards these local coordinates. The equation 2 is Darcy's law, 3 is the mass balance equation and 4 is the expression of appropriate boundary conditions. The variable p denotes the modified fluid pressure p ($p = \frac{\mathrm{p}}{\varrho g}$), g is the gravitational acceleration constant, ϱ is the fluid density, q represents stationary sources/sinks density and z is the elevation, positive upward taken vertical 3-D coordinate expressed in appropriate local coordinates. We require the second rank tensor $\mathbf{K}$ to be symmetric and uniformly positive definite on each α_ℓ. Λ_D is a part of $\partial\mathcal{S}$ where Dirichlet's type boundary conditions is given and similary Λ_N is a part of $\partial\mathcal{S}$ where Neumann's type boundary condition is given. Of course $\partial\mathcal{S} = \overline{\Lambda_D \cup \Lambda_n}$ holds.

3 Function Spaces

We start from $L^2(\alpha_\ell)$, $\|u\|_{0,\alpha_\ell} = (\int_{\alpha_\ell} u^2 \; dS)^{\frac{1}{2}}$ and $\mathbf{L}^2(\alpha_\ell) = L^2(\alpha_\ell) \times L^2(\alpha_\ell)$ in order to introduce

$$L^2(\mathcal{S}) \equiv \prod_{\ell \in L} L^2(\alpha_\ell)\,,\; \mathbf{L}^2(\mathcal{S}) \equiv L^2(\mathcal{S}) \times L^2(\mathcal{S})\,. \tag{5}$$

We begin with classical Sobolev space $H^1(\alpha_\ell)$ of scalar functions with square integrable weak derivatives, $H^1(\alpha_\ell) = \{\varphi \in L^2(\alpha_\ell);\; \nabla\varphi \in \mathbf{L}^2(\alpha_\ell)\}$, $\|\varphi\|_{1,\alpha_\ell} = (\int_{\alpha_\ell}[\varphi^2 + \nabla\varphi\cdot\nabla\varphi]\,dS)^{\frac{1}{2}}$, so as to introduce

$$\begin{aligned} H^1(\mathcal{S}) \equiv \{ v \in L^2(\mathcal{S})\,;\; v|_{\alpha_\ell} \in H^1(\alpha_\ell) \quad \forall \ell \in L\,, \\ (v|_{\alpha_i})|_f = (v|_{\alpha_j})|_f \quad \forall\, f = \overline{\alpha_i} \textstyle\bigcap \overline{\alpha_j}\,,\; i,j \in L\,\}\,. \end{aligned} \tag{6}$$

We note that this is possible even for the investigated geometrical situation. We then have the spaces $H^{\frac{1}{2}}(\partial\mathcal{S})$ and $H^{-\frac{1}{2}}(\partial\mathcal{S})$ and the surjective continuous trace operator $\gamma : H^1(\mathcal{S}) \to H^{\frac{1}{2}}(\partial\mathcal{S})$ as in the standard planar case.

We denote as $\mathbf{H}(div, \alpha_\ell)$ the Hilbert space of vector functions with square integrable weak divergences, $\mathbf{H}(div,\alpha_\ell) = \{\mathbf{v} \in \mathbf{L}^2(\alpha_\ell)\,;\; \nabla\cdot\mathbf{v} \in L^2(\alpha_\ell)\}$, $\|\mathbf{u}\|_{\mathbf{H}(div,\alpha_\ell)} = (\|\mathbf{u}\|^2_{0,\alpha_\ell} + \|\nabla\cdot\mathbf{u}\|^2_{0,\alpha_\ell})^{\frac{1}{2}}$. We can define now

$$\begin{aligned} \mathbf{H}(div,\mathcal{S}) \equiv \{ \mathbf{v} \in \mathbf{L}^2(\mathcal{S})\,;\; \mathbf{v}|_{\alpha_\ell} \in \mathbf{H}(div,\alpha_\ell) \quad \forall \ell \in L\,, \\ \textstyle\sum_{i \in I_f} \langle \mathbf{v}|_{\alpha_i}\cdot\mathbf{n}_i, \varphi_i\rangle = 0\} \\ \forall f \text{ such that } |I_f| \geq 2\,,\; I_f = \{i \in L\,;\; f \subset \partial\alpha_i\}\,,\; \forall\varphi_i \in H^1_{\partial\alpha_i \setminus f}\,. \end{aligned} \tag{7}$$

