Terms and infinite trees as monads over a signature

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Abstract
In this paper, we prove that the usual construction of terms and infinite trees over a signature is a particular case of a more general construction of monads over the category of sets. In this way, we obtain a family of semantical domains having a tree-like structure and appearing as the completions of the corresponding finite structures. Though it is quite different in its technical developments our construction should be compared with the one of De Bakker and Zucker which is very similar in spirit and motivation. We feel that one outcome of the present approach is that, due to its connection with Lawvere's algebraic theories, it should provide an interesting framework to deal with equational varieties of process algebras.

1 Introduction

A major concern in denotational semantics is to provide a meaning to recursively defined expressions ; then, we must be able to express and to solve systems of equations in various domains. Such a requirement leads us to supply those domains with three fundamental algebraic operations, namely builing up t-uple of elements, extracting an element from such a t-uple and making substitution of a t-uple of elements for a t-uple of variables. Lawvere's algebraic theories ([Law63]) are categories that embody those basic manipulations in a uniform manner. An alternative presentation of algebraic theories is that of monad (or following Manes [Man76] *algebraic theories on monoid form*). Monads found applications in automata theory ([AM75]), tree processing ([Ala75]) they also have been used to study recursion schemes ([BG87]) and more recently Petri Nets ([MM88]) .

In both presentation of algebraic theories term substitution plays a central role and allows for the statement of equations (we shall make this statement precise in the case of monads). In analogy to Elgot's iterative algebraic theories we may define the iterative monads as those monads for which all non degenerate equations (such as x=x) admits a unique solution.

In this paper, we prove that the usual construction of terms and infinite trees over a signature Σ is a particular case of a more general construction of monads over the category of sets. Moreover, the monad corresponding to the infinite trees is proved iterative ; and, as such, constitute a potential semantical domain. In this way, we obtain a family of iterative monads associated to an extended notion of signature called **ω-signature.** As typical example we present the monad of synchronization trees ([Mil80]) but other similar examples of algebraic structures may be considered. Roughly speaking such structures are tree-like and the infinite objects are obtained through a completion process from the finite ones. Though quite different in its technical developments our present construction should be compared with the one of De Bakker and Zucker in [dBZ82]. Actually, in that paper they gave a variety of process domains having such a branching structure and enabling one to deal with various concepts arising in the semantics of concurrency such as *parallel composition*, *synchronization* and *communication*. The novelty of the present approach, compared with De Bakker and Zucker's, is that, due to its connection with Lawvere's algebraic theories, it should moreover provide an interesting framework to deal with equational varieties of process algebras.

Concerning monads we refer to the books *Algebraic Theories* ([Man76]) by E. Manes and *Toposes, Triples and Theories* ([BW85]) by M. Barr and C. Wells ; nevertheless we shall recall along this paper all definitions and results we need about monads. The reader is just expected familiar with the basic notions of category theory such as limit and colimit of diagrams, functor categories and adjunctions.

This paper is organized as follows : section 2 gives a construction of terms and infinite trees for a signature, a signature being there some endofunctor of the category **Set**, providing a generalization of the usual construction where the signature corresponds to a ranked alphabet. In sections 3 and 4 respectively we supply the sets of terms and infinite trees with substitution operations by forming the corresponding monads ; moreover the monad of infinite trees is proved iterative. Section 5 is the conclusion.

2 Terms and infinite trees for a signature

We first recall some definitions from universal algebra. A (finitary) signature or ranked alphabet Σ is given by a set whose elements are operator symbols together with a mapping $a : \Sigma \rightarrow \mathbf{N}$ which assigns to each operator f a natural number a(f) called its arity. We denote Σ_n the set of operators of arity n. If Σ' is another signature, a morphism $\varphi : \Sigma \rightarrow \Sigma'$ of signatures is any mapping from Σ to Σ' that preserves the arities i.e. $f \in \Sigma_n \Rightarrow \varphi(f) \in \Sigma'_n$.

Definition 1 *a Σ-algebra is a pair (D,δ) where D is a non empty set (the carrier or domain of the algebra) and $\delta = \{\delta_f \,;\, f \in \Sigma\}$ is a set of functions $\delta_f : D^{a(f)} \rightarrow D$ associated to each operator f in Σ. And a morphism φ between two Σ-algebras*

(D,δ) and (D',δ') is any mapping between their respective domains such that :

$$\varphi(\delta_f(a_1,\ldots,a_n)) = \delta'_f(\varphi(a_1),\ldots,\varphi(a_n))$$

A ranked alphabet Σ can be interpreted as the functor $\Sigma : \mathbf{Set} \rightarrow \mathbf{Set}$ defined on objects and on arrows as

$$\Sigma A = \amalg_{f\in\Sigma} A^{a(f)} \quad and \quad \Sigma\varphi = \amalg_{f\in\Sigma} \varphi^{a(f)}$$

a Σ-algebra is then any mapping $\Sigma A \xrightarrow{a} A$ and a morphism between two Σ-algebras (A,a) and (B,b) is any mapping $\varphi : A \rightarrow B$ such that : $\varphi \circ a = b \circ \Sigma\varphi$. Σ-algebras and their morphisms constitute a category denoted Σ-**Alg**. Since the coproduct of sets are their disjoint union we get a more intuitive representation of ΣA as $\Sigma A = \{ f[< a_1 >,\ldots,< a_n >] \,/\, f \in \Sigma_n$ and $a_1,\ldots,a_n \in A\}$ (where <,>,[,] are special symbols) and, in this way, $\Sigma\varphi(f[< a_1 >,\ldots,< a_n >]) = f[< \varphi(a_1) >,\ldots,< \varphi(a_n) >]$.

