Domain Decomposition Algorithm for Solving Contact of Elastic Bodies

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Abstract. A nonoverlapping domain decomposition algorithm of Neumann-Neumann type for solving variational inequalities arising from the elliptic boundary value problems in two dimensions with unilateral boundary condition is presented. First, the linear auxiliary problem, where the inequality condition is replaced by the equality condition, is solved. In the second step, the solution of the auxiliary problem is used in a successive approximations method. In these solvers, a preconditioned conjugate gradient method with Neumann-Neumann preconditioner is used for solving the interface problems, while local problems within each subdomain are solved by direct solvers. A convergence of the iterative method and results of computational test are reported.

1 Equilibrium of a system of bodies in contact

We consider a system of elastic bodies decomposed into subdomains each of which occupies, in reference configuration, a domain Ω_i^M in $\mathbb{R}^2$, $i = 1, \ldots, I_M$, $M = 1, \ldots, \mathcal{J}$, with sufficiently smooth boundary $\partial\Omega_i^M$. Suppose that boundary $\bigcup_{M=1}^{\mathcal{J}} \partial\Omega^M$ consists of four disjoint parts Γ_u, Γ_τ, Γ_c and Γ_o and that the displacements $u : \Gamma_u \to \mathbb{R}^2$ and forces $P : \Gamma_\tau \to \mathbb{R}^2$ are given. The part Γ_c denote the part of boundary that may get into unilateral contact with some other subdomain and the part Γ_o denote the part of boundary on that is prescribed the condition of the bilateral contact.

We shall look for the displacements that satisfy the conditions of equilibrium in the set $K = \{v \in V | \, v_n^k + v_n^l \leq 0 \text{ on } \Gamma_c\}$ of all kinematically admissible displacements $v \in V$, $V = \{v \in \mathcal{H}^1(\Omega) | \, v = u_0 \text{ on } \Gamma_u, v_n = 0 \text{ on } \Gamma_o\}$, $\mathcal{H}^1(\Omega) = [H^1(\Omega_1^1)]^2 \times \ldots \times [H^1(\Omega_{I_\mathcal{J}}^\mathcal{J})]^2$ is standard Sobolev space. The displacement $u \in K$ of the system of bodies in equilibrium then minimizes the energy functional $\mathcal{L}(v) = \frac{1}{2}a(v, v) - L(v)$:

$$\mathcal{L}(u) \leq \mathcal{L}(v) \text{ for any } v \in K. \tag{1}$$

Conditions that guarantee existence and uniqueness of the solution may be expressed in terms of coercivity of $\mathcal{L}$ and may be found, for example, in [1].

We define $\Gamma_i^M = \partial\Omega_i^M \setminus \partial\Omega^M$ and the interface $\Gamma = \bigcup_{M=1}^{\mathcal{J}} \bigcup_{i=1}^{I_M} \Gamma_i^M$. Let us introduce

$$T^M = \{j \in \{1, \ldots, I_M\} : \quad \Gamma_c \cap \partial\Omega_j^M = \emptyset\}, \qquad M = 1, \ldots, \mathcal{J}. \tag{2}$$

P.M.A. Sloot et al. (Eds.): ICCS 2002, LNCS 2331, pp. 820–829, 2002.

The number of a separate subset Γ_c is P_c, i.e. $\Gamma_c = \bigcup_{j=1}^{P_c} \Gamma_{cj}$. Further, we denote

$$\Omega^{*j} = \{x \in \bigcup_{M=1}^{\mathcal{J}} \bigcup_{i=1}^{I_M} \Omega_i^M : \quad \partial\Omega_i^M \cap \Gamma_{cj} \neq \emptyset\}, \qquad j = 1, \ldots, P_c, \tag{3}$$

$$\vartheta^j = \{[i, M] : \ \partial\Omega_i^M \cap \Gamma_{cj} \neq \emptyset\}, \qquad j = 1, \ldots, P_c, \tag{4}$$

i.e. $\Omega^{*j} = \bigcup_{[i,M]\in\vartheta^j} \Omega_i^M$, $j = 1, \ldots, P_c$. We suppose that $\Gamma \cap \Gamma_c = \emptyset$ then $V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma$ for trace operator $\gamma : [H^1(\Omega_i^M)]^2 \to [L^2(\partial\Omega_i^M)]^2$. We suppose that $\gamma^{-1} : V_\Gamma \to V$ is arbitrary linear inverse mapping for which

$$\sum_{M=1}^{\mathcal{J}} (\gamma^{-1}\bar{v}^M)_n = 0 \quad \forall \bar{v} \in V_\Gamma \quad \text{on } \Gamma_c. \tag{5}$$