Again, such "local" definition is necessary, since we do not deal with a standard planar case. It naturally expresses the continuity of the normal trace of functions from $\mathbf{H}(div, \mathcal{S})$ even for the given geometrical situation. We have the surjective continuous normal trace operator $\zeta \ : \ \mathbf{u} \in \mathbf{H}(div, \mathcal{S}) \to \mathbf{u} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\partial\mathcal{S})$ as in the standard planar case. We further define the space $\mathbf{H}_{0,N}(div, \mathcal{S}) = \{\mathbf{u} \in \mathbf{H}(div, \mathcal{S}) \ ; \ \langle \mathbf{u} \cdot \mathbf{n}, \varphi \rangle_{\partial\mathcal{S}} = 0 \quad \forall \varphi \in H^1_D(\mathcal{S})\}$. (where $H^1_D(\mathcal{S}) = \{\varphi \in H^1(\mathcal{S}) \ ; \ \gamma\varphi = 0 \text{ on } \Lambda_D \}$) Naturally, the norms on the spaces defined by 5, 6, 7 are given as

$$\| \ \|^2_{\cdot,\mathcal{S}} = \sum_{\ell=1}^{|L|} \| \ \|^2_{\cdot,\alpha_\ell} \,. \tag{8}$$

Remark 31 *Note that definitions 5, 6, 7 are essential. The system $\mathcal{S}$, however consisting of plane polygons, is not planar by oneself. Moreover, one edge can be common to three or more polygons α_ℓ creating the system $\mathcal{S}$.*

4 Weak Mixed Solution

Let us denote $\mathbf{A} = \mathbf{K}^{-1}$ on each α_ℓ, characterizing the medium resistance. Let us now consider such $\tilde{\mathbf{u}}$ that $\tilde{\mathbf{u}} \cdot \mathbf{n} = u_N$ on Λ_N in appropriate sense.

Definition 41 *As a weak mixed solution of the steady saturated fracture flow problem described by 2 – 4, we understand a function $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, $\mathbf{u}_0 \in \mathbf{H}_{0,N}(div, \mathcal{S})$, and $p \in L^2(\mathcal{S})$ satisfying*

$$\begin{aligned}(\mathbf{A}\mathbf{u}_0, \mathbf{v})_{0,\mathcal{S}} - (\nabla \cdot \mathbf{v}, p)_{0,\mathcal{S}} = -\langle \mathbf{v} \cdot \mathbf{n}, p_D \rangle_{\Lambda_D} + (\nabla \cdot \mathbf{v}, z)_{0,\mathcal{S}} - \\ -\langle \mathbf{v} \cdot \mathbf{n}, z \rangle_{\partial\mathcal{S}} - (\mathbf{A}\tilde{\mathbf{u}}, \mathbf{v})_{0,\mathcal{S}} \qquad \forall \, \mathbf{v} \in \mathbf{H}_{0,N}(div, \mathcal{S}) \,,\end{aligned} \tag{9}$$

$$-(\nabla \cdot \mathbf{u}_0, \phi)_{0,\mathcal{S}} = -(q, \phi)_{0,\mathcal{S}} + (\nabla \cdot \tilde{\mathbf{u}}, \phi)_{0,\mathcal{S}} \qquad \forall \phi \in L^2(\mathcal{S}) \,. \tag{10}$$

Our requirements are $A_{ij} \in L^\infty(\mathcal{S})$, $q \in L_2(\mathcal{S})$, $p_D \in H^{\frac{1}{2}}(\Lambda_D)$ and $u_N \in H^{-\frac{1}{2}}(\Lambda_N)$.

