# UNIVERSAL OPTIMAL CLONING OF QUBITS AND QUANTUM REGISTERS \*

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We review our recent work on the universal (i.e. input state independent) optimal quantum copying (cloning) of qubits. We present unitary transformations which describe the optimal cloning of a qubit and we present the corresponding quantum logical network. We also present network for an optimal quantum copying "machine" (transformation) which produces N+1 identical copies from the original qubit. Here again the quality (fidelity) of the copies does not depend on the state of the original and is only a function of the number of copies, N. In addition, we present the machine which universaly and optimally clones states of quantum objects in arbitrary-dimensional Hilbert spaces. In particular, we discuss universal cloning of quantum registers.

### I. INTRODUCTION

The most fundamental difference between classical and quantum information is that while classical information can be copied perfectly, quantum information cannot. In particular, it follows from the no-cloning theorem [1] (see also [2,3]) that one cannot create a perfect duplicate of an arbitrary qubit. For example, using the well-known teleportation protocol [4], one can create a perfect copy of the original qubit but this will be at the expense of the complete destruction of information encoded in the original qubit. In contrast, the main goal of quantum copying is to produce a copy of the original qubit which is as close as possible to the original state while the output state of the original qubit is minimally disturbed.

We have shown recently [5,6] that if one is only interested in producing imperfect copies, then it is possible to make quantum clones of the original qubit. To be specific: The copy machine considered by Wootters and Zurek [1] in their proof of the no-cloning theorem, for example, produces two identical copies at its output, but the quality of these copies depends upon the input state. They are perfect for the basis vectors which we denote as  $|0\rangle$  and  $|1\rangle$ , but, because the copying process destroys the off-diagonal information of the input density matrix, they are poor for input states of the form  $(|1\rangle + e^{i\varphi}|0\rangle)/\sqrt{2}$ , where  $\varphi$  is arbitrary. We have introduced [5–7] a different copying machine, the Universal Quantum Copying Machine (UQCM), which produces two identical copies whose quality is independent of the input state. In addition, its performance is, on average, better than that of the Wootters-Zurek machine, and the action of the machine simply scales the expectation values of relevant observables. This UQCM was shown to be optimal, in the sense that it maximizes the average fidelity between the input and output qubits, by Gisin and Massar [8] and by Brußet al. [9]. Gisin and Massar have also been able to find copying transformations which produce N copies from M originals (where N > M) [8]. In addition, we have proposed quantum logic newtworks for quantum copying machines [7,10], and bounds have been placed on how good copies can be [11,12].

In this talk we will firstly review our original ideas on universal quantum copying of a single qubit (Section II). In Section III we will present a quantum network describing the UQCM. Secondly, in Section IV we will introduce the copying machine which produces N+1 identical copies from the original qubit. The quality (fidelity) of copies does not depend on the state of the original and is only a function of a number N of produced copies. We present a quantum network for the quantum copying machine. We show that this machine is formally described by the same unitary transformation as recently introduced by Gisin and Massar [8]. In Section V we will analyze properties of multiply cloned qubits. Thirdly, in Section VI we show how quantum registers (i.e. systems composed of many entangled qubits) can be universally cloned. To be specific, one approach is to use the single-qubit copiers to copy individually (locally) each qubit. We have shown earlier [13] that in the case of two qubits this local copying will preserve some of quantum correlations between qubits, but as we will show, it does not make a particularly good

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copy of the two-qubit state. As an alternative we propose a copy machine which universally (and optimally) clones quantum states in arbitrary-dimensional Hilbert spaces. This allow us to discuss optimal cloning of quantum registers.

# II. UNIVERSAL QUANTUM COPYING MACHINE

Let us assume we want to copy an arbitrary pure state  $|\Psi\rangle_{a_0}$  which in a particular basis  $\{|0\rangle_{a_0}, |1\rangle_{a_0}\}$  is described by the state vector  $|\Psi\rangle_{a_0}$ 

$$|\Psi\rangle_{a_0} = \alpha |0\rangle_{a_0} + \beta |1\rangle_{a_0}; \qquad \alpha = \sin \vartheta / 2e^{i\varphi}; \quad \beta = \cos \vartheta / 2.$$
 (2.1)

The two numbers which characterize the state (2.1) can be associated with the "amplitude"  $|\alpha|$  and the "phase"  $\varphi$  of the qubit. Even though ideal copying, i.e., the transformation  $|\Psi\rangle_{a_0} \longrightarrow |\Psi\rangle_{a_0}|\Psi\rangle_{a_1}$  is prohibited by the laws of quantum mechanics for an *arbitrary* state (2.1), it is still possible to design quantum copiers which operate reasonably well. In particular, the UQCM [5] is specified by the following conditions.

(i) The state of the original system and its quantum copy at the output of the quantum copier, described by density operators  $\hat{\rho}_{a_0}^{(out)}$  and  $\hat{\rho}_{a_1}^{(out)}$ , respectively, are identical, i.e.,

$$\hat{\rho}_{aa}^{(out)} = \hat{\rho}_{aa}^{(out)} \tag{2.2}$$

(ii) If no a priori information about the in-state of the original system is available, then it is reasonable to require that all pure states should be copied equally well. One way to implement this assumption is to design a quantum copier such that the distances between density operators of each system at the output  $(\hat{\rho}_{a_j}^{(out)})$  where j=0,1 and the ideal density operator  $\hat{\rho}^{(id)}$  which describes the in-state of the original mode are input state independent. Quantitatively this means that if we employ the Bures distance [14]

$$d_B(\hat{\rho}_1, \hat{\rho}_2) = \sqrt{2} \left( 1 - \text{Tr} \sqrt{\hat{\rho}_1^{1/2} \hat{\rho}_2 \hat{\rho}_1^{1/2}} \right)^{1/2}, \tag{2.3}$$

as a measure of distance between two operators, then the quantum copier should be such that

$$d_B(\hat{\rho}_{a_j}^{(out)}; \hat{\rho}_{a_j}^{(id)}) = \text{const.}; \qquad j = 0, 1.$$
 (2.4)

(iii) Finally, we would also like to require that the copies are as close as possible to the ideal output state, which is, of course, just the input state. This means that we want our quantum copying transformation to minimize the distance between the output state  $\hat{\rho}_{a_j}^{(out)}$  of the copied qubit and the ideal state  $\hat{\rho}_{a_j}^{(id)}$ . The distance is minimized with respect to all possible unitary transformations U acting on the Hilbert space  $\mathcal{H}$  of two qubits and the quantum copying machine (i.e.,  $\mathcal{H} = \mathcal{H}_{a_0} \otimes \mathcal{H}_{a_1} \otimes \mathcal{H}_x$ )

$$d_B(\hat{\rho}_{a_j}^{(out)}; \hat{\rho}_{a_j}^{(id)}) = \min \left\{ d_B^{(U)}(\hat{\rho}_{a_j}^{(out)}; \hat{\rho}_{a_j}^{(id)}); \forall U \right\}; \qquad (j = 0, 1).$$
(2.5)

Originally, the UQCM was found by analyzing a transformation which contained two free parameters, and then determining them by demanding that condition (ii) be satisfied, and that the distance between the two-qubit output density matrix and the ideal two-qubit output be input state independent. That the UQCM machine obeys the condition (2.5) has only been shown recently [8,9].