After denoting restrictions $\bar{R}_i^M : V_\Gamma \to \Gamma_i^M$, $L_i^M : L^M \to \Omega_i^M$, $a_i^M(.,.) : a^M(.,.) \to \Omega_i^M$, $V(\Omega_i^M) : V(\Omega^M) \to \Omega_i^M$ and introduction

$$V^0(\Omega_i^M) = \{v \in V \,|\, v = 0 \text{ on } (\bigcup_{M=1}^{\mathcal{J}} \Omega^M) \setminus \Omega_i^M\},$$

we can formulate the theorem 1.

Theorem 1. *A function $u \in K$ is the solution of the global problem (1) if and only if the function u satisfies:*

1.
$$\sum_{M=1}^{\mathcal{J}} \sum_{i=1}^{I_M} \left(a_i^M(u_i^M(\bar{u}), \gamma^{-1}\bar{w}) - L_i^M(\gamma^{-1}\bar{w})\right) = 0 \quad \forall \bar{w} \in V_\Gamma, \ \bar{u} \in V_\Gamma, \tag{6}$$

for the trace $\bar{u} = \gamma u|_\Gamma$ on the interface Γ.

2. *Its rescriction $u_i^M(\bar{u}) \equiv u|_{\Omega_i^M}$ satisfies following conditions:*

a)
$$\begin{gathered} a_i^M(u_i^M(\bar{u}), \phi_i^M) = L_i^M(\phi_i^M) \quad \forall \phi_i^M \in V^0(\Omega_i^M), \\ u_i^M(\bar{u}) \in V(\Omega_i^M), \ \gamma u_i^M(\bar{u})|_{\Gamma_i^M} = \bar{R}_i^M \bar{u}, \end{gathered} \tag{7}$$

for $i \in T^M$, $M = 1, \ldots, \mathcal{J}$,

b)
$$\begin{gathered} \sum_{[i,M]\in\vartheta^j} a_i^M(u_i^M(\bar{u}), \phi_i^M) \geq \sum_{[i,M]\in\vartheta^j} L_i^M(\phi_i^M) \\ \forall \phi \equiv (\phi_i^M, \ [i, M] \in \vartheta^j), \ \phi_i^M \in V^0(\Omega_i^M) \end{gathered} \tag{8}$$

such that

$$u + \phi \in K;$$
$$\gamma u_i^M(\bar{u})|_{\Gamma_i^M} = \bar{R}_i^M \bar{u} \ \ \textit{for} \ \ [i, M] \in \vartheta^j,$$

for $j = 1, \ldots, P_c$.

2 The Schur complements

We now want to write the interface problem (6) in operator form. For this purpose, we first introduce additional notation. We introduce the local trace spaces

$$V_i^M = \{\gamma v|_{\Gamma_i^M} \mid v \in K\} = \{\gamma v|_{\Gamma_i^M} \mid v \in V\} \tag{9}$$

and the extension $Tr_{iM}^{-1} : V_i^M \to V(\Omega_i^M)$ defined by

$$\begin{aligned} &\gamma(Tr_{iM}^{-1}\bar{u}_i^M)|_{\Gamma_i^M} = \bar{u}_i^M, \qquad i = 1, \ldots, I_M,\ M = 1, \ldots, \mathcal{J}, \\ &a_i^M(Tr_{iM}^{-1}\bar{u}_i^M, v_i^M) = 0 \qquad \forall v_i^M \in V^0(\Omega_i^M),\ Tr_{iM}^{-1}\bar{u}_i^M \in V(\Omega_i^M), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \in T^M,\ M = 1, \ldots, \mathcal{J}. \end{aligned} \tag{10}$$

For subdomains $\Omega^{*j},\ j = 1, \ldots, P_c$, we completed definition Tr_{iM}^{-1} with boundary condition

$$\sum_{[i,M]\in\vartheta^j} (Tr_{iM}^{-1}\bar{u}_i^M)_n = 0 \quad \text{on } \Gamma_{cj},\ \text{for } j = 1, \ldots, P_c,$$

i.e.