Theorem 41 *The problem (9), (10) has a unique solution.*

Proofs of this and all following theorems and lemmas can be found in [5].

5 Mixed Finite Element Approximation

Let us suppose a triangulation $\mathcal{T}_h$ of the system $\mathcal{S}$ from now on. We define an index set J_h to number the elements of the triangulation, $|J_h|$ denotes the number of elements. We define a 3-dimensional space $\mathbf{RT}^0(e)$ of vector functions linear on a given element e with the basis $\mathbf{v}_i^e$, $i \in \{1, 2, 3\}$, where

$$\mathbf{v}_1^e = k_1^e \begin{bmatrix} x - \alpha_{11}^e \\ y - \alpha_{12}^e \end{bmatrix}, \quad \mathbf{v}_2^e = k_2^e \begin{bmatrix} x - \alpha_{21}^e \\ y - \alpha_{22}^e \end{bmatrix}, \quad \mathbf{v}_3^e = k_3^e \begin{bmatrix} x - \alpha_{31}^e \\ y - \alpha_{32}^e \end{bmatrix}.$$

Concerning its dual basis, we state classically N_j^e, $j = 1, 2, 3$, $N_j^e(\mathbf{u}_h) = \int_{f_j^e} \mathbf{u}_h \cdot \mathbf{n}_j^e \, dl$, with each functional N_j^e expressing the flux through one edge for $\mathbf{u}_h \in \mathbf{RT}^0(e)$; we have $N_j^e(\mathbf{v}_i^e) = \delta_{ij}$ after appropriate choice of $\alpha_{11}^e - \alpha_{32}^e$, $k_1^e - k_3^e$. The local interpolation operator is then given by

$$\pi_e(\mathbf{u}) = \sum_{i=1}^{3} N_i^e(\mathbf{u})\mathbf{v}_i^e \quad \forall\, \mathbf{u} \in (H^1(e))^2 \,. \tag{11}$$

We start from the Raviart–Thomas space $\mathbf{RT}_{-1}^0(\mathcal{T}_h)$ of on each element linear vector functions without any continuity requirements,

$$\mathbf{RT}_{-1}^0(\mathcal{T}_h) \equiv \{\mathbf{v} \in \mathbf{L}^2(\mathcal{S})\,;\; \mathbf{v}|_e \in \mathbf{RT}^0(e) \quad \forall e \in \mathcal{T}_h\}\,,$$

to define the "continuity assuring" space $\mathbf{RT}_0^0(\mathcal{T}_h)$ by

$$\mathbf{RT}_0^0(\mathcal{T}_h) \equiv \{\mathbf{v} \in \mathbf{RT}_{-1}^0(\mathcal{T}_h)\,;\; \textstyle\sum_{i\in I_f} \mathbf{v}|_{e_i} \cdot \mathbf{n}_{f,\partial e_i} = 0 \;\forall f \text{ such that}$$
$$|I_f| \geq 2\,,\, I_f = \{i \in J_h\,;\, f \subset \partial e_i\} = \mathbf{RT}_{-1}^0(\mathcal{T}_h) \cap \mathbf{H}(div, \mathcal{S})\,.$$

We set furthermore

$$\mathbf{RT}_{0,N}^0(\mathcal{T}_h) \equiv \{\mathbf{v} \in \mathbf{RT}_0^0(\mathcal{T}_h)\,;\; \mathbf{v}\cdot\mathbf{n} = 0 \text{ in } \Lambda_N\} = \mathbf{RT}_{-1}^0(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(div, \mathcal{S})$$

and

$$M_{-1}^0(\mathcal{T}_h) \;\equiv\; \{\phi \in L^2(\mathcal{S})\,;\, \phi|_e \in M^0(e) \quad \forall e \in \mathcal{T}_h\}\,,$$

where $M^0(e)$ is the space of scalar functions constant on a given element e. Looking for the basis, appropriate dual basis, and global interpolation operator for $\mathbf{RT}_0^0(\mathcal{T}_h)$, we have the following definitions and lemmas:

We set $\mathcal{N}_h = \{N_1, N_2, \ldots, N_{I_{\mathcal{N}_h}}\}$ as the dual basis of $\mathbf{RT}_0^0(\mathcal{T}_h)$, where for each border edge f, we have one functional N_f defined by $N_f(\mathbf{u}_h) = \int_f \mathbf{u}_h|_e \cdot \mathbf{n}_{\partial e}\, dl$, and for each inner edge f common to elements $e_1, e_2, \ldots, e_{I_f}$, we have $|I_f| - 1$ functionals given by

$$N_{f,j}(\mathbf{u}_h) = \frac{1}{|I_f|}\int_f \mathbf{u}_h|_{e_1} \cdot \mathbf{n}_{\partial e_1} dl - \frac{1}{|I_f|}\int_f \mathbf{u}_h|_{e_{j+1}} \cdot \mathbf{n}_{\partial e_{j+1}} dl, \quad j = 1, \ldots, |I_f| - 1\,.$$

Lemma 1. *For all $\mathbf{u}_h \in \mathbf{RT}_0^0(\mathcal{T}_h)$, from $N_j(\mathbf{u}_h) = 0 \;\forall\; j = 1, \ldots, I_{\mathcal{N}_h}$ follows that $\mathbf{u}_h = 0$.*

We set $\mathcal{V}_h = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{I_{\mathcal{N}_h}}\}$, where for each border edge f, we have one base function $\mathbf{v}_f$ defined by $\mathbf{v}_f = \mathbf{v}_f^e$ with $\mathbf{v}_f^e$ being the local base function appropriate to the element e and its edge f, and for each inner edge f common to elements $e_1, e_2, \ldots, e_{I_f}$, we have $|I_f| - 1$ base functions given by

$$\mathbf{v}_{f,i} = \sum_{k=1,\; k\neq i+1}^{|I_f|} \mathbf{v}_f^{e_k} - (I_f - 1)\mathbf{v}_f^{e_{i+1}} \quad , \quad i = 1, \ldots, |I_f| - 1\,.$$

Lemma 2. *For the bases $\mathcal{N}_h$ and $\mathcal{V}_h$, $N_j(\mathbf{v}_i) = \delta_{ij}$, $i, j = 1, \ldots, I_{\mathcal{N}_h}$ holds.*

We introduce first a space smoother than $\mathbf{H}(div, \mathcal{S})$, corresponding to the classical $(H^1(\mathcal{S}))^2$,

$$\begin{gathered}\mathbf{H}(grad, \mathcal{S}) = \{v \in \mathbf{L}^2(\mathcal{S})\,;\, \mathbf{v}|_{\alpha_\ell} \in (H^1(\alpha_\ell))^2 \quad \forall \ell \in L\,, \\ \textstyle\sum_{i \in I_f} \mathbf{v}|_{\alpha_i} \cdot \mathbf{n}_{f,\partial\alpha_i} = 0 \\ \forall f \text{ such that } |I_f| \geq 2\,,\, I_f = \{i \in L\,;\, f \subset \partial\alpha_i\}\,,\end{gathered}$$

in order to set the global interpolation operator

$$\pi_h(\mathbf{u}) = \sum_{i=1}^{I_{\mathcal{N}_h}} N_i(\mathbf{u})\mathbf{v}_i \quad \forall\, \mathbf{u} \in \mathbf{H}(grad, \mathcal{S})\,. \tag{12}$$

Lemma 3. *Concerning the local and global interpolation operators given by 11, 12 respectively, we have their equality on each element, i.e.*

$$\pi_h(\mathbf{u})|_e = \pi_e(\mathbf{u}|_e) \quad \forall\, e \in \mathcal{T}_h,\ \forall\, \mathbf{u} \in \mathbf{H}(grad, \mathcal{S})\,.$$