The unitary transformation which implements the UQCM [5] is given by

$$|0\rangle_{a_0}|Q\rangle_x \to \sqrt{\frac{2}{3}}|00\rangle_{a_0a_1}|\uparrow\rangle_x + \sqrt{\frac{1}{3}}|+\rangle_{a_0a_1}|\downarrow\rangle_x$$

$$|1\rangle_{a_0}|Q\rangle_x \to \sqrt{\frac{2}{3}}|11\rangle_{a_0a_1}|\downarrow\rangle_x + \sqrt{\frac{1}{3}}|+\rangle_{a_0a_1}|\uparrow\rangle_x,$$
(2.6)

where  $|+\rangle_{a_0a_1} = (|10\rangle_{a_0a_1} + |01\rangle_{a_0a_1})/\sqrt{2}$ , and satisfies the conditions (2.2-2.5). The system labelled by  $a_0$  is the original (input) qubit, while the other system  $a_1$  represents the qubit onto which the information is copied. This qubit is supposed to be prepared initially in a state  $|0\rangle_{a_1}$  (the "blank paper" in a copier). The states of the copy machine are labelled by x. The state space of the copy machine is two dimensional, and we assume that it is always in the same state  $|Q\rangle_x$  initially. If the original qubit is in the superposition state (2.1) then the reduced density operator of both copies at the output are equal [see condition (2.2)] and they can be expressed as

$$\hat{\rho}_{a_j}^{(out)} = \frac{5}{6} |\Psi\rangle_{a_j} \langle \Psi| + \frac{1}{6} |\Psi_\perp\rangle_{a_j} \langle \Psi_\perp|, \qquad j = 0, 1$$
(2.7)

where  $|\Psi_{\perp}\rangle_{a_j} = \beta^{\star}|0\rangle_{a_j} - \alpha^{\star}|1\rangle_{a_j}$ , is the state orthogonal to  $|\Psi\rangle_{a_j}$ . This implies that the copy contains 5/6 of the state we want and 1/6 of the one we do not.

The density operator  $\rho_{a_i}^{(out)}$  given by Eq.(2.7) can be rewritten in a "scaled" form:

$$\hat{\rho}_{a_j}^{(out)} = s_j \hat{\rho}_{a_j}^{(id)} + \frac{1 - s_j}{2} \hat{1}; \qquad j = 0, 1,$$
(2.8)

which guarantees that the distance (2.3) is input-state independent, i.e. the condition (2.4 is automatically fulfilled. The scaling factor in Eq.(2.8) is  $s_j = 2/3$  (j = 0, 1).

We note that once again that the UQCM copies all input states with the same quality and therefore is suitable for copying when no *a priori* information about the state of the original qubit is available. This corresponds to a uniform prior probability distribution on the state space of a qubit (Poincare sphere). Correspondingly, one can measure the quality of copies by the fidelity  $\mathcal{F}$ , which is equal to the mean overlap between a copy and the input state [8]

$$\mathcal{F} = \int d\Omega_{a_j} \langle \Psi | \hat{\rho}_{a_j}^{(out)} | \Psi \rangle_{a_j}, \tag{2.9}$$

where  $\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin \vartheta / 4\pi$ . It is easy to show that the relation between the fidelity  $\mathcal{F}$  and the scaling factor is  $s = 2\mathcal{F} - 1$ .

#### III. COPYING NETWORK

In what follows we show how, with simple quantum logic gates, we can copy quantum information encoded in the original qubit onto other qubits. The copying procedure can be understood as a "spread" of information via a "controlled" entanglement between the original qubit and the copy qubits. This controlled entanglement is implemented by a sequence of controlled-NOT operations operating on the original qubit and the copy qubits which are initially prepared in a specific state.

In designing a network for the UQCM we first note that since the state space of the copy machine itself is two dimensional, we can consider it to be an additional qubit. Our network, then, will take 3 input qubits (one for the input, one which becomes one the copy, and one for the machine) and transform them into 3 output qubits. In what follows we will denote the quantum copier qubit as  $b_1$  rather than x. The operation of this network is such, that in order to transfer information from the original  $a_0$  qubit to the target qubit  $a_1$  we will need one *idle* qubit  $b_1$  which plays the role of quantum copier.

Before proceeding with the network itself let us specify the one and two-qubit gates from which it will be constructed. Firstly we define a single-qubit rotation  $\hat{R}_{j}(\theta)$  which acts on the basis vectors of qubits as

$$\hat{R}_{i}(\theta)|0\rangle_{i} = \cos\theta|0\rangle_{i} + \sin\theta|1\rangle_{i}; \qquad \hat{R}_{i}(\theta)|1\rangle_{i} = -\sin\theta|0\rangle_{i} + \cos\theta|1\rangle_{i}. \tag{3.1}$$

We also will utilize a two-qubit operator (a two-bit quantum gate), the so-called controlled-NOT gate, which has as its inputs a control qubit (denoted as  $\bullet$  in Fig.1) and a target qubit (denoted as  $\circ$  in Fig.1). The control qubit is unaffected by the action of the gate, and if the control qubit is  $|0\rangle$ , the target qubit is unaffected as well. However, if the control qubit is in the  $|1\rangle$  state, then a NOT operation is performed on the target qubit. The operator which implements this gate,  $\hat{P}_{kl}$ , acts on the basis vectors of the two qubits as follows (k denotes the control qubit and k the target):

$$\hat{P}_{kl}|0\rangle_{k}|0\rangle_{l} = |0\rangle_{k}|0\rangle_{l}; \qquad \hat{P}_{kl}|0\rangle_{k}|1\rangle_{l} = |0\rangle_{k}|1\rangle_{l}; 
\hat{P}_{kl}|1\rangle_{k}|0\rangle_{l} = |1\rangle_{k}|1\rangle_{l}; \qquad \hat{P}_{kl}|1\rangle_{k}|1\rangle_{l} = |1\rangle_{k}|0\rangle_{l}.$$
(3.2)