$$\begin{aligned} &\sum_{[i,M]\in\vartheta^j} a_i^M(Tr_{iM}^{-1}\bar{u}_i^M, v_i^M) = 0 \quad \forall (v_i^M,\ [i,M] \in \vartheta^j) :\ v_i^M \in V^0(\Omega_i^M), \\ &\qquad\qquad \text{so that} \sum_{[i,M]\in\vartheta^j} (v_i^M)_n = 0 \text{ on } \Gamma_{cj}, \quad j = 1, \ldots, P_c. \end{aligned} \tag{11}$$

Definition 1. *The local Schur complement, for $i \in T^M$, $M = 1, \ldots, \mathcal{J}$, is operator $S_i^M : V_i^M \to (V_i^M)^*$ defined by*

$$\langle S_i^M \bar{u}_i^M, \bar{v}_i^M \rangle = a_i^M(Tr_{iM}^{-1}\bar{u}_i^M, Tr_{iM}^{-1}\bar{v}_i^M) \quad \forall \bar{u}_i^M, \bar{v}_i^M \in V_i^M. \tag{12}$$

In matrix form, we have

$$S_i^M \bar{U}_i^M = (\bar{A}_{iM} - B_{iM}^T \overset{\circ}{A}{}_{iM}{}^{-1} B_{iM})\, \bar{U}_i^M, \tag{13}$$

where we decompose the degrees of freedom U_i of u_i into internal degrees of freedom $\overset{\circ}{U}{}_i^M$ and interface degrees of freedom $\bar{U}_i^M$:

$$U_i^M = \left[\overset{\circ}{U}{}_i^M,\ \bar{U}_i^M\right]^T.$$

With this decomposition, the matrix representation of $a_i^M(.,.)$ on $H^1(\Omega_i^M)$ take the form

$$A_{iM} = \begin{bmatrix} \overset{\circ}{A}{}_{iM} & B_{iM} \\ B_{iM}^T & \bar{A}_{iM} \end{bmatrix}. \tag{14}$$

Definition 2. *The combined local Schur complement, for subdomains Ω^{*j}, $j = 1, \ldots, P_c$, is operator*

$$S_{*j}: \ (V_i^M, [i,M] \in \vartheta^j) \to (V_i^M, [i,M] \in \vartheta^j)^*, \quad j = 1, \ldots, P_c,$$

defined by

$$\langle S_{*j}(\bar{u}_i^M, [i,M] \in \vartheta^j), (\bar{v}_i^M, [i,M] \in \vartheta^j)\rangle = \sum_{[i,M]\in\vartheta^j} a_i^M(u_i^M(\bar{u}_i^M), Tr_{iM}^{-1}\bar{v}_i^M)$$
$$\forall(\bar{v}_i^M, \ [i,M] \in \vartheta^j) \in (V_i^M, [i,M] \in \vartheta^j), \tag{15}$$

where $u_i^M(\bar{u}_i^M)$ is the solution of the problem (8) and $\bar{R}_i^M \bar{u} \equiv \bar{u}_i^M, \ [i,M] \in \vartheta^j$.

Lemma 1. *The condition (6) for the function $\bar{u}$ on interface Γ is equivalent to the condition (16):*

$$\sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} \langle S_i^M \bar{u}_i^M, \bar{w}_i^M\rangle + \sum_{j=1}^{P_c} \langle S_{*j}(\bar{u}_i^M, [i,M] \in \vartheta^j), (\bar{w}_i^M, [i,M] \in \vartheta^j)\rangle =$$
$$= \sum_{M=1}^{\mathcal{J}} \sum_{i\in I_M} L_i^M(Tr_{iM}^{-1}\bar{w}_i^M), \quad \forall \bar{w} \in V_\Gamma, \ \textit{where } \bar{w}_i^M = \bar{R}_i^M \bar{w}, \ \bar{u}_i^M = \bar{R}_i^M \bar{u}, \tag{16}$$

by using the local Schur complements.