Lemma 4. *Even for the considered special function spaces and their finite dimensional subspaces, we have*

$$\begin{array}{ccc} \mathbf{H}(grad, \mathcal{S}) & \xrightarrow{\text{div}} & L^2(\mathcal{S}) \\ \big\downarrow \pi_h & & \big\downarrow P_h \\ \mathbf{RT}_0^0(\mathcal{T}_h) & \xrightarrow{\text{div}} & M_{-1}^0(\mathcal{T}_h) \end{array}\ , \tag{13}$$

i.e. the commutativity diagram property, where π_h is the global interpolation operator defined in (12), and P_h is the $L^2(\mathcal{S})$-orthogonal projection onto $M_{-1}^0(\mathcal{T}_h)$.

Definition 51 *As the lowest order Raviart–Thomas mixed approximation of the the problem (9), (10), we understand functions $\mathbf{u}_{0,h} \in \mathbf{RT}_{0,N}^0(\mathcal{T}_h)$ and $p_h \in M_{-1}^0(\mathcal{T}_h)$ satisfying*

$$\begin{gathered}(\mathbf{A}\mathbf{u}_{0,h}, \mathbf{v}_h)_{0,\mathcal{S}} - (\nabla \cdot \mathbf{v}_h, p_h)_{0,\mathcal{S}} = -\langle \mathbf{v}_h \cdot \mathbf{n}, p_D \rangle_{\Lambda_D} + (\nabla \cdot \mathbf{v}_h, z)_{0,\mathcal{S}} - \\ -\langle \mathbf{v}_h \cdot \mathbf{n}, z \rangle_{\partial\mathcal{S}} - (\mathbf{A}\tilde{\mathbf{u}}, \mathbf{v}_h)_{0,\mathcal{S}} \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}_{0,N}^0(\mathcal{T}_h)\,,\end{gathered} \tag{14}$$

$$-(\nabla \cdot \mathbf{u}_{0,h}, \phi_h)_{0,\mathcal{S}} = -(q, \phi_h)_{0,\mathcal{S}} + (\nabla \cdot \tilde{\mathbf{u}}, \phi_h)_{0,\mathcal{S}} \qquad \forall \phi_h \in M_{-1}^0(\mathcal{T}_h)\,. \tag{15}$$

Theorem 51 *The problem (14), (15) has a unique solution.*

6 Error Estimates and Hybridization of the Mixed Method

If the solution $(\mathbf{u}_0, p)$ of (9), (10) is such that $(\mathbf{u}_0, p) \in \mathbf{H}(grad, \mathcal{S}) \times H^1(\mathcal{S})$ and $\nabla \cdot \mathbf{u}_0 \in H^1(\mathcal{S})$ and if $(\mathbf{u}_{0,h}, p_h)$ is the solution of (14), (15), then

$$\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{\mathbf{H}(div,\mathcal{S})} + \|p - p_h\|_{0,\mathcal{S}} \leq Ch(|p|_{1,\mathcal{S}} + |\mathbf{u}_0|_{1,\mathcal{S}} + |\nabla \cdot \mathbf{u}_0|_{1,\mathcal{S}})\,,$$

where the constant C does not depend on h and $|\varphi|_{1,\mathcal{S}} = \|\nabla\varphi\|_{0,\mathcal{S}}$, $|\mathbf{u}|^2_{1,\mathcal{S}} = \sum_{i=1}^{2} |\mathbf{u}_i|^2_{1,\mathcal{S}}$ (see [4], Theorem 13.2).