We can decompose the quantum copier network into two parts. In the first part the copy  $(a_1)$  and the idle  $(b_1)$  qubits are prepared in a specific state  $|\Psi\rangle_{a_1b_1}^{(prep)}$ . Then in the second part of the copying network the original information from the original qubit  $a_0$  is redistributed among the three qubits. That is the action of the quantum copier can be described as a sequence of two unitary transformations

$$|\Psi\rangle_{a_0}^{(in)}|0\rangle_{a_1}|0\rangle_{b_1} \longrightarrow |\Psi\rangle_{a_0}^{(in)}|\Psi\rangle_{a_1b_1}^{(prep)} \longrightarrow |\Psi\rangle_{a_0a_1b_1}^{(out)}.$$
(3.3)

The network for the quantum copying machine is displayed in Fig. 1.

## A. Preparation of quantum copier

Let us first look at the preparation stage. Prior to any interaction with the input qubit we have to prepare the two quantum copier qubits  $(a_1 \text{ and } b_1)$  in a specific state  $|\Psi\rangle_{a_1b_1}^{(prep)}$ 

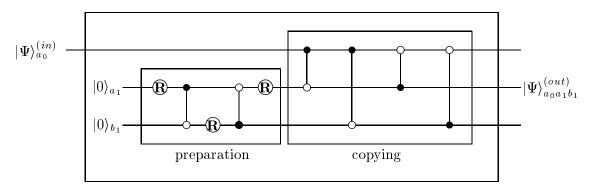
$$|\Psi\rangle_{a_1b_1}^{(prep)} = \frac{1}{\sqrt{6}} \left( 2|00\rangle_{a_1b_1} + |01\rangle_{a_1b_1} + |11\rangle_{a_1b_1} \right), \tag{3.4}$$

which can be prepared by a simple quantum network (see the "preparation" box in Fig.1) with two controlled-NOTs  $\hat{P}_{kl}$  and three rotations  $\hat{R}(\theta_j)$ , i.e.

$$|\Psi\rangle_{a_1b_1}^{(prep)} = \hat{R}_{a_1}(\theta_3)\hat{P}_{b_1a_1}\hat{R}_{b_1}(\theta_2)\hat{P}_{a_1b_1}\hat{R}_{a_1}(\theta_1)|0\rangle_{a_1}|0\rangle_{b_1},\tag{3.5}$$

with the rotation angles defined as [7]

$$\cos 2\theta_1 = \frac{1}{\sqrt{5}}; \qquad \cos 2\theta_2 = \frac{\sqrt{5}}{3}; \qquad \cos 2\theta_3 = \frac{2}{\sqrt{5}}.$$
 (3.6)



quantum copier

FIG. 1. Graphical representation of the UQCM network. The logical controlled-NOT  $\hat{P}_{kl}$  given by Eq.(3.2) has as its input a control qubit (denoted as  $\bullet$ ) and a target qubit (denoted as  $\circ$ ). The action of the single-qubit operator  $\mathbf{R}$  is specified by the transformation (3.1). We separate the preparation of the quantum copier from the copying process itself. The copying, i.e. the transfer of quantum information from the original qubit, is performed by a sequence of four controlled-NOTs. We note that the amplitude information from the original qubit is copied in the obvious direction in an XOR or the controlled-NOT operation. Simultaneously, the phase information is copied in the opposite direction making the XOR a simple model of quantum non-demolition measurement and its back-action.

### B. Quantum copying

Once the qubits of the quantum copier are properly prepared then the copying of the initial state  $|\Psi\rangle_{a_0}^{(in)}$  of the original qubit can be performed by a sequence of four controlled-NOT operations (see Fig. 1)

$$|\Psi\rangle_{a_0a_1b_1}^{(out)} = \hat{P}_{b_1a_0}\hat{P}_{a_1a_0}\hat{P}_{a_0b_1}\hat{P}_{a_0a_1}|\Psi\rangle_{a_0}^{(in)}|\Psi\rangle_{a_1b_1}^{(prep)}.$$
(3.7)

When this operation is combined with the preparation stage, we find that the basis states of the original qubit  $(a_0)$  are copied as described by Eq.(2.6) with  $|\uparrow\rangle_x \equiv |0\rangle_{b_1}$  and  $|\downarrow\rangle_x \equiv |1\rangle_{b_1}$ . When the original qubit is in the superposition state (2.1) then the state vector of the three qubits after the copying has been performed reads

$$|\Psi\rangle_{a_0a_1b_1}^{(out)} = |\Phi_0\rangle_{a_0a_1}|0\rangle_{b_1} + |\Phi_1\rangle_{a_0a_1}|1\rangle_{b_1},\tag{3.8}$$

with

$$|\Phi_0\rangle_{a_0a_1} = \alpha\sqrt{\frac{2}{3}}|00\rangle_{a_0a_1} + \beta\frac{1}{\sqrt{3}}|+\rangle_{a_0a_1}; \quad |\Phi_1\rangle_{a_0a_1} = \beta\sqrt{\frac{2}{3}}|11\rangle_{a_0a_1} + \alpha\frac{1}{\sqrt{3}}|+\rangle_{a_0a_1}. \tag{3.9}$$

From this it follows that at the output of the quantum copier we find a pair of entangled qubits in a state described by the density operator

$$\hat{\rho}_{a_0 a_1}^{(out)} = |\Phi_0\rangle_{a_0 a_1} \langle \Phi_0| + |\Phi_1\rangle_{a_0 a_1} \langle \Phi_1|. \tag{3.10}$$

Each of the copy qubits at the output of the quantum copier has a reduced density operator  $\hat{\rho}_{a_j}^{(out)}$  (j=0,1) given by Eq.(2.8). The Bures distance  $d_B(\hat{\rho}_{a_j}^{(out)};\hat{\rho}_{a_j}^{(id)})$  (j=0,1) between the output qubit and the ideal qubit is constant and it reads

$$d_B(\hat{\rho}_{a_j}^{(out)}; \hat{\rho}_{a_j}^{(id)}) = \sqrt{2} \left( 1 - \sqrt{\frac{5}{6}} \right). \tag{3.11}$$

We note, that the idle qubit after the copying is performed is in a state

$$\hat{\rho}_{b_1}^{(out)} = \frac{1}{3} \left( \hat{\rho}_{b_1}^{(id)} \right)^{\mathrm{T}} + \frac{1}{3} \hat{1}, \tag{3.12}$$

where the superscript T denotes the transpose.