We rewrite the condition (16) in the form

$$S_0\bar{U} + S_{KON}\bar{U} = F, \tag{17}$$

where

$$S_0 = \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} (\bar{R}_i^M)^T S_i^M \bar{R}_i^M, \quad S_{KON} = \sum_{j=1}^{P_c} \bar{R}_{*j}^T S_{*j} \bar{R}_{*j},$$

$$F = \sum_{M=1}^{\mathcal{J}} \sum_{i\in I_M} (\bar{R}_i^M)^T (Tr_{iM}^{-1})^T L_i^M$$

and

$$\bar{R}_{*j}\bar{u} = (\bar{R}_i^M \bar{u}, \ [i,M] \in \vartheta^j)^T, \quad \bar{u} \in V_\Gamma, \quad \forall j = 1, \ldots, P_c.$$

By reason that operator S_{KON} is nonlinear, we solve the equation (17) successive aproximations method. We choose the solution of the auxiliary linear problem as an initial aproximation $\bar{U}^0$. In the auxiliary problem we replace the set K by

$$K^0 = \{v \in V | \sum_{[i,M]\in\vartheta^j} (v_i^M)_n = 0 \quad \text{on } \Gamma_{cj}\}$$

and we obtain

$$u_0 = \arg\min_{v\in K^0} \mathcal{L}(v),$$
$$\bar{U}^0 = \gamma u^0|_\Gamma.$$

Now we come back to the equation (17) and we compute $\bar{U}^k$as the solution of the linear problem

$$S_0\bar{U}^k = F - S_{KON}\bar{U}^{k-1}, \qquad k = 1, 2, \ldots. \tag{18}$$

3 The linearized problem

We solve the variational equation

$$u^0 \in K^0, \quad D\mathcal{L}(u^0, v) = 0 \quad \forall v \in K^0. \tag{19}$$

For problem (19) we can describe the analogy of theorem 1 with one different in case 2b) where inequality is replaced by equality on K^0. For solution of this variational equality we define combined local Schur complement S^0_{*j}, $j = 1, \ldots, P_c$ same as in definition 2.

Definition 3. *We define a global Schur complement:*

$$S = \sum_{j=1}^{P_c} \bar{R}^T_{*j} S^0_{*j} \bar{R}_{*j} + \sum_{M=1}^{\mathcal{J}} \sum_{i \in T^M} (\bar{R}^M_i)^T S^M_i \bar{R}^M_i \tag{20}$$

and the condition (6) on the interface Γ has form

$$S\bar{U} = F \tag{21}$$

in dual space $(V_\Gamma)^*$.

The equation (21) we solve by a conjugate gradient method with Neumann-Neumann preconditioner. This method does not require the explicit construction of the local Schur complement matrix S^M_i but does require an efficient preconditioner $\mathcal{M}^{-1}$. Its inverse $(S^M_i)^{-1}$, resp. $(S^0_{*j})^{-1}$ simply consists in associating to the generalized derivative $g \in (V^M_i)^*$ the trace $\gamma\phi^M_i$ on Γ^M_i of the solution ϕ^M_i of the corresponding Neumann problem.

Definition 4. *We define an injection*

$$\begin{aligned} D^M_i : &\quad V^M_i \to V_\Gamma, \quad i \in T^M, \quad M = 1, \ldots, \mathcal{J}, \\ D_{*j} : &\quad (V^M_i, [i, M] \in \vartheta^j) \to V_\Gamma, \; D_{*j} = (D^M_i, [i, M] \in \vartheta^j), \quad j = 1, \ldots, P_c, \end{aligned} \tag{22}$$

such that on each interface degree of freedom is

$$D^M_i \bar{v}(P_l) = \bar{v}(P_k) \frac{\varrho^M_i}{\varrho_T}, \quad i = 1, 2, \ldots I_M, \; M = 1, \ldots, \mathcal{J}, \tag{23}$$

if the lth degree of freedom of V_Γ corresponds to the kth degree of freedom of V^M_i and

$$D^M_i \bar{v}(P_l) = 0, \quad \textit{if not.} \tag{24}$$

Here ϱ^M_i is a local measure of the stiffness of subdomain Ω^M_i (for example an average Young modulus on Ω^M_i) and

$$\varrho_T = \sum_{P_l \in \bar{\Omega}^M_j} \varrho^M_j \tag{25}$$

is the sum of ϱ^M_j on all subdomains na Ω^M_j containing P_l.

The Neumann-Neumann precoditioner supposes that the solution of each local Neumann problem is uniquely defined, whereas rigid body motions are possible. This weakness can be fixed by replacing $(S_i^M)^{-1}$, resp. $(S_{*j}^0)^{-1}$ by a regularized inverse $(\tilde{S}_i^M)^{-1}$, resp. $(\tilde{S}_{*j}^0)^{-1}$. We introduce on each subdomain Ω_i^M, resp. Ω^{*j} a small local coarse space Z_i^M, resp. Z^{*j} containing all rigid body motion.

