# QUANTUM EFFECTS IN ALGORITHMS 

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#### Abstract

We discuss some seemingly paradoxical yet valid effects of quantum physics in information processing. Firstly, we argue that the act of "doing nothing" on part of an entangled quantum system is a highly non-trivial operation and that it is the essential ingredient underlying the computational speedup in the known quantum algorithms. Secondly, we show that the watched pot effect of quantum measurement theory gives the following novel computational possibility: suppose that we have a quantum computer with an on/off switch, programmed ready to solve a decision problem. Then (in certain circumstances) the mere fact that the computer would have given the answer if it were run, is enough for us to learn the answer, even though the computer is in fact not run.


## 1 Introduction

Many recent developments in quantum computation are motivated by existing results in theoretical computer science, adapted and rewritten in a quantum context. This includes much of the recent work on quantum error correcting codes (see for example [3, ©, 5]) and also the idea of using the Fourier transform to determine periodicity, which underlies many of the known quantum algorithms [1]. There are relatively few results (such as (7) with no classical analogue, motivated intrinsically from considerations of physics. This is a curious situation considering that the entire subject of quantum computation derives from differences between the classical and quantum laws of physics. Apart from the computer science benefits of providing more efficient computation, an important fundamental aspect of the subject is the insight that it might provide for a deeper understanding of the quantum laws and their origins. Computer science and information theory provide an entirely new conceptual framework for considering this question of physics. Thus we will consider the question: what are the essential physical effects that give rise to the known computational speedups? And is it possible to use other differences between quantum and classical physics for novel computational possibilities?

## 2 Quantum Information Processing and Entanglement

It is often said that the power of quantum computation derives from the superposition principle - the ability to do different computations in parallel in superposition, and combine the results with cleverly arranged interferences. But this explanation is not precise enough because classical waves also exhibit superposition and any effect of superposition can be mimicked by a classical wave system. However there is an essential difference between classical and quantum superposition, which lies in the different way that the two physical theories describe composite systems [2].

Consider $n$ two-level systems. In the classical case we may for example think of each system as comprising the two lowest energy modes of vibration of a string with fixed endpoints together with all superpositions. According to the laws of classical mechanics, the total state space of the composite system is the Cartesian product of the $n$ subsystem spaces. Thus no matter how much the strings interact in their physical evolution, the total state is always a product state of the $n$ separate systems. Hence we can say that the information needed to describe the total state grows linearly with $n$ (being $n$ times the information needed to describe a single subsystem).

In contrast, according to the laws of quantum mechanics the total space is the tensor product of the subsystem spaces and a general state may be written as

$$
\begin{equation*}
\left|\psi_{n}\right\rangle=\sum_{i_{1}, . ., i_{n}=0}^{1} a_{i_{1} \ldots i_{n}}\left|i_{1}\right\rangle \ldots\left|i_{n}\right\rangle \tag{1}
\end{equation*}
$$

Thus generally we will have $O\left(2^{n}\right)$ superposition components present and the information needed to describe the total state will grow exponentially with $n$. The novel quantum effect here - the passage from Cartesian to tensor product - is precisely the phenomenon of entanglement i.e. the ability to superpose general product states.

As stated above, quantum entanglement can be readily mimicked by classical wave systems: instead of taking $n$ two-level systems, we consider a single classical wave system with $2^{n}$ levels, allowing general superpositions of all these levels, and merely interpret these as entangled states via a chosen mathematical isomorphism between $\otimes^{n} V_{2}$ and $V_{2^{n}}$ (where $V_{k}$ is a $k$-dimensional vector space). However this mathematical isomorphism is not a valid correspondence for considerations of complexity (i.e. in which we assess the utilisation of physical resources): if the $2^{n}$ classical levels are, say, equally spaced energy modes, then to produce a general state in $V_{2^{n}}$ we will need to invest an amount of energy exponential in $n$, whereas a general state in $\otimes^{n} V_{2}$ will require only a linear amount of energy (as at most, each of the $n$ two-level systems will need to be excited). The essential point here is that entanglement allows one to construct exponentially large superpositions with only linear physical resources and this cannot be achieved with classical superposition.