Finally we note that the flow of quantum information in our network can be effectively controlled by the choice of the preparation of the quantum copier. That is, we can a priori decide which qubits at the output will be clones of the original qubit [6,7].

### IV. MULTIPLE COPYING

Here we present a generalization of the transformation (2.6) to the case when a set of N copy qubits  $a_j$  (j = 1, ..., N) are produced out of the original qubit  $a_0$ . We also present a simple quantum network which realizes this multiple quantum copying  $1 \to 1 + N$  [10].

We already know that ideal multiple copying of the form  $|\Psi\rangle_{a_0} \longrightarrow |\Psi\rangle_{a_0} |\Psi\rangle_{a_1}...|\Psi\rangle_{a_N}$  does not exist. But, as we shall show, one can generalize the copying procedure described in Section III, and find a transformation such that

$$\hat{\rho}_{a_0}^{(out)} = \hat{\rho}_{a_j}^{(out)}, \qquad j = 1, ..., N,$$
(4.1)

with the distances  $d_B$  [see Eq.(2.3)] which do not depend on the initial state (2.1) of the original qubit.

To find the  $1 \longrightarrow 1 + N$  network we assume the following:

- (1) We assume that the information from the original qubit is copied to N copy qubits  $a_j$  which are initially prepared in the state  $|N;0\rangle_{\vec{a}} \equiv |0\rangle_{a_1}...|0\rangle_{a_N}$  (here the subscript  $\vec{a}$  is a shorthand notation indicating that  $|N;0\rangle_{\vec{a}}$  is a vector in the Hilbert space of N qubits  $a_j$ ).
- (2) To implement multiple quantum copying we need to associate an *idle* qubit  $b_j$  with each copy qubit,  $a_j$ . These N idle qubits, which play the role of the copying machine itself, are initially prepared in the state  $|N;0\rangle_{\vec{b}} \equiv |0\rangle_{b_1}...|0\rangle_{b_N}$ .
- (3) Prior to the transfer of information from the original qubit, the copy and the idle qubits have been prepared in a specific state  $|\Psi\rangle_{\vec{a}\vec{b}}^{(prep)}$ . Once this is done the copying is performed by a simple sequence of controlled-NOT operations.

### A. Preparation of the quantum copier

In order to find the explicit form for the quantum network for  $1 \to 1 + N$  copying we introduce normalized state vectors  $|N;k\rangle_{\vec{a}}$  describing a *symmetric N*-qubit state with k qubits in the state  $|1\rangle$  and (N-k) qubits in the state  $|0\rangle$ . These states are orthonormalized, i.e.  $_{\vec{a}}\langle N;l|N;k\rangle_{\vec{a}}=\delta_{k,l}$ , and have the property

$$|N;l\rangle_{\vec{a}} = \sqrt{\frac{N-l}{N}}|0\rangle_{a_m}|N-1;l\rangle_{a_1...a_{m-1}a_{m+1}...a_N} + \sqrt{\frac{l}{N}}|1\rangle_{a_m}|N-1;l-1\rangle_{a_1...a_{m-1}a_{m+1}...a_N}.$$
(4.2)

As we have already said, we assume that the copy+idle qubits are initially prepared in the state

$$|\Psi\rangle_{\vec{n}\vec{b}}^{(in)} = |N;0\rangle_{\vec{a}}|N;0\rangle_{\vec{b}}.$$
(4.3)

By performing a sequence of local rotations **R** and controlled-NOT operations analogous to Eq.(3.5) we can obtain the state  $|\Psi\rangle_{\vec{a}\vec{b}}^{(prep)}$  [15]

$$|\Psi\rangle_{\vec{a}\vec{b}}^{(prep)} = \sum_{k=0}^{N} \left[ e_k |N; k\rangle_{\vec{a}} + f_k |N; k-1\rangle_{\vec{a}} \right] |N; k\rangle_{\vec{b}}, \tag{4.4}$$

where

$$e_k = \sqrt{\frac{2}{N+2}} \frac{\binom{N}{k}}{\binom{N+1}{k}}; \qquad f_k = \sqrt{\frac{k}{N-k+1}} e_k. \tag{4.5}$$

Once the copying machine is prepared in the state  $|\Psi\rangle_{\vec{a}\vec{b}}^{(prep)}$  we can start to copy information from the original qubit  $a_0$ .

# B. Copying of information

To describe the copying network we firstly introduce an operator  $\hat{Q}_{a_0\vec{a}}$  which is a product of the controlled-NOTs defined by Eq.(3.2) with  $a_0$  being a control qubit and  $a_j$  (j = 1, ..., N) being targets:

$$\hat{Q}_{a_0 \vec{a}} \equiv \hat{P}_{a_0 a_N} \hat{P}_{a_0 a_{N-1}} ... \hat{P}_{a_0 a_1}. \tag{4.6}$$

We also introduce the operator  $\hat{Q}_{\bar{a}a_0}$  describing the controlled-NOT process with  $a_0$  playing the role of the target qubit, i.e.

$$\hat{Q}_{\vec{a}a_0} \equiv \hat{P}_{a_N a_0} \hat{P}_{a_{N-1}a_0} \dots \hat{P}_{a_1 a_0}. \tag{4.7}$$

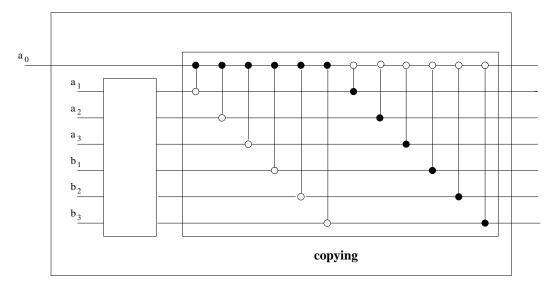


FIG. 2. Graphical representation of the network for the  $1 \to 1 + N$  copying. The logical controlled-NOT  $\hat{P}_{kl}$  given by Eq.(3.2) has as its input a control qubit (denoted as  $\bullet$ ) and a target qubit (denoted as  $\circ$ ). We separate the preparation of the quantum copier from the copying process itself. The copying, i.e. the transfer of quantum information from the original qubit, is performed by a sequence of controlled-NOTs as described by Eq.(4.9).