More generally, if F is an endofunctor in a category $\mathcal{C}$ we let F-**Alg**, F-co-**Alg** and F-**fp** denote the categories whose objects are respectively the F-algebras (i.e. arrows $FA \xrightarrow{a} A$), F-co-algebras (i.e. arrows $A \xrightarrow{a} FA$) and F-fixed-points (i.e. isomorphisms $FA \xrightarrow{a} A$) and whose arrows from (A,a) to (B,b) are those arrows $\varphi : A \rightarrow B$ such that $\varphi \circ a = b \circ F\varphi$ for algebras and fixed-points and $F\varphi \circ a = b \circ \varphi$ for co-algebras.
Our purpose, in this section, is to provide a generalization of the construction of terms and infinite trees corresponding to a signature, a signature being now some endofunctor of the category **Set**. For this, to any endofunctor Σ of **Set** we associate an endofunctor $\mathcal{F}$ of the functor category $\mathcal{C} = Func(\mathbf{Set},\mathbf{Set})$. $\mathcal{F}$ is the endofunctor of $\mathcal{C}$ defined on objects (functors F) and on arrows (natural transformations τ) by : [1]

$$\mathcal{F}F = 1_{\mathcal{C}} + \Sigma F \quad and \quad (\mathcal{F}\tau)_X = 1_X + \Sigma\tau_X$$

We recall the category of sets admits initial and terminal elements being respectively the empty set $\emptyset$ and any singleton set, say $\{\Omega\}$. Let 0 and 1 denote the constant endofunctors of **Set** whose respective values are $\emptyset$ and $\{\Omega\}$. They are the initial and terminal objects of $\mathcal{C}$, moreover $\mathcal{C}$ is, as the category **Set**, complete and co-complete. Let Δ and ∇ be the following chains on $\mathcal{C}$:

$$\Delta = 0 \xrightarrow{\varphi_0} \mathcal{F}0 \xrightarrow{\mathcal{F}\varphi_0} \mathcal{F}^2 0 \ldots \mathcal{F}^n 0 \xrightarrow{\mathcal{F}^n\varphi_0} \mathcal{F}^{n+1}0 \ldots$$

$$\nabla = 1 \xleftarrow{\psi_0} \mathcal{F}1 \xleftarrow{\mathcal{F}\psi_0} \mathcal{F}^2 1 \ldots \mathcal{F}^n 1 \xleftarrow{\mathcal{F}^n\psi_0} \mathcal{F}^{n+1}1 \ldots$$

(where φ_0 and ψ_0 are uniquely defined by universality of 0 and 1)
and let $(T,\jmath) = colim(\Delta)$ and $(T^\infty,\pi) = lim(\nabla)$ be their colimiting cone and

[1] If x is an object in a category we let 1_x stands for the identity arrow in x ; in particular $1_{\mathcal{C}}$ is the identity functor from $\mathcal{C}$ to itself.

limiting cone respectively. Concerning limits and colimits in functor categories we recall from [Sch72] the following result : if $\mathcal{D}$ is complete (resp. co-complete) the functor category $Func(\mathcal{C},\mathcal{D})$ is also complete (co-complete) and the constructioon is *pointwise* which means that for every object c of $\mathcal{C}$ the evaluation functor $E_c : Func(\mathcal{C},\mathcal{D}) \to \mathcal{D}$ defined by $E_c(F) = F(c)$ and $E_c(\tau) = \tau_c$ preserves the limits (co-limits). It follows that $(TX, \jmath_X) = colim(\Delta X)$ and $(T^\infty X, \pi_X) = lim(\nabla X)$ where ΔX and ∇X stands for the chains gotten by evaluating Δ and ∇ at X. [2]
We observe that :

$$\Delta X = \emptyset \xrightarrow{\varphi_{0,X}} \mathcal{F}_X\emptyset \xrightarrow{\mathcal{F}_X(\varphi_{0,X})} \mathcal{F}_X^2\emptyset \ldots \mathcal{F}_X^n\emptyset \xrightarrow{\mathcal{F}_X^n(\varphi_{0,X})} \mathcal{F}_X^{n+1}\emptyset \ldots$$

$$\nabla X = \{\Omega\} \xleftarrow{\psi_{0,X}} \mathcal{F}_X\{\Omega\} \xleftarrow{\mathcal{F}_X(\psi_{0,X})} \mathcal{F}_X^2\{\Omega\} \ldots \mathcal{F}_X^n\{\Omega\} \xleftarrow{\mathcal{F}_X^n(\psi_{0,X})} \mathcal{F}_X^{n+1}\{\Omega\} \ldots$$

where $\mathcal{F}_X$ is the endofunctor of **Set** defined on sets and mappings as :

$$\mathcal{F}_X Y = X + \Sigma Y \quad \mathcal{F}_X\varphi = 1_X + \Sigma\varphi$$

Now let us have a look to TX and T^∞X in the particular case where Σ is the functor associated to a ranked alphabet. First, we notice that in the chain

$$\Delta X = \emptyset \xrightarrow{\varphi_{0,X}} X + \Sigma\emptyset \xrightarrow{\varphi_{1,X}} X + \Sigma(X + \Sigma\emptyset) \xrightarrow{\varphi_{2,X}} \ldots$$

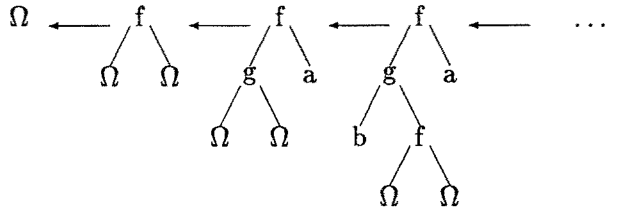
each set is an actual subset of its follower and the mappings $\varphi_{i,X}$ are the inclusion maps. The colimit TX is then their set-theoretic union i.e. the *least* set (regarding inclusion) containing X and such that if $f \in \Sigma_n$ and $t_1, \ldots, t_n \in TX$ then $f[t_1, \ldots, t_n]$ is also an element of TX. We then meet the usual definition of the set of terms corresponding to a signature. Now, as $T^\infty X$ is the limit of the chain

$$\nabla X = 1 \xleftarrow{\psi_0} X + \Sigma 1 \xleftarrow{\psi_1} X + \Sigma(X + \Sigma 1) \xleftarrow{\psi_2} \ldots$$

an element of $T^\infty X$ is a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{F}_X^n 1$ and for every integer n one has $u_n = \psi_{n,X}(u_{n+1})$. Note that the limiting cone $\pi_X : T^\infty X \to \nabla X$ satisfies the following :

$$\begin{aligned} \pi_{X,0}(t) &= \Omega \\ \pi_{X,n+1}(< x >) &= < x > \\ \pi_{X,n+1}(f[t_1, \ldots, t_n]) &= f[\pi_{X,n}(t_1), \ldots, \pi_{X,n}(t_n)] \end{aligned}$$

We then have the usual construction of infinite tree, where a tree is represented by the sequence of its n^{th}-sections.