In the sense described above the state $\left|\psi_{n}\right\rangle$ can encode an exponentially large amount of information. This would be of little consequence if we could not process the information in a suitably efficient way. Fortunately the laws of quantum physics allow precisely this possibility, which appears to be at the heart of the computational speedup exhibited by the known quantum algorithms. Suppose we apply a one-qubit gate $U$ to the first qubit of the
entangled state $\left|\psi_{n}\right\rangle$. This would count as just one step of quantum information processing but to compute the result classically (say by matrix multiplication) we would calculate the new amplitudes by

$$
\begin{equation*}
\tilde{a}_{j_{1} \ldots j_{n}}=\sum_{i=0}^{1} U_{j_{1} i} a_{i j_{2} \ldots j_{n}} \tag{2}
\end{equation*}
$$

where $U_{j i}$ is the unitary matrix for $U$. Now, this computation involves exponentially many steps: the $2 \times 2$ matrix multiplication of $U$ needs to be performed successively $2^{n-1}$ times for all possible values of the indices $j_{2} \ldots j_{n}$. Although the action of $U$ on qubit 1 is a physically simple operation, it is represented mathematically as a tensor product $U \otimes I_{2} \otimes \cdots \otimes I_{2}$ (where $I_{2}$ is the identity matrix which represents "doing nothing" on qubits 2 to $n$ ) and hence mathematically it becomes an exponentially large unitary operation. Thus because of the tensor product rule we can (somewhat enigmatically) state the principle:
(P1): The physical act of doing nothing on part of an entangled composite system is a highly nontrivial operation. It leads to an exponential information processing benefit if used in conjunction with performing an operation on another (small) part of the system.
Indeed it is difficult to process the quantum information by only a "small amount". Eq. (22) illustrates that any small local operation (addressing a small part of the system) will generally correspond to an exponentially large processing operation from a classical point of view. Intuitively this reflects the denseness of the exponential quantum information stored within the linear resources.

One may object to (P1), claiming that surely the information processing gain arises from the local operation that is actually performed (e.g. $U$ above) rather than from the part that is not performed (e.g. the $(n-1)$ identity operations above)! To see that this is not the case consider our row of $n$ qubits and suppose now that $U$ operates on the first $k$ qubits (so $U$ is a $2^{k} \times 2^{k}$ matrix). Let us compare the number of steps required to perform this transformation in the classical and quantum contexts respectively. It is known that any $d \times d$ unitary matrix may be programmed on a quantum computer in $O\left(d^{2}\right)$ steps [8, 13] so the quantum implementation of $U$ will require $O\left(\left(2^{k}\right)^{2}\right)$ steps. Classically, direct matrix multiplication for a $d \times d$ matrix requires $O\left(d^{2}\right)$ steps. For $U$ we have $d=2^{k}$ and the multiplication must be performed $2^{n-k}$ times. Thus the classical implementation will require $O\left(\left(2^{k}\right)^{2} 2^{n-k}\right)$ steps. Hence the ratio of quantum computing effort to classical computing effort is $O\left(2^{k} / 2^{n}\right)$. This ratio decreases if either $n$ is held fixed and $k$ is decreased, or $k$ is held fixed and $n$ is increased. In either case we are increasing the proportion of "doing nothing" and this is giving rise to an increased information processing benefit.

The Fourier transform is a fundamental ingredient [1], 17, 18] in most of the known quantum algorithms which exhibit a super-classical computational speedup. This includes the algorithms of Deutsch [10], Simon [11], Shor (12, 13] and Grover (15]. Using the mathematical formalism of the fast Fourier transform (FFT) [14], the unitary transformation that is the Fourier transform can be implemented exponentially more efficiently in a quantum context [6] than in any known classical context. For example, for the group of integers modulo $2^{n}$ the classical fast Fourier transform algorithm runs in time $O\left(n 2^{n}\right)$ whereas its quantum implementation runs in time $O\left(n^{2}\right)$. An analysis of the implementation of the FFT algorithm in the quantum context, given in detail in [6], shows that the achieved exponential
speedup may be entirely attributed to the influence of the principle ( $\mathbf{P} \mathbf{1}$ ). This appears to be an essential feature of the speedup exhibited by all known quantum algorithms.