Now we find the  $1 \longrightarrow 1 + N$  copying network to be

$$|\Psi\rangle_{a_0}^{(in)}|N;0\rangle_{\vec{a}}|N;0\rangle_{\vec{b}} \longrightarrow |\Psi\rangle_{a_0}^{(in)}|\Psi\rangle_{\vec{a}\vec{b}}^{(prep)} \longrightarrow |\Psi\rangle_{a_0\vec{a}\vec{b}}^{(out)}, \tag{4.8}$$

where the (2N+1) qubit output of the copying process is described by the state vector  $|\Psi\rangle_{a_0\vec{a}\vec{b}}^{(out)}$  which is defined as

$$|\Psi\rangle_{a_0\vec{d}\vec{b}}^{(out)} = \hat{Q}_{\vec{b}a_0}\hat{Q}_{\vec{a}a_0}\hat{Q}_{a_0\vec{b}}\hat{Q}_{a_0\vec{a}}|\Psi\rangle_{a_0}^{(in)}.|\Psi\rangle_{\vec{d}\vec{b}}^{(prep)}.$$
(4.9)

This last equation describes a simple quantum network when firstly the original qubit controls the target qubits of the quantum copier. Then the qubits  $\vec{a}$  and  $\vec{b}$  "control" the state of the original qubit via another sequence of controlled-NOTs (see Fig.2). In this way one can produce out of a single original qubit a set of quantum clones.

# V. PROPERTIES OF COPIED QUBITS

Using the explicit expression for the output state  $|\Psi\rangle_{a_0\vec{a}\vec{b}}^{(out)}$  we find that the original and the copy qubits at the output of the quantum copier are in the same state described by the density operator

$$\hat{\rho}_{a_j}^{(out)} = s^{(N)} \hat{\rho}_{a_j}^{(id)} + \frac{1 - s^{(N)}}{2} \hat{1}; \qquad j = 0, 1, ..., N,$$
(5.1)

where the scaling factor  $s^{(N)}$  depends on the number N of copies, i.e.

$$s^{(N)} = \frac{1}{3} + \frac{2}{3(N+1)},\tag{5.2}$$

which corresponds to the fidelity  $\mathcal{F} = 2/3 + 1/3(N+1)$ . We see that this result for N=1 reduces to the case of the UQCM discussed in Section III. We also note that in the limit  $N \to \infty$ , i.e. when an infinite number of copies is *simultaneously* produced via the generalization of the UQCM, the copy qubits still carry information about the original qubit, because their density operators are given by the relation

$$\hat{\rho}_{a_j}^{(out)} = \frac{1}{3}\hat{\rho}_{a_j}^{(id)} + \frac{1}{3}\hat{1}; \qquad j = 0, 1, ..., \infty,$$
(5.3)

which corresponds to the fidelity  $\mathcal{F}=2/3$ . This is the optimal fidelity achievable when an *optimal* measurement is performed on a single qubit [16,17]. From this point of view one can consider quantum copying as a transformation of quantum information into classical information [8]. This also suggests that quantum copying can be utilized to obtain novel insights into the quantum theory of measurement [e.g., a simultaneous measurement of conjugated observables on two copies of the original qubit; or a specific realization of the generalized (POVM) measurement performed on the original qubit.

# Comment 1

We note that if the original qubit is copied sequentially by a system of N copying machines of the type  $1 \to 1+1$  (each machine copies two outcomes of the previous copier) then  $2^N$  copies of the original qubit in the limit  $N \to \infty$  are in the state  $\hat{\rho}_{a_j}^{(out)} = \hat{1}/2$ . In this case the copied qubits do not carry information about the original qubit, while all idle qubits are in the state (3.12).

### Comment 2

Using the copying transformation (4.9) we find that the basis vectors  $|0\rangle_{a_0}$  and  $|1\rangle_{a_0}$  of the original qubit are transformed as [compare with Eq.(2.6)]

$$|0\rangle_{a_{0}}|\Psi\rangle_{\vec{a}\vec{b}}^{(in)} \to \sum_{k=0}^{N} \lambda_{k}^{(N+1)}|N+1;k\rangle_{a_{0}\vec{a}}|N;k\rangle_{\vec{b}};$$

$$|1\rangle_{a_{0}}|\Psi\rangle_{\vec{a}\vec{b}}^{(in)} \to \sum_{k=0}^{N} \lambda_{N-k}^{(N+1)}|N+1;k+1\rangle_{a_{0}\vec{a}}|N;k\rangle_{\vec{b}},$$
(5.4)

where  $\lambda_k^{(N+1)} = [2(N+1-k)/(N+1)(N+2)]^{1/2}$ . We clearly see that the set of N+1 completely symmetric orthonormal states  $|N;k\rangle_{\vec{b}}$  (with k=0,1,...,N) of the idle qubits  $b_j$  plays the role of a set of basis vectors of the abstract quantum copier and in this form the transformation (5.4) describes the action of the quantum copier as discussed by Gisin and Massar [8]. These authors have also shown that transformation (5.4) describes the optimal input-state independent  $1 \to 1 + N$  quantum copier.

# Comment 3

We note that idle qubits  $b_j$  after the copying is performed are always in the state

$$\hat{\rho}_{b_j}^{(out)} = \frac{1}{3} \left( \hat{\rho}_{b_j}^{(id)} \right)^{\mathrm{T}} + \frac{1}{3} \hat{1}, \qquad j = 1, ..., N,$$
(5.5)

irrespective of the number of copies created from the original qubit.

### A. Inseparability of cloned qubits

We first recall that a density operator of two subsystems is inseparable if it cannot be written as a convex sum

$$\hat{\rho}_{xy} = \sum_{m} w^{(m)} \hat{\rho}_x^{(m)} \otimes \hat{\rho}_y^{(m)}. \tag{5.6}$$

Inseparability is one of the most fundamental quantum phenomena. It is required for a violation of Bell's inequality (to be specific, a separable system always satisfy Bell's inequality, but the contrary is not necessarily true). Distant parties cannot prepare an inseparable state from a separable one if they only use local operations and classical communication. In the case of two spins-1/2 we can utilize the Peres-Horodecki theorem [18,19] which states that the positivity of the partial transposition of a state is necessary and sufficient for its separability.