[2] There is a slight abuse of notation justified by the isomorphisms $Func(\omega, Func(\mathbf{Set},\mathbf{Set})) \cong Func(\mathbf{Set}, Func(\omega,\mathbf{Set}))$ and $Func(\omega^{op}, Func(\mathbf{Set},\mathbf{Set})) \cong Func(\mathbf{Set}, Func(\omega^{op},\mathbf{Set}))$

We usually assume that Σ owns at least one operator of arity 0 otherwise the set of closed terms (i.e. $T\emptyset$) should be empty ; under this hypothesis we may prove that the set of term TX is a dense (in a topological sense) subset of $T^{\infty}X$. This result will generalize if we make some additional assumptions ; those considerations lead to the following definition that summarize the hypothesis on Σ which are necessary so as to make our construction work.

Definition 2 *A signature Σ is any endofunctor of* **Set** *such that for any set X, ΣX is a non-empty set ; and, if X is a subset of Y with inclusion map $i : X \to Y$, ΣX is a subset of ΣY with inclusion map Σi.*

3 The term monad over an ω-signature

As we stressed in the introduction, in both presentations of algebraic theories term substitution plays a central role ; it is modelled by composition in Lawvere's algebraic theories and by a natural transformation (called structure map : μ) in a monad . Incidently, another natural transformation (called embedding of generators : η) allows substitutions to take place without explicit mention to variables.

Definition 3 *a monad on a category $\mathcal{C}$ is a triple (T, η, μ) where T is an endofunctor on $\mathcal{C}$ and $\eta : I \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ are natural transformations (I is the identity endofunctor of $\mathcal{C}$) satisfying the following commuting diagrams :*

$$\begin{tikzcd} TTT \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

Those axioms, that should be compared with the axioms of a monoid (the associative law and the two unit laws), provide the minimal conditions expected for a substitution operation namely being associative and well-behaved regarding the embedding of elements.

In order to supply T and T^{∞} with monad structures, we make the additional assumption that the functor $\mathcal{F}$, associated to Σ, is both ω-co-continuous and ω^{op}-continuous ; we shall say, in such a case, that Σ is an ω-**signature**. We know (see [SP82]) that, under those hypothesis, we obtain an initial $\mathcal{F}$-algebra and a terminal $\mathcal{F}$-co-algebra whose respective carriers are T and T^{∞}. More precisely, $\Phi : \Delta \to \mathcal{F}T$ defined by $\Phi_n = \mathcal{F}j_n \circ \mathcal{F}^n\varphi_0$ is a co-cone ; let φ be the mediating arrow $\Phi = \varphi \circ j$; thanks to ω-co-continuity of $\mathcal{F}$ φ is an isomorphism and (T, φ^{-1}) is the initial $\mathcal{F}$-algebra. In the same way, we obtain $\psi : \mathcal{F}(T^{\infty}) \to T^{\infty}$ making (T^{∞}, ψ^{-1}) the terminal $\mathcal{F}$-co-algebra ; moreover, (T, φ^{-1}) and (T^∞, ψ) are, as well, the initial and terminal elements of the category of $\mathcal{F}$-fixed points. Now, for our particular case, we observe that $\varphi^{-1} : \mathcal{F}T = I + \Sigma T \longrightarrow T$ splits into $\varphi^{-1} = [\eta, \sigma]$ where $\eta : I \dot{\rightarrow} T$ and $\sigma : \Sigma T \dot{\rightarrow} T$. In the same way $\psi = [\eta^\infty, \sigma^\infty]$ where $\eta^\infty : I \dot{\rightarrow} T^\infty$ and $\sigma^\infty : \Sigma T^\infty \dot{\rightarrow} T^\infty$. Since all previous constructions were made componentwise, $(TX, [\eta_X, \sigma_X])$ and $(T^\infty X, [\eta_X^\infty, \sigma_X^\infty])$ are actually the respective initial $\mathcal{F}_X$-algebra and terminal $\mathcal{F}_X$-co-algebra. So

$$\boxed{(TX, [\eta_X, \sigma_X]) \textit{ is the initial } \mathcal{F}_X \textit{ algebra}}$$

Spelled out, for any $\mathcal{F}_X$-algebra $(Y, [\alpha, \beta])$ there exists a unique mapping $\psi : T_X \rightarrow Y$ such that

$$\begin{tikzcd} X + \Sigma TX \arrow[r, "{[\eta_X, \sigma_X]}"] \arrow[d, "1_X + \Sigma\psi"'] & TX \arrow[d, "\psi"] \\ X + \Sigma Y \arrow[r, "{[\alpha, \beta]}"] & \mathrm{Y} \end{tikzcd}$$ commutes.

In other words, splitting this diagram into two parts, for a given set Y and a couple of mappings $\alpha : X \rightarrow Y$ and $\beta : \Sigma Y \rightarrow Y$ there exists a unique mapping $\psi : T_X \rightarrow Y$ such that the two following diagrams commute.

$$\begin{tikzcd} \mathrm{X} \arrow[r, "\eta_X"] \arrow[dr, "\alpha"'] & TX \arrow[d, "\psi"] \\ & \mathrm{Y} \end{tikzcd} \qquad \begin{tikzcd} \Sigma TX \arrow[r, "\sigma_X"] \arrow[d, "\Sigma\psi"'] & TX \arrow[d, "\psi"] \\ \Sigma\,\mathrm{Y} \arrow[r, "\beta"] & \mathrm{Y} \end{tikzcd}$$