The general trick to upgrade the original preconditioner then consists in adding to the initial local contribution ϕ_i^M, resp. ϕ_j a "bad" $z_i^M \in Z_i^M$, resp. $z_j \in Z^{*j}$ which is chosen in order to get the smallest difference $(\mathcal{M}^{-1} - S^{-1})$.

We suppose that L satisfies the invariance property

$$\langle L, D_i^M \gamma z_i^M \rangle = 0 \qquad \forall z_i^M \in Z_i^M, \quad i \in T^M, \quad M = 1, \ldots, \mathcal{J}, \tag{26}$$

$$\langle L, D_{*j} \gamma z_j \rangle \equiv \sum_{[i,M] \in \vartheta^j} \langle L, D_i^M \gamma z_i^M \rangle = 0 \qquad \forall z_j \in Z^{*j}, \quad j = 1, \ldots, P_c. \tag{27}$$

We introduce a closed orthogonal complement space $Q(\Omega_i^M)$ of Z_i^M in $V(\Omega_i^M)$ and a closed orthogonal complement space $Q(\Omega^{*j})$ of Z^{*j} in $\hat{V}_j$ where

$$\hat{V}_j = \{(v_i^M, [i,M] \in \vartheta^j) | v_i^M \in V(\Omega_i^M), \sum_{[i,M] \in \vartheta^j} (v_i^M)_n = 0 \text{ on } \Gamma_{cj}\}.$$

Let then $\phi_i^{0M} \in Q(\Omega_i^M)$ be the particular solution of the variational problem defined by

$$a_i^M(\phi_i^{0M}, v_i^M) = \langle L, D_i^M(\gamma v_i^M)|_{\Gamma_i^M} \rangle \qquad \forall v_i^M \in V(\Omega_i^M) \tag{28}$$

and $\phi_{*j}^0 = (\phi_i^{0M}, [i,M] \in \vartheta^j) \in Q(\Omega^{*j})$ be the particular solution of the variational problem defined by

$$\sum_{[i,M] \in \vartheta^j} a_i^M(\phi_i^{0M}, v_i^M) = \sum_{[i,M] \in \vartheta^j} \langle L, D_i^M(\gamma v_i^M)|_{\Gamma_i^M} \rangle \qquad \forall v_j \in \hat{V}_j. \tag{29}$$

Equations (28), (29) are well posed varitional problems set on $Q(\Omega_i^M)$, $Q(\Omega^{*j})$.

Definition 5. *We define Neumann-Neumann preconditioner $\mathcal{M}^{-1}(z^0)$ by*

$$\mathcal{M}^{-1}(z^0)L = \sum_{M=1}^{\mathcal{J}} \sum_{i=1}^{I_M} D_i^M \gamma(\phi_i^{0M} + z_i^{0M})|_{\Gamma_i^M}, \tag{30}$$

with the solution z_i^{0M} of the minimization problem

$$z^0 = \arg\min_{z \in \Pi Z} \langle \underbrace{S(\mathcal{M}^{-1}(z) - S^{-1})L, (\mathcal{M}^{-1}(z) - S^{-1})L}_{J(z)} \rangle, \tag{31}$$

$$\Pi Z \equiv (\bigotimes_{i \in T^M, M=1,\ldots,\mathcal{J}} (Z_i^M)) \times (\bigotimes_{j=1,\ldots,P_c} (Z^{*j})).$$

4 Successive approximations method

Now we solve, by the successive approximations method, the equation (18). We must effectively compute the solution $\bar{U}^k$ of the linear problem

$$S_0 \bar{U}^k = b^k, \tag{32}$$

with

$$S_0 = \sum_{M=1}^{\mathcal{J}} \sum_{i \in T^M} (\bar{R}_i^M)^T S_i^M \bar{R}_i^M, \qquad b^k = F - S_{KON} \bar{U}^{k-1},$$

$$F = \sum_{M=1}^{\mathcal{J}} \sum_{i \in I_M} (\bar{R}_i^M)^T (Tr_{iM}^{-1})^T L_i^M, \qquad S_{KON} = \sum_{j=1}^{P_c} \bar{R}_{*j}^T S_{*j} \bar{R}_{*j}.$$

The equation (32) we solve by a preconditioned conjugate gradient method.