### Comment 4

The two-qubit density operator  $\hat{\rho}_{a_m a_n}^{(out)}$  (here m, n = 0, 1, ..., N and  $m \neq n$ ) associated with the output state  $|\Psi\rangle_{a_0 \vec{a} \vec{b}}^{(out)}$  [see Eq.(4.9)] in the basis  $|11\rangle_{a_m a_n}, |10\rangle_{a_m a_n}, |01\rangle_{a_m a_n}, |00\rangle_{a_m a_n}$  is described by the matrix

$$\hat{\rho}_{a_{m}a_{n}}^{(out)} = \frac{1}{6} \begin{pmatrix} \frac{(3N+5)|\beta|^{2} + (N-1)|\alpha|^{2}}{N+1} & \frac{\alpha^{*}\beta(N+3)}{N+1} & \frac{\alpha^{*}\beta(N+3)}{N+1} & 0\\ \frac{\alpha\beta^{*}(N+3)}{N+1} & 1 & 1 & \frac{\alpha^{*}\beta(N+3)}{N+1} \\ \frac{\alpha\beta^{*}(N+3)}{N+1} & 1 & 1 & \frac{\alpha^{*}\beta(N+3)}{N+1} \\ 0 & \frac{\alpha\beta^{*}(N+3)}{N+1} & \frac{\alpha\beta^{*}(N+3)}{N+1} & \frac{(3N+5)|\alpha|^{2} + (N-1)|\beta|^{2}}{N+1} \end{pmatrix}.$$

$$(5.7)$$

From Eq.(5.7) we find that the eigenvalues  $\vec{E} = \{E_1, E_2, E_3, E_4\}$  of the partially transposed matrix  $(\hat{\rho}_{a_m a_n}^{(out)})^{T_2}$  are input-state independent and read:

$$\vec{E} = \left\{ \frac{1}{6}, \frac{1}{6}, \frac{1}{3} + \frac{\sqrt{2(5+4N+N^2)}}{6(N+1)}, \frac{1}{3} - \frac{\sqrt{2(5+4N+N^2)}}{6(N+1)} \right\}.$$
 (5.8)

Using the Peres-Horodecki theorem [18,19] we can conclude that the two copied qubits at the output of the copier are inseparable only in the case N=1. In this case on of the eigenvalues (5.8) is negative, which is the necessary and sufficient condition for the inseparability of the matrix (5.7). For N>1 all pairs of copied qubits at the output of the quantum copier are separable (i.e., the eigenvalues given by Eq.(5.8) are positive).

# Comment 5

To quantify how the "quantum copier" (i.e., the idle qubits) is entangled with the original and the copy qubits at the output, we evaluate the parameter  $\xi^{(N)} = \text{Tr}[\hat{\rho}_{\vec{b}}^{(out)}]^2$ , which quantifies the purity of the quantum copier. If  $\xi = 1$ , then the copier (i.e., the subsystem of the whole system  $a_0\vec{a}\vec{b}$ ) is in a pure state. Otherwise (i.e., when  $\xi < 1$ ) it is in an impure state. If the whole system is in a pure state, i.e.  $\text{Tr}[\hat{\rho}_{a_0\vec{a}\vec{b}}^{(out)}]^2 = 1$ , then  $\xi$  quantifies the degree of entanglement between the two subsystems. From Eq.(4.9) we find

$$\xi^{(N)} = \frac{1}{N+1} \frac{2(2N^2 + 7N + 6)}{3(N+2)^2},\tag{5.9}$$

from which it follows that in the limit of large N

$$\xi^{(N)} \simeq \frac{4}{3(N+1)}.\tag{5.10}$$

The lower bound  $\xi_{\min}$  of the purity parameter  $\xi$  of an arbitrary quantum system in the N+1 dimensional Hilbert space (i.e., this is the size of the Hilbert space of the quantum copier) is

$$\xi_{\min} = \frac{1}{N+1}.\tag{5.11}$$

We see that for all values of N the parameter  $\xi^{(N)}$  is very close to its lower bound, i.e. the quantum copier and the copies are highly entangled. To understand the nature of this entanglement, we briefly consider the  $1 \to 1+1$  quantum copying. In this case, we can evaluate the density operator  $\hat{\rho}_{a_1b_1}^{(out)}$  which in matrix form can be written as:

$$\hat{\rho}_{a_1b_1}^{(out)} = \frac{1}{6} \begin{pmatrix} 4|\beta|^2 + |\alpha|^2 & \alpha\beta^* & 2\alpha^*\beta & 2\\ \alpha^*\beta & |\beta|^2 & 0 & 2\alpha^*\beta\\ 2\alpha\beta^* & 0 & |\alpha|^2 & \alpha\beta^*\\ 2 & 2\alpha\beta^* & \alpha^*\beta & 4|\alpha|^2 + |\beta|^2 \end{pmatrix}.$$
 (5.12)

For  $\alpha$  and  $\beta$  real, the eigenvalues of the corresponding partially transposed matrix do not depend on these parameters and they read:

$$\vec{E} = \left\{ \frac{1}{3}, \frac{2}{3}, \frac{1 - \sqrt{17}}{12}, \frac{1 + \sqrt{17}}{12} \right\}. \tag{5.13}$$

We see that one of the eigenvalues is negative which means that each copy qubit (i.e., either  $a_0$  or  $a_1$ ) and the idle qubit are quantum-mechanically entangled. In the case when  $\alpha$  and  $\beta$  are complex, the eigenvalues of the partially transposed matrix associated with the matrix Eq.(5.12) do depend on  $\alpha$  and  $\beta$  and one of the eigenvalues is always negative. So these qubits are quantum-mechanically entangled.

# VI. CLONING OF QUANTUM REGISTERS

In what follows we will propose a copy machine which copies universally higher dimensional systems. We shall be particularly interested in how the quality of the copies scales with the dimensionality, M, of the system being copied. What we find is that the fidelity of the copies decreases with M, as expected, but, somewhat surprisingly, does not go to zero as M goes to infinity.

Let us now consider a quantum system prepared in a pure state which is described by the vector

$$|\Phi\rangle_{a_0} = \sum_{i=1}^{M} \alpha_i |\Psi_i\rangle_{a_0} \tag{6.1}$$

in an M-dimensional Hilbert space spanned by M orthonormal basis vectors  $|\Psi_i\rangle_{a_0}$  (i=1,...,N). The complex amplitudes  $\alpha_i$  are normalized to unity, i.e.  $\sum |\alpha_i|^2 = 1$ . In particular, one can consider  $M=2^m$  where m is the number of qubits in a given quantum register. One can generalize the no-cloning theorem which has been proven for spin-1/2 particles (qubits) by Wootters and Zurek [1] for arbitrary quantum systems. That is, there does not exist a unitary transformation such that the state given in Eq. (6.1) can be ideally cloned (copied), i.e. it is impossible to find a unitary transformation such that  $|\Phi\rangle_{a_0} \longrightarrow |\Phi\rangle_{a_0}|\Phi\rangle_{a_1}$ .