So we have an adjunction **Set** $\overset{<F,U>}{\longleftrightarrow}$ Σ-**Alg** where U is the forgetful functor ; F sends X to the Σ-algebra (TX, σ_X) and is defined on arrows as follows : given a mapping $\varphi : X \rightarrow Y$, $F\varphi$ is the unique morphism of Σ-algebras from (TX, σ_X) to (TY, σ_Y) such that $UF\varphi \circ \eta_X = \eta_Y \circ \varphi$. Since $T\varphi$ is the underlying mapping of a Σ-algebra morphism from (TX, σ_X) to (TY, σ_Y) (by naturality of σ) and that $T\varphi \circ \eta_X = \eta_Y \circ \varphi$ (by naturality of η) it follows that $T\varphi = UF\varphi$; and then $T = UF$. Now, we know (see, for example [BW85]) that to each adjunction (U, F, η, ϵ) (where F is left adjoint to U with unit η and counit ϵ) corresponds the monad $(UF, \eta, U\epsilon F)$. So if we let $\mu = U\epsilon F$, the triple $(T = UF, \eta, \mu)$ so obtained is a monad ; we shall call it **the term monad over the signature** Σ.
As particular case if (Y,y) is a Σ-algebra, there exists a unique T-algebra (Y, y^*) extending it in the following sense

$$
\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[dr, "1_Y"'] & TY \arrow[d, "y^*"] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma TY \arrow[r, "\sigma_Y"] \arrow[d, "\Sigma y^*"'] & TY \arrow[d, "y^*"] \\
\Sigma Y \arrow[r, "y"] & Y
\end{tikzcd}
$$

We shall call y^* the **inductive extension** of y.
We recall that a **T**-algebra is a T-algebra such that evaluation commutes with substitution ; more precisely

Definition 4 *If* $\mathbf{T}=(T,\eta,\mu)$ *is a monad on a category* $\mathcal{C}$ *a* **T**-*algebra is a pair (A,a) where A is an object of* $\mathcal{C}$ *and* $a : TA \rightarrow A$ *, an arrow (its structure map) such that both following diagrams commute.*

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "a"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
TTA \arrow[r, "\mu_A"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"] & A
\end{tikzcd}
$$

As for Σ-algebras, morphisms of **T**-algebras (A,a) and (B,b) are any arrow $\varphi : A \rightarrow B$ such that $\varphi \circ a = b \circ T\varphi$. **T**-algebras and their morphisms constitute a category denoted $\mathcal{C}^{\mathbf{T}}$; it is the Eilenberg-Moore category associated to **T**. Now we prove the

Proposition 1 *for any set Y,* $\mu_Y = U\epsilon FY : TTY \dot{\rightarrow} TY$ *is the inductive extension of* $\sigma_Y : \Sigma TY \dot{\rightarrow} TY$*. Moreover, if (Y,y) is a* Σ*-algebra then* (Y,y^*) *is a* **T**-*algebra.*

Proof
A set X and a Σ-algebra $\mathcal{Y}$=(Y,y) being given, let us denote $g^\sharp$: FX $\rightarrow \mathcal{Y}$ the morphism of Σ-algebras corresponding, via the adjunction, to the mapping $g : X \rightarrow Y$.

$$
\boxed{
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & TX \arrow[d, "U(g^\sharp)"] \arrow[dr, "Tg"] & \\
& Y & TY \arrow[l, "U\epsilon\mathcal{Y}"]
\end{tikzcd}
}
\quad
\boxed{
\begin{tikzcd}
FX = (TX,\sigma_X) \arrow[d, "g^\sharp"'] \arrow[dr, "Fg"] & \\
(Y,y) = \mathcal{Y} & (TY,\sigma_Y) = FU\mathcal{Y} \arrow[l, "\epsilon\mathcal{Y}"]
\end{tikzcd}
}
$$

The component of the counit $\epsilon : FU \dot{\rightarrow}$ I of the adjunction in $\mathcal{Y}$ satisfies : $g^\sharp = \epsilon\mathcal{Y} \circ Fg$ and thus $Ug^\sharp = U\epsilon\mathcal{Y} \circ Tg$. Now if we take X=Y and $g = 1_Y$ we obtain the inductive extension of y as $y^* = Ug^\sharp = U\epsilon\mathcal{Y}$ and, in the particular case where $\mathcal{Y} = (TY,\sigma_Y) = FY$, we have $\sigma_Y^* = U\epsilon FY = \mu_Y$.
Now applying the naturality of the co-unit to the Σ-morphism $1_Y^\sharp : (TY,\sigma_Y) \rightarrow \mathcal{Y}$ leads to $1_Y^\sharp \circ \epsilon_{(TY,\sigma_Y)} = \epsilon_{\mathcal{Y}} \circ FU(1_Y^\sharp)$ and then, by applying U, $y^* \circ U\epsilon_{FTY} = U\epsilon_{\mathcal{Y}} \circ Ty^*$ i.e. $y^* \circ \mu_{TY} = y^* \circ Ty^*$. $\square$

Remark : On one hand a **T**-algebra is a Σ-algebra, more precisely we have a natural transformation $\tau = \sigma \circ \Sigma\eta : \Sigma \dot{\longrightarrow} T$ and, corresponding to it a functor $U^\tau : \mathbf{Set}^\mathbf{T} \longrightarrow \Sigma\text{-}\mathbf{Alg}$ defined by $U^\tau(A,a) = \Sigma A \xrightarrow{\tau_A} TA \xrightarrow{a} A$ and $U^\tau(f) = f$ on **T**-algebras and arrows respectively. On the other hand we just prove that if (A,a) is a Σ-algebra its inductive extension is a **T**-algebra ; actually we have a functor called the **Eilenberg Moore comparison functor** $\phi : \Sigma\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}^\mathbf{T}$ defined by $\phi(A,a) = (A,a^*)$ and $\phi(f) = f$. Those two functors are inverse isomorphisms (as readily verified). Axioms for **T**-algebras impose evaluation to respect compositionality and that is why a **T**-algebra is the same data as a Σ-algebra.