Definition 6. *We define an injection*

$$D_i^M : \quad V_i^M \to V_\Gamma, \quad i \in T^M, \quad M = 1, \ldots, \mathcal{J}, \tag{33}$$

such that on each interface degree of freedom is

$$D_i^M \bar{v}(P_l) = \bar{v}(P_k), \quad \text{if } P_k \in \Gamma_i^M \subset \partial\Omega^{*j} \text{ for any } j \in \{1, \ldots, P_c\}, \tag{34}$$

$$D_i^M \bar{v}(P_l) = \bar{v}(P_k) \frac{\varrho_i^M}{\varrho_T}, \quad \text{for } P_k \in \Gamma_i^M \not\subset \partial\Omega^{*j} \quad \forall j = 1, \ldots, P_c, \tag{35}$$

if the lth degree of freedom of V_Γ corresponds to the kth degree of freedom of V_i^M and

$$D_i^M \bar{v}(P_l) = 0, \quad \text{if not.} \tag{36}$$

Let $\phi_i^{0M} \in Q(\Omega_i^M)$ be the particular solution of the variational problem defined by (28). Similarly to the linearized problem we define a preconditioner.

Definition 7. *We define Neumann-Neumann preconditioner $\mathcal{M}_o^{-1}$ by*

$$\mathcal{M}_o^{-1}(z^0) L = \sum_{M=1}^{\mathcal{J}} \sum_{i \in T^M} D_i^M \gamma(\phi_i^{0M} + z_i^{0M})|_{\Gamma_i^M}, \tag{37}$$

with the solution z_i^{0M} of the minimization problem

$$z^0 = \arg \min_{z \in \Pi_o Z} \langle S_0(\mathcal{M}_o^{-1}(z) - S_0^{-1})L, (\mathcal{M}_o^{-1}(z) - S_0^{-1})L \rangle, \tag{38}$$

$$\Pi_o Z \equiv \bigotimes_{i \in T^M, M=1,\ldots,\mathcal{J}} (Z_i^M).$$

We introduce the coarse trace space

$$V_{oH} = \sum_{M=1}^{\mathcal{J}} \sum_{i \in T^M} D_i^M \gamma Z_i^M, \tag{39}$$

a set $V_{oH}^{\perp} \subset (V_\Gamma)^*$ given by

$$L \in V_{oH}^{\perp} \;\Leftrightarrow\; \langle L, z \rangle = 0 \qquad \forall z \in V_{oH}.$$

A convergence theorem requires to introduce some definitions. Let Θ be an ortogonal complement of V_{oH} in V_Γ. We introduce seminorms

$$|\bar{R}_{*j}\bar{v}|_{a_{*j}} = \sqrt{\sum_{[i,M]\in\vartheta^j} a_i^M (Tr_{iM}^{-1}\bar{R}_i^M\bar{v}, Tr_{iM}^{-1}\bar{R}_i^M\bar{v})}, \qquad j = 1, \ldots, P_c.$$

Lemma 2. *The expression*

$$\|\bar{u}\|_Q^2 = \langle S_0\bar{u}, \bar{u}\rangle$$

is a norm on Θ where

$$Q = \bigotimes_{i,M:\; i\in T^M;\, M=1,\ldots,\mathcal{J}} Q(\Omega_i^M).$$

Definition 8. *Let $\mathcal{T} : \Theta \to \Theta$ be a mapping defined by*

$$\langle S_0(\mathcal{T}\bar{y}), \bar{v}\rangle = \langle F - S_{KON}(\bar{y}), \bar{v}\rangle \qquad \forall \bar{v} \in \Theta. \tag{40}$$

Theorem 2. *Assume that there exists a constant $\lambda < \frac{1}{\sqrt{2P_c}}$ such that the following condition hold:*

$$|\bar{R}_{*j}\bar{u}|_{a_{*j}} \le \lambda\|\bar{u}\|_Q, \quad \forall \bar{u} \in \Theta, \quad \forall j \in \{1, \ldots, P_c\}. \tag{41}$$

Then the mapping $\mathcal{T}$ is the contraction on Θ. If $\bar{U}^0 \in \Theta$ then the sequence of the iterations $\bar{U}^k$, computed by (32), are convergent and the limit is a fixed point $\bar{U}$ of the mapping $\mathcal{T}$. The following error estimate holds

$$\|\bar{U}^k - \bar{U}\|_Q \le \frac{(2\lambda^2 P_c)^k}{1 - 2\lambda^2 P_c}\|\bar{U}^0 - \mathcal{T}\bar{U}^0\|_Q.$$

5 Numerical experiments

In this section, we illustrate the practical behavior of our algorithm on solution of the geomechanical model problem describing loaded tunnel which is crossing by the deep fault and based on the geomechanical theory and models having connection with radioactive waste repositories (see [3]). The introduced algorithm has been implemented in MPI version 1.2.0 by using FORTRAN 77 compiler. A geometry of the problem is in Fig. 1.