Following our previous discussion we can ask whether a universal cloning transformation exists which will generate two imperfect copies from the original state,  $|\Phi\rangle_{a_0}$ . The quality of the cloning should not depend on the particular state (in the given Hilbert space) which is going to be copied. This input-state independence (invariance) of the cloning can be formally expressed as

$$\hat{\rho}_{a_j}^{(out)} = s\hat{\rho}_{a_j}^{(id)} + \frac{1-s}{M}\hat{1},\tag{6.2}$$

where  $\hat{\rho}_{a_j}^{(id)} = |\Phi\rangle_{a_0} a_0 \langle \Phi|$  is the density operator describing the original state which is going to be copied. This scaling form of Eq.(6.2) guarantees that the Bures distance (2.3) between the input and the output density operators is input-state independent.

The quantum copying machine we shall use is itself an M dimensional quantum system, and we shall let  $|X_i\rangle_x$  (i=1,...,M) be an orthonormal basis of the copying machine Hilbert space. This copier is initially prepared in a particular state  $|X\rangle_x$ . The action of the cloning transformation can be specified by a unitary transformation acting on the basis vectors of the tensor product space of the original quantum system  $|\Psi_i\rangle_{a_0}$ , the copier, and an additional M-dimensional system which is to become the copy (which is initially prepared in an arbitrary state  $|0\rangle_{a_1}$ ). We make the Ansatz

$$|\Psi_{i}\rangle_{a_{0}}|0\rangle_{a_{1}}|X\rangle_{x} \longrightarrow c|\Psi_{i}\rangle_{a_{0}}|\Psi_{i}\rangle_{a_{1}}|X_{i}\rangle_{x} + d\sum_{j\neq i}^{M} (|\Psi_{i}\rangle_{a_{0}}|\Psi_{j}\rangle_{a_{1}} + |\Psi_{j}\rangle_{a_{0}}|\Psi_{i}\rangle_{a_{1}})|X_{j}\rangle_{x}; \qquad i = 1, ..., M,$$

$$(6.3)$$

and we assume that the coefficients c and d are real. From the unitarity of the transformation in Eq. (6.3) it follows that c and d satisfy the relatation

$$c^2 + 2(M-1)d^2 = 1. (6.4)$$

Using this transformation we find that the particles  $a_0$  and  $a_1$  at the output of the cloner are in the same state (have the same reduced density matrixes), which is described by the density operator

$$\hat{\rho}_{a_k}^{(out)} = \sum_{i=1}^{M} |\alpha_i|^2 \left( c^2 + (M-2)d^2 \right) |\Psi_i\rangle \langle \Psi_i| + \sum_{\substack{i,j=1\\i\neq j}}^{M} \alpha_i \alpha_j^* \left( 2cd + (M-2)d^2 \right) |\Psi_i\rangle \langle \Psi_j| + d^2 \hat{1}.$$
 (6.5)

Now our task is to find the values for c and d such that the density operator in Eq. (6.5) takes the scaled form of Eq. (6.2). This directly guarantees the universality of the transformation (6.3), i.e. the fidelity of the cloning does not depend on the initial states of the particle which is going to be cloned.

Comparing these two equations we find that c and d must satisfy the equation

$$c^2 = 2cd. (6.6)$$

Taking into account the normalization condition in Eq. (6.4) we find that

$$c^2 = \frac{2}{(M+1)}; \qquad d^2 = \frac{1}{2(M+1)};$$
 (6.7)

from which it follows that the scaling factor s is

$$s = c^{2} + (M - 2)d^{2} = \frac{(M + 2)}{2(M + 1)}.$$
(6.8)

If M=2, then the transformation in Eq. (6.3) reduces to the copying transformation for qubits given by Eq.(2.6). From earlier results of Gisin and Massar [8] then optimality of the transformation (6.3) for M=2 directly follows. At the moment we are not able to prove rigorously that the cloning transformation (6.3) is optimal for arbitrary M>2. Nevertheless, we have performed numerical tests which suggest that the cloning transformation (6.3) is optimal.

We first note that the scaling factor, which describes the quality of the copy, is a decreasing function of M. This is not surprising, because a quantum state in a large dimensional space contains more quantum information than one in a small dimensional one (e. g. a state in a 4 dimensional space contains information about 2 qubits while a state in a 2 dimensional one describes only a single qubit), so that as M increases one is trying to copy more and more quantum information. On the other hand, it is interesting to note that in the limit  $M \to \infty$ , i.e. in the case when the Hilbert space of the given quantum system is infinite dimensional (e.g. quantum-mechanical harmonic oscillator), the cloning can still be performed efficiently with the scaling factor equal to 1/2.

In order to confirm that the quality of the copies which the copying transformation in Eq. (6.3) produces is inputstate independent (i.e. all states are cloned equally well) we evaluate the Bures distance (2.3). In our particular case we find, that the distance between  $\hat{\rho}_{a_k}^{(out)}$  and  $\hat{\rho}_{a_k}^{(id)}$  depends only on the dimension of the Hilbert space M, but not on the state which is cloned, i.e.

$$d_B(\hat{\rho}_{a_k}^{(out)}, \hat{\rho}_{a_k}^{(id)}) = \sqrt{2} \left( 1 - \sqrt{\frac{M+3}{2(M+1)}} \right)^{1/2}.$$
 (6.9)

The Bures distance in Eq. (6.9) is maximal when states in the infinite-dimensional Hilbert space are cloned, and in that case we find

$$\lim_{M \to \infty} d_B(\hat{\rho}_{a_k}^{(out)}, \hat{\rho}_{a_k}^{(id)}) = \sqrt{2 - \sqrt{2}}.$$
(6.10)

This means that even for an infinite-dimensional system, reasonable cloning can be performed, which is reflected in the fact that the corresponding scaling factor s is equal to 1/2.

Now we evaluate the von Neumann entropy  $S = -\text{Tr}\hat{\rho} \ln \hat{\rho}$  of the outure state (6.5):

$$S = \ln[2(M+1)] - \frac{M+3}{2(M+1)}\ln(M+3). \tag{6.11}$$

This is an increasing function of M which implies that the copy states are becoming more and more mixed as M increases. In the limit  $M \to \infty$  we find that

$$S \simeq \ln \sqrt{M}. \tag{6.12}$$

This can be compared to the maximum value of the von Neumann entropy in an M-dimensional Hilbert space which is equal to  $S = \ln M$ .