We can summarize our results in the following

Proposition 2 *Let Σ be a signature, the forgetful functor $U : \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}$ has a left adjoint F. We define the term monad over the signature Σ as the monad (T,η,μ) which results from that adjunction. If $FX = (TX, \sigma_X)$ the arrows σ_X are the components of a natural transformation $\sigma : \Sigma T \dot{\to} T$ and the component of μ at a set X is the unique arrow making both following diagrams commute.*

$$\begin{tikzcd}
TX \arrow[r, "\eta_{TX}"] \arrow[dr, "1_{TX}"'] & TTX \arrow[d, "\mu_X"] & \Sigma TTX \arrow[r, "\sigma_{TX}"] \arrow[d, "\Sigma\mu_X"'] & TTX \arrow[d, "\mu_X"] \\
& TX & \Sigma TX \arrow[r, "\sigma_X"] & TX
\end{tikzcd}$$

This provides an inductive definition *of μ.*

When Σ is the functor associated to a ranked alphabet, this inductive definition is equivalent to :

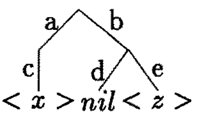
$$\begin{aligned}
\mu_X(< t >) &= t \\
\mu_X(f[t_1,\ldots,t_n]) &= f[\mu_X(t_1),\ldots,\mu_X(t_n)]
\end{aligned}$$

Another example is the following. If A is a set of actions we define ΣX as the set of all finite subsets of $A \times X$, it extends into a functor (we define Σ on mappings by extension). The corresponding monad of term is (S_A, η, μ) where the set of synchronization trees $S_A(X)$ is the least set verifying :

- $x \in X \implies < x > \in S_A(X)$
- any *finite* subset of $A \times S_A(X)$ is an element of $S_A(X)$
  (in particular the empty set is a synchronization tree sometimes denoted *nil*)

$S_A(X)$ is the set of finitely branching, non-deterministic and unordered finitary trees whose branches are labelled by elements in A and leaves by elements in X. For example :

$$\{< a, \{< c, < x >>\} >; < b, \{< d, \emptyset >; < e, < z >>\} >\} \quad = $$



Its mapping function is defined for $\varphi : X \to Y$ by $S_A(\varphi)(< x >) = < \varphi(x) >$ and if
$T = \{< a_i, T_i > \ / \ i \in I\} \subset S_A(X)$ then $S_A(\varphi)(T) \ = \ \{< a_i, S_A(\varphi)(T_i) > \ / \ i \in I\}$
The embedding of generators $\eta_X : X \to S_A(X)$ is given by : $\eta_X(x) = < x >$
and the structure map $\mu_X : S_A(S_A(X)) \to S_A(X)$ by $\mu_X(< t >) \ = \ t$ and if
$T = \{< a_i, T_i > \ / \ i \in I\} \subset S_A(S_A(X))$ then $\mu_X(T) \ = \ \{< a_i, \mu_X(T_i) > \ / \ i \in I\}$

Since the forgetful functor $U : \Sigma$-**Alg** $\longrightarrow$ **Set** has a left adjoint an ω-signature is a particular case of **input process** as defined by Manes ([Man76]) ; the following proposition then follows from the similar and more general result on input processes. (see also [Ala75])

Proposition 3 *the term monad (T, η, μ) is the free monad generated by Σ ; that is to say, the natural transformation $\tau = \sigma \circ \Sigma\eta : \Sigma \dot{\to} T$ is such that for every monad (T', η', μ') and natural transformation $\lambda : \Sigma \dot{\longrightarrow} T'$ there exists a unique monad morphism φ such that $\lambda = \varphi \circ \tau$.*

4 The monad of infinite trees

Concerning T^∞ we already have an embedding of generators (η^∞) it remains to define the structure map : $\mu^\infty : T^\infty T^\infty \dot{\longrightarrow} T^\infty$. For this, we define a morphism of ω^{op}-chains $\xi_X : \nabla T^\infty X \longrightarrow \nabla X$ as follows :

$$\begin{tikzcd}
\{\Omega\} \arrow[d, "\xi_{X,0}"'] & T^\infty X + \Sigma\{\Omega\} \arrow[l, "\psi_{T^\infty X,0}"'] \arrow[d, "\xi_{X,1}"] & \cdots \arrow[l] & (\nabla T^\infty X)_n \arrow[d, "\xi_{X,n}"] & T^\infty X + \Sigma(\nabla T^\infty X)_n \arrow[l, "\psi_{T^\infty X,n}"'] \arrow[d, "{[\pi_{X,n+1}\ ;\ \Sigma\xi_{X,n}]}"] \\
\{\Omega\} & X + \Sigma\{\Omega\} \arrow[l, "\psi_{X,0}"] & \cdots \arrow[l] & (\nabla X)_n & X + \Sigma(\nabla X)_n \arrow[l, "\psi_{X,n}"]
\end{tikzcd}$$

- $\xi_{X,0} = 1_{\{\Omega\}}$ is the identity mapping on $\{\Omega\}$
- for every integer n : $\xi_{X,n+1} = [\pi_{X,n+1} \ ; \ \Sigma\xi_{X,n}]$.

proving that ξ_X is a morphism of ω^{op}-chains amounts to proving that each elementary square commutes which is an easy verification.
We then define $\mu^\infty_X : T^\infty T^\infty X \longrightarrow T^\infty X$ as the mediating morphism between the cone $\xi_X \circ \pi_{T^\infty X} : T^\infty T^\infty \longrightarrow \nabla X$ and the limiting cone $\pi_X : T^\infty X \longrightarrow \nabla X$. ξ_X is clearly natural in X, the naturality of μ^∞ then follows from the fact that the components of π are limiting cones and consequently left-cancelable arrows of the category $Func(\omega^{op}, \mathbf{Set})$.

$$\begin{tikzcd}
\nabla T^\infty X \arrow[rrr, "\xi_X"] \arrow[ddd, "\nabla T^\infty f"'] & & & \nabla X \arrow[ddd, "\nabla f"] \\
& T^\infty T^\infty X \arrow[ul, "\pi_{T^\infty X}"'] \arrow[r, "\mu^\infty_X"] \arrow[d, "T^\infty T^\infty f"'] & T^\infty X \arrow[ur, "\pi_X"] \arrow[d, "T^\infty f"] & \\
& T^\infty T^\infty Y \arrow[dl, "\pi_{T^\infty Y}"] \arrow[r, "\mu^\infty_Y"] & T^\infty Y \arrow[dr, "\pi_Y"'] & \\
\nabla T^\infty Y \arrow[rrr, "\xi_Y"] & & & \nabla Y
\end{tikzcd}$$

And we prove :

Proposition 4 *$(T^\infty, \eta^\infty, \mu^\infty)$ is a monad.*

In order to establish this result, a first stage consists in verifying by diagram chasing the following lemma.