The full (exponentially large) amount of information embodied in the identity of a quantum state $\left|\psi_{n}\right\rangle$ is termed "quantum information". The formalism of quantum mechanics places an extraordinary limitation on the above entanglement-related benefits of quantum information storage and processing: quantum measurement theory implies severe restrictions on the accessibility of the quantum information in the state. For example, according to Holevo's theorem [9] we can obtain at most $n$ bits of information about the identity of an unknown state $\left|\psi_{n}\right\rangle$ of $n$ qubits by any physical means whatever. This bound is the same as the information capacity of a classical system with the same number of levels. Thus, curiously, natural physical evolution in quantum physics corresponds to a super-fast processing of (quantum) information at a rate that cannot be matched by any classical means, but then, most of the processed information cannot be read! It is a remarkable fact that these two effects do not anull each other - the small amounts of information that are possible to obtain about the identity of the final processed state do not coincide with the particular meagre kinds of information processing that can be achieved by classical computation on the input running for a similar length of time. This disparity directly entails the computational speedup possibilities of quantum computation.

## 3 Counterfactual Quantum Computation

We have argued above that the information processing benefits seen in the known quantum algorithms all rest on some specific features of quantum entanglement. However these features do not exhaust all the ways in which quantum physics differs from classical physics. In an effort to find new quantum algorithms we might ask whether other non-classical features of quantum physics may be exploited for novel computational possibilities (not necessarily just a speedup of computation). Quantum measurement theory (c.f. the inaccessibility of quantum information mentioned above) provides further non-classical aspects of the quantum formalism and these are also related to controversial interpretational issues. We will now describe a novel computational possibility which we call "counterfactual quantum computation", based on properties of quantum measurement.

A counterfactual effect may be defined as an observable physical effect E whose outcome depends on an event A that might conceivably have happened but in fact did not happen i.e. E is affected by the mere existence of A as a valid possible alternative even though A did not actually occur. Classical physics does not allow physically observable counterfactual effects but quantum physics does, at least in the sense described below. Their surprising and somewhat paradoxical occurrence in quantum mechanics has been highlighted in Penrose [20] (see especially $\S \S 5.2,5.3,5.7,5.8,5.9,5.18)$.

Suppose that we have a quantum computer which has been programmed ready to solve a decision problem. The computer also has an on/off switch, initially set in position off. We will show that in certain circumstances, the mere fact that the computer would have given the result of the computation if it were run, is sufficient to cause a physically measureable effect from which we can learn the result, even though the computer is in fact not run! Our
method is based on the so-called Elizur-Vaidman bomb testing problem [21] and the essential idea may be clarified by considering the operation of a simple Mach-Zender interferometer, which we discuss first.

Consider the Mach-Zender interferometer as shown in the following diagram.


Here $H 1$ and $H 2$ are beam splitters and $M 1$ and $M 2$ are rigid perfect mirrors. The action of each beamsplitter is taken to be the following (written in terms of the states labelled at $H 2$ ). For horizontal photons

$$
\begin{equation*}
|U\rangle \rightarrow \frac{1}{\sqrt{2}}(|F\rangle+|G\rangle) \tag{3}
\end{equation*}
$$

and for vertical photons

$$
\begin{equation*}
|L\rangle \rightarrow \frac{1}{\sqrt{2}}(|F\rangle-|G\rangle) \tag{4}
\end{equation*}
$$

A photon enters at $|A\rangle$ and is separated into a superposition $\frac{1}{\sqrt{2}}(|L\rangle+|U\rangle)$ of upper and lower paths. In the absence of the measuring instrument $\mathcal{M}$ the two beams coherently interfere at $H 2$ and according to eqs. (3) and (4) the result is $|F\rangle$. Thus the photon is always registered in detector $\mathcal{F}$ and never in detector $\mathcal{G}$.