Intending to hybridize the mixed approximation, we define two sets of edges,

$$\Lambda_h = \cup_{e\in\mathcal{T}_h}\partial e \quad , \quad \Lambda_{h,D} = \cup_{e\in\mathcal{T}_h}\partial e - \Lambda_D \, .$$

If $f \in \Lambda_h$, we define first the space $M^0(f)$ of functions constant on this edge and finally

$$M^0_{-1}(\Lambda_{h,D}) \equiv \{\mu_h : \Lambda_h \to R;\ \mu_h|_f \in M^0(f) \quad \forall f \in \Lambda_h \, , \\ \mu_h|_f = 0 \quad \forall f \in \Lambda_D\} \, .$$

It now follows immediately that if $\mathbf{v}_h \in \mathbf{RT}^0_{-1}(\mathcal{T}_h)$, then $\mathbf{v}_h \in \mathbf{RT}^0_{0,N}(\mathcal{T}_h)$ if and only if

$$\sum_{e\in\mathcal{T}_h} \langle \mathbf{v}_h \cdot \mathbf{n}, \lambda_h\rangle_{\partial e\,\cap\Lambda_{h,D}} = 0 \quad \forall\lambda_h \in M^0_{-1}(\Lambda_{h,D})\, ,$$

which allows us to state the hybrid version of the lowest order Raviart–Thomas mixed method:

Definition 61 *As the lowest order Raviart–Thomas mixed-hybrid approximation of the the problem (9), (10), we understand functions* $\mathbf{u}_{0,h} \in \mathbf{RT}^0_{-1}(\mathcal{T}_h)$, $p_h \in M^0_{-1}(\mathcal{T}_h)$ *and* $\lambda_h \in M^0_{-1}(\Lambda_{h,D})$ *satisfying*

$$\sum_{e\in\mathcal{T}_h} \{(\mathbf{A}\mathbf{u}_{0,h}, \mathbf{v}_h)_{0,e} - (\nabla\cdot\mathbf{v}_h, p_h)_{0,e} + \langle \mathbf{v}_h\cdot\mathbf{n}, \lambda_h\rangle_{\partial e\,\cap\Lambda_{h,D}}\} = \\ = \sum_{e\in\mathcal{T}_h} \{-\langle \mathbf{v}_h\cdot\mathbf{n}, p_D\rangle_{\partial e\,\cap\Lambda_D} + (\nabla\cdot\mathbf{v}_h, z)_{0,e} - \langle \mathbf{v}_h\cdot\mathbf{n}, z\rangle_{\partial e} - (\mathbf{A}\tilde{\mathbf{u}}, \mathbf{v}_h)_{0,e}\} \\ \forall\, \mathbf{v}_h \in \mathbf{RT}^0_{-1}(\mathcal{T}_h)\, , \tag{16}$$

$$-\sum_{e\in\mathcal{T}_h} (\nabla\cdot\mathbf{u}_{0,h}, \phi_h)_{0,e} = -\sum_{e\in\mathcal{T}_h} \{(q, \phi_h)_{0,e} - (\nabla\cdot\tilde{\mathbf{u}}, \phi_h)_{0,e}\} \\ \forall\phi_h \in M^0_{-1}(\mathcal{T}_h)\, , \tag{17}$$

$$\sum_{e\in\mathcal{T}_h} \langle \mathbf{u}_{0,h}\cdot\mathbf{n}, \mu_h\rangle_{\partial e\,\cap\Lambda_{h,D}} = \sum_{e\in\mathcal{T}_h} \{\langle u_N, \mu_h\rangle_{\partial e\cap\Lambda_N} - \langle \tilde{\mathbf{u}}\cdot\mathbf{n}, \mu_h\rangle_{\partial e\,\cap\Lambda_{h,D}}\} \\ \forall\mu_h \in M^0_{-1}(\Lambda_{h,D})\, . \tag{18}$$

Due to the previously mentioned, the triple $\mathbf{u}_{0,h}$, p_h, λ_h surely exist and is unique, $\mathbf{u}_{0,h}$ and p_h are moreover at the same time the unique solutions of (14), (15); moreover, the multiplier λ_h is an approximation of the trace of p on all edges from $\Lambda_{h,D}$. Consequently, all error estimates are valid also for the mixed-hybrid solution triple $\mathbf{u}_{0,h}, p_h, \lambda_h$. Thus, we have the following theorem:

Theorem 61 *The problem (16) – (18) has a unique solution.*

7 Model Problem

We consider a simple model problem with the system $\mathcal{S}$ viewed in Figure 1,

$$\mathcal{S} = \overline{\alpha_1} \bigcup \overline{\alpha_2} \bigcup \overline{\alpha_3} \bigcup \overline{\alpha_4} \setminus \partial\mathcal{S} \,,$$

$$\mathbf{u} = -\left(\nabla p + \nabla z\right) \quad \text{in} \quad \mathcal{S} \,,$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \mathcal{S} \,,$$

$$\begin{aligned} p = 0 \quad \text{in} \quad \Lambda_1 \quad , \quad p = 0 \quad \text{in} \quad \Lambda_2 \\ \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{in} \quad \Lambda_3 \quad , \quad \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{in} \quad \Lambda_4 \\ p = \sin\left(\frac{\pi x_1}{2X}\right) \sinh\left(\frac{\pi(A+B)}{2X}\right) + S\cdot A \quad \text{in} \quad \Lambda_5 \quad , \quad p = S \cdot y_1 \quad \text{in} \quad \Lambda_6 \\ \\ p = 0 \quad \text{in} \quad \Lambda_7 \quad , \quad p = 0 \quad \text{in} \quad \Lambda_8 \\ \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{in} \quad \Lambda_9 \quad , \quad \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{in} \quad \Lambda_{10} \\ p = \sin\left(\frac{\pi x_4}{2X}\right) \sinh\left(\frac{\pi(B+B)}{2X}\right) \quad \text{in} \quad \Lambda_{11} \quad , \quad p = 0 \quad \text{in} \quad \Lambda_{12} \,. \end{aligned} \tag{19}$$

The exact solutions in α_1 can be easily found as

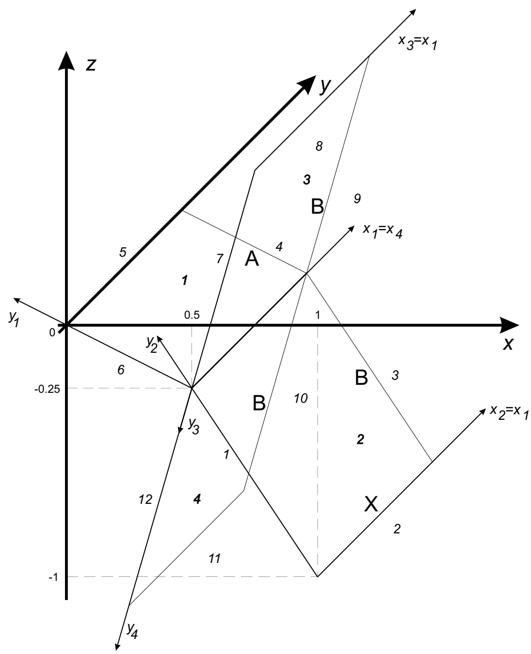

Fig. 1. Considered Domain for the Model Problem

$$\begin{aligned}
p_{\alpha_1} &= \sin\Big(\frac{\pi x_1}{2X}\Big)\sinh\Big(\frac{\pi(y_1+B)}{2X}\Big) + S\cdot y_1\,,\\
\mathbf{u}_{\alpha_1} &= \Big(-\frac{\pi}{2X}\cos\Big(\frac{\pi x_1}{2X}\Big)\sinh\Big(\frac{\pi(y_1+B)}{2X}\Big),\\
&\quad -\frac{\pi}{2X}\sin\Big(\frac{\pi x_1}{2X}\Big)\cosh\Big(\frac{\pi(y_1+B)}{2X}\Big) - S - \nabla z^y_{\alpha_1}\Big)\,,
\end{aligned}$$

where $\nabla z_{\alpha_1} = (0, \nabla z^y_{\alpha_1})$, $S + \nabla z^y_{\alpha_1} = \nabla z^y_{\alpha_2}$.