Lemma 1 *For every set X the following three diagrams (that we shall name respectively $monad_1(X)$, $monad_2(X)$ and $monad_3(X)$) commute.*

$$\begin{tikzcd}
\Sigma T^\infty T^\infty X \arrow[r, "\sigma^\infty_{T^\infty X}"] \arrow[d, "\Sigma\mu^\infty_X"'] & T^\infty T^\infty X \arrow[d, "\mu^\infty_X"] \\
\Sigma T^\infty X \arrow[r, "\sigma^\infty_X"] & T^\infty X
\end{tikzcd}
\qquad
\begin{tikzcd}
T^\infty X \arrow[r, "\eta^\infty_{T^\infty X}"] \arrow[dr, "1_{T^\infty X}"'] & T^\infty T^\infty X \arrow[d, "\mu^\infty_X"] & T^\infty X \arrow[l, "T^\infty\eta^\infty_X"'] \arrow[dl, "1_{T^\infty X}"] \\
& T^\infty X &
\end{tikzcd}$$

Proof :
Consider the diagram (in $Func(\omega^{op}, \mathbf{Set})$)

$$\begin{tikzcd}
T^\infty X + \Sigma T^\infty T^\infty X \arrow[rr, "{[\eta^\infty_{T^\infty X}, \sigma^\infty_{T^\infty X}]}"] \arrow[dr, "{[1_{T^\infty X}, \sigma^\infty_X \circ \mu^\infty_X]}"'] \arrow[ddd, "1_{T^\infty X} + \Sigma\pi_{T^\infty X}"'] & {} \arrow[d, phantom, "(4)"] & T^\infty T^\infty X \arrow[dl, "\mu^\infty_X"] \arrow[ddd] \\
{} \arrow[dr, phantom, "(3)"] & T^\infty X \arrow[d, "\pi_X"] \arrow[dr, phantom, "(2)"] & \\
& \nabla X \arrow[d, phantom, "(1)"] & {} \\
T^\infty X + \Sigma \nabla T^\infty X \arrow[ur, "{[\pi_X, \psi_X \circ \Sigma\xi_X]}"] \arrow[rr, "\psi_{T^\infty X}"'] & {} & \nabla T^\infty X \arrow[ul, "\xi_X"']
\end{tikzcd}$$

Since $\psi_{X,n} \circ \pi_{X,n+1} = \pi_{X,n}$ it follows $[\pi_{X,n}; \psi_{X,n} \circ \Sigma\xi_{X,n}] = \psi_{X,n} \circ [\pi_{X,n+1}; \Sigma\xi_{X,n}] = \xi_{X,n} \circ \psi_{T^\infty X,n}$ i.e. (1) commutes. (2) commute by definition of μ^∞ and since $\psi_X \circ \Sigma\pi_X = \pi_X \circ \sigma_X^\infty$

$$
\begin{aligned}
[\pi_X; \psi_X \circ \Sigma\xi_X] \circ (1_{T^\infty X} + \Sigma\pi_{T^\infty X}) &= [\pi_X; \psi_X \circ \Sigma\xi_X \circ \pi_{T^\infty X})] \\
&= [\pi_X; \psi_X \circ \Sigma(\pi_X \circ \mu_X^\infty)] \\
&= [\pi_X; \pi_X \circ \sigma_X^\infty \circ \Sigma\mu_X^\infty] \\
&= \pi_X \circ [1_{T^\infty X}; \sigma_X^\infty \circ \Sigma\mu_X^\infty]
\end{aligned}
$$

i.e. (3) commutes. Since diagrams (1), (2) and (3) commute, as well as the outer rectangle and because π_X is a left-cancellable arrow in $Func(\omega_{op}, \mathbf{Set})$ (as a limiting cone) it follows that diagram (4) commutes i.e. $monad_1(X)$ and $monad_2(X)$ commute. Now, consider the following diagram :

$$
\begin{tikzcd}
T^\infty X \arrow[rr, "1_{T^\infty X}"] \arrow[dr, "T^\infty\eta_X^\infty"'] \arrow[dddd, "\pi_X"'] \arrow[rr, phantom, shift right=3, "monad_3(X)"] & & T^\infty X \arrow[dddd, "\pi_X"] \\
& T^\infty T^\infty X \arrow[ur, "\mu_X^\infty"'] \arrow[dd, "\pi_{T^\infty X}"] \arrow[d, phantom, shift right=8, "(7)"] \arrow[d, phantom, shift left=8, "(6)"] & \\
& \ & \\
& \nabla T^\infty X \arrow[dr, "\xi_X"] \arrow[d, phantom, "(5)"] & \\
\nabla X \arrow[ur, "\nabla\eta_X^\infty"] \arrow[rr, "\nabla 1_X"'] & \ & \nabla X
\end{tikzcd}
$$

We prove $\xi_X \circ \nabla\eta_X^\infty = \nabla 1_X$ by induction, (6) commutes by definition of μ^∞ ; (7) and the outer rectangle commute by naturality of π. As previously, it follows that the upper triangle (i.e. $monad_3(X)$) commutes.

□

Now,to deduce the associativity law from $monad_1$ we shall need some topological arguments expressing that TX is (in a topological sense) a dense subset of $T^\infty X$. Firstly we need to define the embedding of TX into $T^\infty X$; since $(T, [\eta, \mu])$ and $(T^\infty, [\eta^\infty, \mu^\infty])$ are the respective initial and terminal $\mathcal{F}$-fixed points we know there exists a unique natural transformation $\alpha : T \stackrel{\cdot}{\longrightarrow} T^\infty$ such that

$$
\begin{tikzcd}
X + \Sigma TX \arrow[r, "{[\eta_X, \sigma_X]}"] \arrow[d, "1_X + \Sigma\alpha_X"'] & TX \arrow[d, "\alpha_X"] \\
X + \Sigma T^\infty X \arrow[r, "{[\eta_X^\infty, \sigma_X^\infty]}"] & T^\infty X
\end{tikzcd}
\qquad \text{commutes.}
$$

In other words, splitting this diagram into two parts, both following diagrams (respectively named $morph_1(X)$ and $morph_2(X)$) commute.