Consider now a nondestructive measurement device $\mathcal{M}$ placed in the lower arm, which registers whether or not the photon passed along that arm. The initial state of $\mathcal{M}$ is $\left|M_{0}\right\rangle$
and if a photon is registered it becomes an orthogonal state $\left|M_{1}\right\rangle$. Following the photon we now have

$$
\begin{equation*}
|A\rangle \rightarrow \frac{1}{\sqrt{2}}(|U\rangle+|L\rangle)\left|M_{0}\right\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|U\rangle\left|M_{0}\right\rangle+|L\rangle\left|M_{1}\right\rangle\right) \tag{5}
\end{equation*}
$$

and the last state may be thought of as the "collapsed" mixture of $|U\rangle\left|M_{0}\right\rangle$ or $|L\rangle\left|M_{1}\right\rangle$, each with probability half. Thus the interference at $H 2$ is spoilt and we always have a $50 / 50$ probability of registering the photon in either $\mathcal{F}$ or $\mathcal{G}$.

Suppose now that the photon is registered in $\mathcal{G}$ and the measurement instrument is seen to be in state $\left|M_{0}\right\rangle$. (This event occurs with probability $\frac{1}{4}$.) Thus the photon has been registered absent in the lower arm and the measurement instrument, having thus apparently done nothing, remains in state $\left|M_{0}\right\rangle$. Yet the photon is seen at $\mathcal{G}$, which is forbidden in the absence of $\mathcal{M}$ ! Although $\mathcal{M}$ apparently does nothing, it cannot be removed, since then the photon can never register in $\mathcal{G}$. This is our fundamental counterfactual effect: we can say that the photon can be registered in $\mathcal{G}$ because if the photon would have gone along the lower path, it would have been detected, even though it did not, in fact, go along the lower arm (since it was not seen by $\mathcal{M}$ ).

We can use this effect for computational advantage as follows. Consider an idealised quantum computer which is an isolated physical system containing an on/off switch, a set of program/data registers denoted by the state |comp $\rangle$ and an output register. The on/off switch is a two-level system with basis states $\mid$ on $\rangle$ and $\mid$ off $\rangle$ and the output register is a two-level system with basis states $|0\rangle$ and $|1\rangle$. The program/data registers are set up ("programmed") to solve some given decision problem together with its input (e.g. it might be programmed to test for primality together with a given input integer.) The output register, initially in state $|0\rangle$ will be set by the computation to $|0\rangle$ or $|1\rangle$ according to the answer of the decision problem. The length $T$ of the computation is a known function of the input. The time evolution of the computer for time $T$ is given by

$$
\begin{aligned}
\mid \text { on }\rangle|\operatorname{comp}\rangle|0\rangle & \rightarrow \mid \text { on }\rangle \mid \text { comp }\rangle|r\rangle \\
\mid \text { off }\rangle|\operatorname{comp}\rangle|0\rangle & \rightarrow \mid \text { off }\rangle|\operatorname{comp}\rangle|0\rangle
\end{aligned}
$$

Here $r=0$ or 1 is the (initially unknown) result of the computation and the computation will run only if the switch is set to "on". The result is written into the output register and all program/data registers are returned to their initial state.

Heuristically we will relate this scenario to the interferometer as follows. $\mathcal{M}$ is the quantum computer with $\left|M_{0}\right\rangle$ and $\left|M_{1}\right\rangle$ being the states $|0\rangle$ and $|1\rangle$ of the output register. The photon is the on/off switch and the two paths are delayed by a time $T$ for the photon to eventually arrive at $H 2$. Thus if $r=0$ the running of the computation makes no distinction between the paths and the photon is always seen in $\mathcal{F}$. If $r=1$ the computation (if it ran) would distinguish the two paths and we will see the photon at $\mathcal{G}$ with probability $\frac{1}{2}$. As before, with probability $\frac{1}{4}$ the photon will register at $\mathcal{G}$ (so that we are sure that $r=1$ ) and the output register will be seen to be in state $|0\rangle$. Thus the computation has not run, yet we have learnt the result!