Material parameters: 2 regions with Young's modulus $E = 0.52^{10}$[Pa] and Poisson's ratio $\nu = 0,18$.

Boundary conditions: Prescribed displacement $(-2,5 \times 10^{-2}, 0)$ [m] on 3-4. Pressure $0,5 \times 10^{7}$[Pa] on 1-4. Bilateral contact boundary: 1-2 and 2-3. Unilateral contact boundary: 5-6 and 7-8.

Discretization statistics: 12 subdomains, 5501 nodes, 9676 elements, 10428 unknowns, 89 unilateral contact conditions, 466 interface elements.

Convergence statistics: 19 iterations of the PCG algorithm for the auxiliary problem 14 iterations of the successive approximations method, total 38 iterations of the PCG algorithm for the original problem.

Fig. 2 represents detail of deformations and Fig. 3 demonstrates detail of principal stresses in model problem.

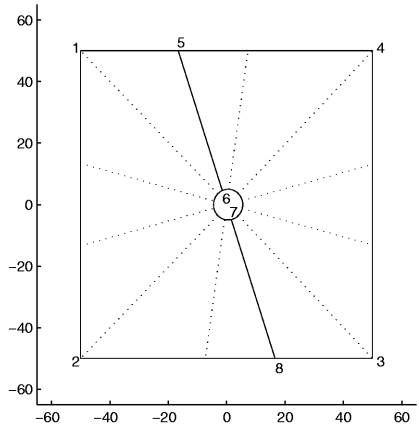

Fig. 1. A geometry of the problem

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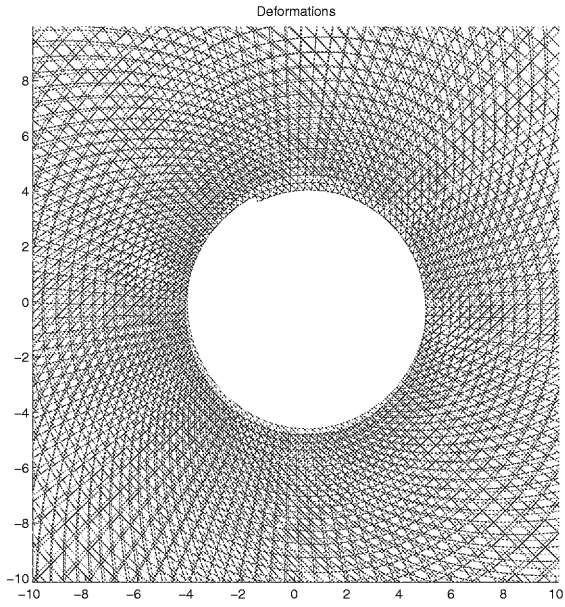

Fig. 2. A detail of deformations in model problem (enlarge factor=10)

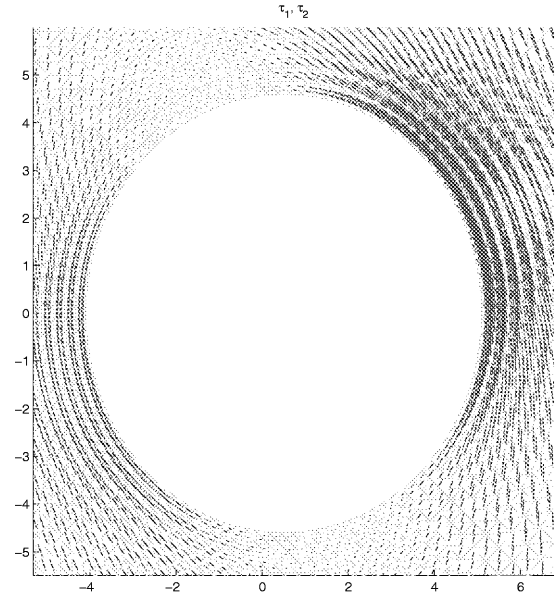

Fig. 3. A detail of principal stresses in neighbourhood of the tunnel. Figure show that maximal pressure is along right-hand side of the tunnel.