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\eta_X^\infty"'] & TX \arrow[d, "\alpha_X"] \\
& T^\infty X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma TX \arrow[r, "\sigma_X"] \arrow[d, "\Sigma\alpha_X"'] & TX \arrow[d, "\alpha_X"] \\
\Sigma T^\infty X \arrow[r, "\sigma_X^\infty"] & T^\infty X
\end{tikzcd}
$$

Lemma 2 *The inductive extension $\nu_X : TT^\infty X \longrightarrow T^\infty X$ of $\sigma_X^\infty : \Sigma T^\infty X \longrightarrow T^\infty X$ is given by $\nu_X = \mu_X^\infty \circ \alpha_{T^\infty X}$.*

Actually both diagrams below commute since (1) is $morph_1(T^\infty X)$, (2) is $morph_2(T^\infty$
(3) is $monad_2(X)$ and (4) is $monad_1(X)$.

$$
\begin{tikzcd}
T^\infty X \arrow[r, "\eta_{T^\infty X}"] \arrow[dr, "\eta^\infty_{T^\infty X}"'] \arrow[ddr, "1_{T^\infty X}"'] & TT^\infty X \arrow[d, "\alpha_{T^\infty X}"] \arrow[dl, phantom, "(1)"] \\
 & T^\infty T^\infty X \arrow[d, "\mu_X^\infty"] \arrow[dl, phantom, "(3)"] \\
 & T^\infty X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma TT^\infty X \arrow[r, "\sigma_{T^\infty X}"] \arrow[d, "\Sigma\alpha_{T^\infty X}"'] \arrow[dr, phantom, "(2)"] & TT^\infty X \arrow[d, "\alpha_{T^\infty X}"] \\
\Sigma T^\infty T^\infty X \arrow[r, "\sigma^\infty_{T^\infty X}"] \arrow[d, "\Sigma\mu_X^\infty"'] \arrow[dr, phantom, "(4)"] & T^\infty T^\infty X \arrow[d, "\mu_X^\infty"] \\
\Sigma T^\infty X \arrow[r, "\sigma_X^\infty"] & T^\infty X
\end{tikzcd}
$$

Definition 5 *A mapping $f : T^\infty X \longrightarrow T^\infty Y$ is said* **iterative** *(w.r.t. T^∞) if there exists a chain morphism $F : \nabla X \longrightarrow \nabla Y$ such that f is the mediating morphism between the cone $F \circ \pi_X : T^\infty X \longrightarrow \nabla Y$ and the limiting cone $\pi_Y : T^\infty Y \longrightarrow \nabla Y$.*

For example μ_X^∞ is iterative for any set X, $T^\infty f$ is iterative for any mapping $f : X \rightarrow Y$; note, moreover that the composite of two iterative mappings is, as well, iterative.

Then we can state the following lemma which says that two iterative mappings that agree on finite trees are equal (we call finite trees those elements of $T^\infty X$ which are image of some term through α_X)

Lemma 3 (Density lemma) *if $f, g : T^\infty X \longrightarrow T^\infty Y$ are two iterative mappings such that $f \circ \alpha_X = g \circ \alpha_X$ then $f = g$.*

Sketch of proof : we recall that the elements of $T^\infty X$ are the sequences $(u_n)_{n \in \mathrm{N}}$ such that $u_n \in \mathcal{F}_X^n 1 = (\nabla X)_n$ and for every integer n, $u_n = \psi_n(u_{n+1})$; we supply $T^\infty X$ with an ultrametric distance as follows :

$$d(u, v) = \inf\{2^{-n} \; ; \; u_n \neq v_n\}$$

We verify that $(T^\infty X, d)$ is a complete metric space, that the finite trees constitute a dense subset of $T^\infty X$ (for the metric topology) and that an iterative mapping is continuous for that topology. □

Proof of the proposition : Consider the diagram

$$
\begin{tikzcd}
T^\infty T^\infty T^\infty X \arrow[drr, "\mu^\infty_{T^\infty X}"] \arrow[ddd, "T^\infty \mu_X^\infty"'] & & \\
(4) & TT^\infty T^\infty X \arrow[ul, "\alpha_{T^\infty T^\infty X}"'] \arrow[r, "\nu_{T^\infty X}"] \arrow[d, "T\mu_X^\infty"'] \arrow[u, phantom, "(3)"] \arrow[dr, phantom, "(1)"] & T^\infty T^\infty X \arrow[d, "\mu_X^\infty"] \\
 & TT^\infty X \arrow[r, "\nu_X"] \arrow[dl, "\alpha_{T^\infty X}"] \arrow[d, phantom, "(2)"] & T^\infty X \\
T^\infty T^\infty X \arrow[urr, "\mu_X^\infty"'] & &
\end{tikzcd}
$$

$monad_1(X)$ told us that $\mu^{\infty}_X : (T^{\infty}T^{\infty}X, \sigma^{\infty}_{T^{\infty}X}) \longrightarrow (T^{\infty}X, \sigma^{\infty}_X)$ is a morphism of Σ-algebras ; thanks to the Eilenberg-Moore comparison functor $\phi : \Sigma\text{-}\mathbf{Alg} \longrightarrow \mathbf{Set}^{\mathbf{T}}$ we know that μ^{∞}_X is, as well, a morphism of $\mathbf{T}$-algebras between the corresponding inductive extensions i.e. (1) commutes. (2) and (3) commute thanks to lemma 2 and (4) commutes because α is a natural transformation. Then it follows :

$$(\mu^{\infty}_X \circ \mu^{\infty}_{T^{\infty}X}) \circ \alpha_{T^{\infty}T^{\infty}X} \quad = \quad (\mu^{\infty}_X \circ T^{\infty}\mu^{\infty}_X) \circ \alpha_{T^{\infty}T^{\infty}X}$$