More formally in terms of states of the computer, we first set the on/off switch to the
superposition:

$$
\begin{equation*}
\left.\left.\left(\frac{|\mathrm{off}\rangle+|\mathrm{on}\rangle}{\sqrt{2}}\right) \right\rvert\, \text { comp }\right\rangle|0\rangle \tag{6}
\end{equation*}
$$

and then allow time $T$ (the computation time) to elapse yielding the state

$$
\begin{equation*}
\left.\frac{1}{\sqrt{2}}(\mid \text { off }\rangle|\mathrm{comp}\rangle|0\rangle+|\mathrm{on}\rangle|\mathrm{comp}\rangle|r\rangle\right) \tag{7}
\end{equation*}
$$

Next rotate the state of the switch by

$$
\left.\left.\left.\left.\left.\mid \text { off }\rangle \rightarrow \frac{1}{\sqrt{2}}(\mid \text { off }\rangle+\mid \text { on }\right\rangle\right) \quad \mid \text { on }\right\rangle \rightarrow \frac{1}{\sqrt{2}}(\mid \text { off }\rangle-\mid \text { on }\right\rangle\right)
$$

This yields the state

$$
\begin{gather*}
\left.\left.\left.\frac{1}{\sqrt{2}}\left(\left.\frac{(\mid \text { off }\rangle+\mid \text { on }\rangle)}{\sqrt{2}} \right\rvert\, \text { comp }\right\rangle|0\rangle+\frac{(\mid \text { off }\rangle-\mid \text { on }\rangle)}{\sqrt{2}} \right\rvert\, \text { comp }\right\rangle|r\rangle\right) \\
\left.\left.\left.=\frac{1}{\sqrt{2}}(\mid \text { off }\rangle \frac{(|0\rangle+|r\rangle)}{\sqrt{2}}+\mid \text { on }\right\rangle \frac{(|0\rangle-|r\rangle)}{\sqrt{2}}\right) \mid \text { comp }\right\rangle \tag{8}
\end{gather*}
$$

Here $r=0$ or 1 according to the (as yet unknown) result of the computation. Next we measure the switch to see if it is on or off. Note that if $r=0$ then we never see "on" and if $r=1$ we see "on" with probability $1 / 2$. Suppose that we see "on". Then we know that the result of the computation must certainly be $r=1$. We then examine the output register which will show $|0\rangle$ with probability $1 / 2$. If this happens then the computation has not been run (because if it had, then the output register must show $|1\rangle$ ). Overall, if the result is actually $r=1$ then with probability $1 / 4$ we learn the correct result (and know it is correct) with no computation having taken place!

Note that if the actual solution of the decision problem is $r=0$ then we will never ascertain this from the above procedure because if $r=0$ then the output register will always show 0 and the switch will always be finally seen to be "off". But this outcome also arises for $r=1$ with probability $\frac{1}{4}$ and we cannot a posteriori distinguish the two possible causes. Correspondingly, if the actual solution is $r=1$ then with probability $\frac{1}{4}$ will we fail to ascertain this.

The above description of the process represented by eqs. ( 6 ) to ( 8 ) involves some delicate interpretational issues. For example, a many-worlds adherent might object that initially the switch was set in an equal superposition of being on and off, so even in the subsequent case of "no computation taking place" the computer actually did run in another "parallel universe" so we cannot claim to get the result for free. One may, to some extent, counter this objection as follows: suppose that when the result is really $r=1$, the computer is also designed to explode at the end of the computation, if it is run. Then using the above procedure, in $m y$ world I learn that $r=1$ and the computer remains unexploded, available to do another run. I do not really care if it self-destructs in some "other universe"!

The counterfactual quantum computation procedure above may be considerably improved (using a method inspired by the improvements to the Elizur-Vaidman problem given
in [22]) to essentially eliminate the deficencies noted above. As described below, we will achieve the following:
For any given $\epsilon>0$
(i) If the result is $r=0$, we will learn this with probability 1 but some computation will have taken place.
(ii) If the result is $r=1$, we will learn this with probability $1-\epsilon$ with no computation having taken place.