The following table gives pressure, velocity, and pressure trace approximation errors in the first fracture α_1. There is the expected $O(h)$ convergence in pressure and velocity, but only $O(h^{\frac{1}{2}})$ in pressure trace in $\|\cdot\|_{0,\Lambda_{h,D}}$ norm. All the computations were done in double precision on a personal computer, the resulting symmetric indefinite systems of linear equations were solved by the solver GI8 of the Institute of Computer Science, Academy of Sciences of the Czech Republic, see [2]. This is based on the sequential elimination onto a system with Schur's complement and subsequent solution of this system by the specially preconditioned conjugate gradients method. The solver accuracy was set to 10^{-8}.

Table 1. Pressure, Velocity, and Pressure Trace Errors in α_1 for the Model Problem

N	triangles	$\|p - p_h\|_{0,\mathcal{S}}$	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(div,\mathcal{T}_h)}$	$\|\lambda - \lambda_h\|_{0,\Lambda_{h,D}}$
2	8×4	0.4445	1.2247	1.4973
4	32×4	0.2212	0.6263	1.0562
8	128×4	0.1102	0.3150	0.7509
16	512×4	0.0550	0.1577	0.5332
32	2048×4	0.0275	0.0789	0.3779
64	8192×4	0.0138	0.0394	0.2676
128	32768×4	0.0069	0.0197	0.1893
256	131072×4	0.0034	0.0099	0.1339

8 Example of real-world problem

Results of model problem presented in previous section prooved correctness of mathematical model as well as correctness of its numerical implementation. Therefore we have tried to use the model for solving a real-world problem.

The problem was based on measurements in the boreholes PTP–3 and PTP–4 situated in Krun Hory mountains, Czech Republic. Results of these measurements have given us data for creating computer approximation of fractured environment in rock massif and, consequently, for its discretizing to FEM/FVM mesh. Then, boundary contition was set and calculation has been started.

Example of results of such calculation is shown at figure 2. Mesh presented of this figure covers volume 5x10x10 meter. It consists of approx. 200 fractures and 3000 triangle elements.

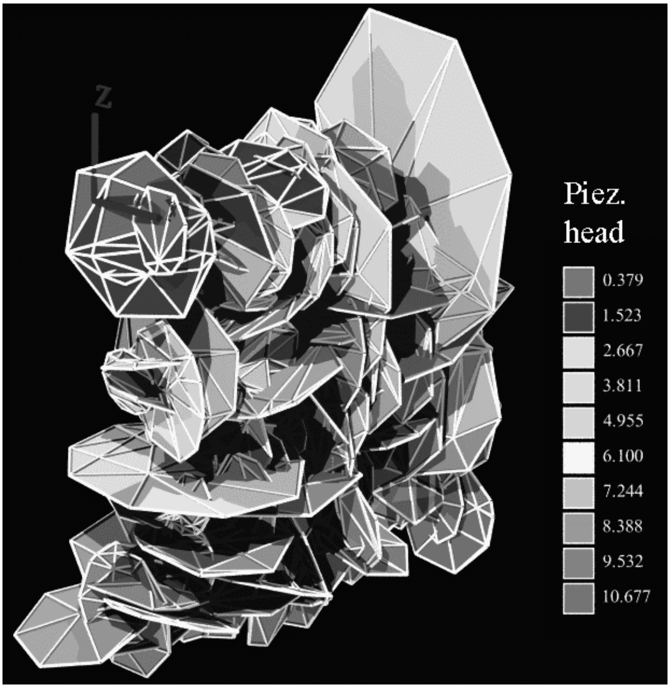

Fig. 2. Example of real-world problem calculation

9 Conclusion

Mathematical model of groundwater flow was described. This model is based on assumption, that flow in particular fracture can be approximated by Darcy's law. Mixed-hybride approximation of solution of problem was introduced and error estimation for such approximation was derived. Practical tests prooved correctness of presented approach.

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