Since $\mu^{\infty}_X \circ \mu^{\infty}_{T^{\infty}X}$ and $\mu^{\infty}_X \circ T^{\infty}\mu^{\infty}_X$ are iterative mappings thanks to the density lemma they are equal. Which completes the proof □

We recall that a morphism of monads $\alpha : (T, \eta, \mu) \longrightarrow (T', \eta', \mu')$ is a natural transformation $\alpha : T \dot{\rightarrow} T'$ preserving the structure i.e. verifying

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\alpha"] \\ & T' \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "\mu"] \arrow[d, "\alpha\alpha"'] & T \arrow[d, "\alpha"] \\ T'T' \arrow[r, "\mu'"] & T' \end{tikzcd}$$

where $\alpha\alpha = \alpha T' \circ T\alpha = T'\alpha \circ \alpha T$ is the vertical composition of natural transformations . Now we can state the following

Proposition 5 *$\alpha : T \dot{\longrightarrow} T^{\infty}$ is a monad morphism.*
Proof :

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\eta^{\infty}_X"'] \arrow[r, phantom, shift right=4, "(1)"] & TX \arrow[d, "\alpha_X"] \\ & T^{\infty}X \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "\mu_X"] \arrow[d, "T\alpha_X"] \arrow[dd, bend right=60, "\alpha\alpha_X"'] \arrow[dr, phantom, "(2)"] & TX \arrow[d, "\alpha_X"] \\ TT^{\infty}X \arrow[r, "\nu_X"] \arrow[d, "\alpha_{T^{\infty}X}"] \arrow[dr, phantom, "(4)"] & T^{\infty}X \\ T^{\infty}T^{\infty}X \arrow[ur, "\mu^{\infty}_X"'] & \end{tikzcd}$$

with (3) labelling the region between $\alpha\alpha_X$ and $\alpha_{T^{\infty}X} \circ T\alpha_X$.

(1) is $morph_1(X)$, (2) follows from $morph_2(X)$ thanks to the Eilenberg Moore comparison functor, (3) is the definition of the vertical composition of natural transformations and (4) is lemma 2. □

To end this section, we prove that the monad of infinite trees is iterative which means, in analogy with Elgot's terminology, that all non degenerated equations have a unique solution. To make this precise, we first give some definitions. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad over $\mathbf{Set}$, for a given valuation $v : X \rightarrow TY$ we let $v^* = \mu_Y \circ Tv : TX \rightarrow TY$. $v^*(t)$ is the term t in which each occurrence of a variable $x \in X$ has been replaced by its value $v(x) \in TY$; so we shall sometimes denote $v^*(t)$ as t[v]. If $x \in X$ and $u \in TY$ let v=(u/x) denote the valuation $v : X \rightarrow TY$ such that v(x)=u and v(y)=$\eta(y)$ for $y \neq x$.

Definition 6 *A monad* $\mathbf{T} = (T, \eta, \mu)$ *is said to be* **algebraically closed** *whenever* x *is a variable in* X, t *a term in* TX *and* $t \neq < x >$ *there exists a term* $v \in T(X\backslash\{x\})$ *such that* $t[v/x] = v$. *If moreover this solution is unique* $\mathbf{T}$ *is said to be* **iterative.**

Proposition 6 *For every* ω*-signature* Σ, *the monad* $(T^\infty, \eta^\infty, \mu^\infty)$ *of infinite trees over that signature is iterative.*

Sketch of proof : with the hypothesis of the above definition, the mapping $\mathbf{subs}_{x,t} : T^\infty X \to T^\infty X$ defined by $\mathbf{subs}_{x,t}(v) = t[v/x]$ is contractive and then admits, thanks to Banach's fixed point theorem, a unique fixpoint in $T^\infty X$. $\square$

5 Conclusion

The restriction to ω-signature may probably be weakened if, in our construction, we admit chains of a sufficiently large ordinal. But this hypothesis was essential for the density lemma ; will we still be able to define substitution for infinite objects in that case ?

To conclude we note that if Σ is an ω-signature, we can deal with equational varieties of Σ-algebras defined as follows. Let (D,δ) be a $\mathbf{T}$-algebra (where $\mathbf{T}$ is the term monad corresponding to Σ) and $v : V \to D$ be a mapping (called a valuation). Let $v_\delta^* = \delta \circ Tv : TV \longrightarrow D$, it assigns each term in TV to its *value* in the *valued interpretation* (D,δ;v). Now let a Σ*-equation* be a pair (e_1, e_2) of elements in TV, we say that an interpretation (D, δ) *satisfies* (e_1, e_2) if, for any valuation v: V $\to$ D, one has $v_\delta^*(e_1) = v_\delta^*(e_2)$. Defining an *equational presentation* as a pair (Σ,E) where Σ is an ω-signature and E a set of Σ-equations, we say that a Σ-algebra is a (Σ,E)-algebra when it satisfies every equation in E ; the class of all (Σ,E)-algebras is called a *variety of algebras.* Let (Σ,E)-**Alg** denote the full subcategory of Σ-**Alg** corresponding to the (Σ,E)-algebras. Thanks to a categorical version of Birkhoff's theorem due to Hatcher [Hat70] and Herrlich and Ringel [HR72] used here in the particular case where the base category is **Set** we deduce that the forgetful functor from (Σ,E)-**Alg** to **Set** admits a left adjoint and the category of (Σ,E)-algebras is isomorphic as a category of sets with structure to the category of $\mathbf{T}_{\Sigma,E}$-algebras where $\mathbf{T}_{\Sigma,E}$ is the monad induced by this adjunction. And moreover we are able to characterize the full subcategory of $\mathbf{T}^\infty$-algebras verifying a set E of Σ-equations as an equational variety (of $\mathbf{Set}^{\mathbf{T}^\infty}$) by intersecting (a pull-back construction) the two subcategories (Σ, E)-**Alg** and $\mathbf{Set}^{\mathbf{T}^\infty}$ of Σ-algebras. For example we can describe a variety of process algebras by mean of equations over finite synchronization trees and consider the free algebras corresponding to that set of axioms. We feel that an interesting outcome of the construction described in the present paper is to set up a link between the models (e.g. process algebras) and the construction of domains those models are based upon.

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