Thus for the many-worlds adherent, the universe in which the computation takes place can be made to occur with arbitrarily small amplitude $O(\sqrt{\epsilon})$ (in the case that $r=1$ ), which considerably weakens his/her/its objection. Recall that many basic results in information theory and computer science are formulated in an asymptotic framework which allows an arbitrarily small failure of some desired property. This occurs for example in the distinction between the complexity classes P and BPP 16] (the latter allowing an arbitrarily small probability of a false result) and Shannon's source coding theorem having not perfect fidelity, but fidelity $1-\epsilon$ (for any $\epsilon>0$ ) for the signals reconstructed from their coded compressed versions. Thus if some undesirable result can be made to occur with arbitrarily small (although non-zero) probability then FAPP it may be ignored. 19]

The improved counterfactual scheme exploits the so-called quantum watched pot effect (or quantum Zeno effect) and it goes as follows. We note first that the state $\mid$ comp $\rangle$ will never become entangled with the other registers so we omit it, writing the action of the computer as

$$
\begin{array}{lll}
\mid \text { off }\rangle|0\rangle & \rightarrow & \mid \text { off }\rangle|0\rangle  \tag{9}\\
\mid \text { on }\rangle|0\rangle & \rightarrow & \mid \text { on }\rangle
\end{array}|r\rangle
$$

Choose an angle $\alpha=\frac{\pi}{2 N}$ for $N$ sufficiently large (c.f. later). Then perform the following five operations:
(a) Rotate the switch by angle $\alpha$.
(b) Allow the running time $T$ to elapse.
(c) Read the output register. If it shows 0 then continue. If it shows 1 then discard the state and start again from the beginning.
Remark. (a) and (b) will result in the evolution

$$
\begin{equation*}
\mid \text { off }\rangle|0\rangle \rightarrow(\cos \alpha \mid \text { off }\rangle+\sin \alpha \mid \text { on }\rangle)|0\rangle \rightarrow \cos \alpha \mid \text { off }\rangle|0\rangle+\sin \alpha \mid \text { on }\rangle|r\rangle \tag{10}
\end{equation*}
$$

If $r=0$ then the output will always show 0 and (c) will result in the state $(\cos \alpha|\mathrm{off}\rangle+$ $\sin \alpha \mid$ on $\rangle)|0\rangle$ with probability 1 . If $r=1$ then (c) will result in the collapsed state $\mid$ off $\rangle|0\rangle$ obtained with (high) probability $\cos ^{2} \frac{\pi}{2 N}$. To complete the procedure we:
(d) Repeat (a), (b) and (c) a further $N-1$ times.
(e) Finally measure the switch to see if it is on or off (assuming that all stages have been kept in (c) and (d)).

We claim that in (e), if the switch is seen to be "on" then $r$ is certainly 0 (and some computation has been done), and if the switch is seen to be "off", then $r$ is certainly 1 and no computation has taken place. In the latter case the probability of keeping all stages is $\left(\cos ^{2} \frac{\pi}{2 N}\right)^{N}$ which tends to 1 as $N \rightarrow \infty$. Thus by choosing $N$ to be sufficiently large we can make the probability of success greater than $1-\epsilon$ for any given $\epsilon$.

To see that our claim is correct, note that if $r=0$ then the switch is just successively rotated from $|\mathrm{off}\rangle$ to $|\mathrm{on}\rangle$ in $N$ stages and it never entangles with the output register. If $r=1$ then the state is repeatedly collapsed to $\mid$ off $\rangle|0\rangle$ so that no computation takes place in any stage (because if it did, the output register would show the result $r=1$ ). Indeed the waiting in (b) acts as a measurement of "on" versus "off" for the switch (if $r=1$ ) and in this case, we are just freezing the switch in its |off $\rangle$ state by frequent repeated measurement. This is the quantum watched pot effect.

Note that according to (i), if $r=0$ then this result is not learnt "for free". A natural question is whether or not there is a counterfactual scheme which yields the information of either result ( $r=0$ or 1 ) with no computation having taken place. The procedure described above may be readily modified to provide a scheme with the following properties: with probability $1-\epsilon$ we learn the result and for either outcome, be it $r=0$ or $r=1$, it is obtained for "free" with probability $\frac{1-\epsilon}{2}$. We also learn whether or not the produced result was obtained for "free". It remains an open question whether or not each of the two results may be obtained for "free" with high probability $1-\epsilon$, or indeed, whether the sum of these two probabilities can be made to exceed 1 .

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