Using the transformation in Eq. (6.3) we can also find the state of the copy machine after the cloning has been performed

$$\hat{\rho}_x^{(out)} = 2d^2 \left(\hat{\rho}_x^{(id)}\right)^T + 2d^2\hat{1},\tag{6.13}$$

i. e. the copier is left in a state proportional to the transposed state of the original quantum system. The von Neumann entropy of the copier at the output reflects the degree of entanglement between the copies and the copier. As expected, this entropy does not depend on the state to be copied and is just a function of the dimension of the Hilbert space, i.e.

$$S = \ln(M+1) - \frac{2\ln 2}{M+1}. (6.14)$$

This is again and increasing function of M which reflects the fact that the copies and the copier become increasingly correlated as M increases. On the other hand, it follows from the Araki-Lieb theorem that the maximum value of the entropy of two entangled subsystems with the dimension of the smaller subsystem being M, is equal to  $S_{max} = \ln M^2$ , which shows that the cloner and the clones are far from being maximally entangled.

### A. Local vs.nonlocal cloning

Finally, we would like to compare two methods of copying of quantum registers. In particular, we consider cloning of an entangled state of two qubits. We assume that the two qubits are prepared in the state

$$|\Phi\rangle_{a_0b_0} = \alpha |00\rangle_{a_0b_0} + \beta |11\rangle_{a_0b_0},$$
(6.15)

where, for simplicity, we have taken  $\alpha$  and  $\beta$  to be real, and  $\alpha^2 + \beta^2 = 1$ . First, we shall consider the case in which each of the two qubits  $a_0$  and  $b_0$  is copied *locally* by two independent quantum copiers [13]. Each of these two copiers is described by the transformation in Eq. (6.3) with M = 2. Next, we shall consider a *nonlocal* cloning of the two-qubit state in Eq. (6.15) when this system is cloned via the unitary transformation in Eq. (6.3) with M = 4, i.e. the cloner in this case can act non-locally on the two qubits. Our chief task will be analyze how inseparability is cloned in these two scenarios, but we shall also examine the quality of the copies which are produced in the two cases. From the Peres-Horodecki theorem it follows that the state in Eq. (6.15) is inseparable for all values of  $\alpha^2$  such that  $0 < \alpha^2 < 1$ .

Now suppose that each of the two original qubits  $a_0$  and  $b_0$  is cloned by two independent local cloners  $X_I$  and  $X_{II}$ , each described by the transformation in Eq. (6.3) with M=2. The cloner  $X_I$  ( $X_{II}$ ) generates out of qubit  $a_0$  ( $b_0$ ) two qubits  $a_0$  and  $a_1$  ( $b_0$  and  $b_1$ ). After we perform trace over the two cloners we obtain a four-qubit density operator  $\hat{\rho}_{a_0a_1b_0b_1}^{(out)}$  which also describes two nonlocal two-qubit systems, i.e.  $\hat{\rho}_{a_0b_1}$  and  $\hat{\rho}_{a_1b_0}$ . These two two-qubit systems are the clones of the original two-qubit register (6.15) and they are described by the density operators [13]

$$\hat{\rho}_{a_0b_1}^{(out)} = \hat{\rho}_{a_1b_0}^{(out)} = \frac{24\alpha^2 + 1}{36}|00\rangle\langle00| + \frac{24\beta^2 + 1}{36}|11\rangle\langle11| + \frac{5}{36}(|01\rangle\langle01| + |10\rangle\langle10|) + \frac{4\alpha\beta}{9}(|00\rangle\langle11| + |11\rangle\langle00|). \tag{6.16}$$

We first note that this density matrix cannot be expressed in the scaled form of Eq. (6.2), and that the quality fo the copies depends on the input state. Therefore, this procedure does not produce a universal quantum copy machine. From the Peres-Horodecki theorem we immediately find that the density operators in Eq. (6.16) are inseparable if

$$\frac{1}{2} - \frac{\sqrt{39}}{16} \le \alpha^2 \le \frac{1}{2} + \frac{\sqrt{39}}{16}.\tag{6.17}$$

This proves that for a restricted set of pure two-qubit states (6.15), those which satisfy the condition in Eq. (6.17), it is possible to *locally* copy them so that their original inseparability is (partially) preserved.

Let us now see what happens when we copy the entire two-qubit register at once. We would like to determine whether the set of original two-qubit states (see Eq. (6.15)), which after the cloning exhibit inseparability, is larger (i.e., the restriction of the form given in Eq. (6.17) is weaker) than when a nonlocal cloning is performed. To do so, we introduce four basis vectors  $|\Psi_1\rangle = |00\rangle$ ;  $|\Psi_2\rangle = |01\rangle$ ;  $|\Psi_3\rangle = |10\rangle$ ; and  $|\Psi_4\rangle = |11\rangle$ , so that the original two-qubit state

in Eq. (6.15) is expressed as  $|\Phi\rangle = \alpha |\Psi_1\rangle + \beta |\Psi_4\rangle$ . The copying is now performed according to the transformation in Eq. (6.3) with M=4. We find that each of the two pairs of two-qubit copies at the output of the copier is described by the same density operator

$$\hat{\rho}_{a_0b_1}^{(out)} = \frac{6\alpha^2 + 1}{10}|00\rangle\langle 00| + \frac{6\beta^2 + 1}{10}|11\rangle\langle 11| + \frac{1}{10}(|01\rangle\langle 01| + |10\rangle\langle 10|) + \frac{3\alpha\beta}{5}(|00\rangle\langle 11| + |11\rangle\langle 00|). \tag{6.18}$$

Here the fidelity of copying is input-state independent. Moreover, the quality of the cloned register is higher than that in Eq. (6.16). Again, using the Peres-Horodecki theorem we find that the density operator in Eq. (6.18) is inseparable if

$$\frac{1}{2} - \frac{\sqrt{2}}{3} \le \alpha^2 \le \frac{1}{2} + \frac{\sqrt{2}}{3}.\tag{6.19}$$

We conclude that quantum inseparability can be copied better (i.e. for much larger range of the parameter  $\alpha$ ) by using a nonlocal copier than when two local copiers are used.

### VII. CONCLUSIONS

We have presented the universal optimal quantum copying machine which optimally clones a single original qubit to N+1 qubits. We have found a simple quantum network which realizes this quantum copier. In addition we have presented a universal cloner for quantum registers. We have numerically tested the optimality of this cloner, but the rigorous proof has still be presented.

Quantum copiers can be effectively utilized in various processes designed for manipulation with quantum information. In particular, quantum copiers can be used for an optimal eavesdropping [20]; they can be applied for realization of the optimal generalized (POVM) measurements [21], or they can be utilized for storage and retrieval of information in quantum computers [22].

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