AN ALGEBRAIC MODEL FOR THE STORAGE OF DATA IN PARALLEL MEMORIES

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ABSTRACT

The use of SIMD computers requires efficient schemes of storage of data in order to have conflict-free access in parallel computation. In this paper we restate the problem of finding such schemes in an algebraic context. This approach supplies simple statements and proofs of main results on the subject, and allows further development of it.

Index terms: Parallel memories, conflict-free access, skewing schemes.

1. INTRODUCTION

The use of parallel processing in large computers leads to the organization of primary memories in parallel units, in order to allow the access to multiple data in every memory cycle.

The effective utilization of such organization depends heavily on being able to arrange the data in the memory banks so that certain subsets of data can be fetched simultaneously without conflict. In this context, a typical problem arises when a two dimensional matrix is to be stored in such a way that all vectors of interest (rows, columns, diagonals, etc.) can be retrieved in one memory cycle, see [1].

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Any storage scheme of a set of data into a set of memory banks is called skewing scheme, and such a scheme allows simultaneous access to the elements of a subset of data if and only if they are stored in different memory units. In [4], the existence and properties of skewing schemes that supply conflict-free access to prescribed subsets of data was related to the study of plane tessellations. This geometrical approach was also used in [4-7].

In this work we restate the problem in a more general context introducing algebraic tessellations of Abelian groups. Within this framework, all definitions and statements are given in a simple and concise way, allowing further development of the subject. The paper is organized as follows. In Section 2 we give the basic definitions and results concerning algebraic tessellations of Abelian groups. In Section 3, a general definition of skewing scheme identifying each datum with an element of a group is given. Valid and proper schemes relative to a subset of data are characterized as a generalization of known results. Finally, in Section 4, the important class of periodic skewing schemes is set up in this new context, and the results of the general case are particularized.

Through the paper, $G=(G,+)$ denotes an Abelian group. We write $[g]$ for the equivalence class of an element g of G modulo a given subgroup, and, if T is a subset of G, $[T]$ denotes the set of classes of the elements of T. In this case, it will always be supposed that all the elements of T belong to different classes. If T, H are subsets of G, their sum $T+H$ is the set $\{t+h : t\in T,\ h\in H\}$. When each element in $T+H$ can be uniquely written as a sum of an element of T and an element of H, we say that the sum is direct, and we represent it as $T\oplus H$.

2. ALGEBRAIC TESSELLATIONS OF AN ABELIAN GROUP

In this section we introduce the basic definitions and results that are used later. The notion of "tessellation" defined below generalizes its usual meaning.

2.1. Definition: Let T be a subset of a commutative group G. We will say that T *pseudotessellates* G *by* H if there exists a subset H of G such that the sum $T+H$ is direct. If $T\oplus H=G$, we say that T *tessellates* G *by* H, or that the family of subsets $\{T+h : h\in H\}$ is a *tessellation* of G.

From this definition, some basic results follow.

2.2. Lemma: *The following statements are equivalent*

(a) T (pseudo)tessellates G by H.

(b) H (pseudo)tessellates G by T.

(c) T (pseudo)tessellates G by $g+H$ for any g in G.

(d) T (pseudo)tessellates G by $-H$ (if T is a finite subset).

Proof: By the symmetry of the Definition 2.1, (a) and (b) are equivalent. On the other hand, for a given g in G, $g+t+h=g+t'+h' \Leftrightarrow t+h=t'+h'$ so (a) is equivalent to (c). For the last equivalence, we have $t-h=t'-h' \Leftrightarrow t+h'=t'+h$, so T pseudotessellates G by H iff T pseudotessellates G by $-H$. Finally, suppose that T is a finite subset, say $T=\{t_1,\dots,t_q\}$, and that $T\oplus H=G$. By (c) and (b), we can suppose that $0\in T$. Let g be any element in G. For each $i=1,\dots,q$, there exists an unique h_i in H and $\sigma(t_i)\in T$ such that $g+t_i=h_i+\sigma(t_i)$. Notice that, if for some indices i, j we have $\sigma(t_i)=\sigma(t_j)$, then $h_i-t_i=h_j-t_j$, or $h_i+t_j=h_j+t_i$, and the sum $T+H$ is not direct. Then, σ must be a permutation of T, and $\sigma(t_i)=0$ for some t_i. Then, $g+t_i=h_i$, so that $g=h_i-t_i$. Since the choice of g was arbitrary, $G=(-T)\oplus H$, or, equivalently, $-G=G=T\oplus(-H)$, and T tessellates G by $-H$. ■

Notice that in the last equivalence of the above lemma, the finiteness of T is not necessary in the case of pseudotessellation. When H is a proper subgroup of G, the family of cosets of H in G can be viewed as a tessellation of the group. A slight generalization of this important case is typified in the next definition.

2.3. Definition: We say that T (pseudo)tessellates G by H *periodically* if there exists a nontrivial subgroup $H'<G$ and a subset $H^*\subset G$ such that $H=H'\oplus H^*$. The subgroup H' is said to be the *period* of the tessellation.

According to this definition, if H is a subgroup of G, a transversal T to H tessellates G periodically with period H ($H^*=\{0\}$), and T can be viewed as the quotient group G/H.

2.4. Lemma: *Let H' be a non-trivial subgroup of G, $H^*\subset G$, and $H=H'\oplus H^*$. Let $T\subset G$ such that it has no two elements in the same class modulo H'. Then T pseudotessellates (tessellates) G by H with period H' iff $[T]$ pseudotessellates (tessellates) G/H' by $[H^*]$.*

Proof: There exist $t_1,t_2 \in T$, and $h^*_1,h^*_2 \in H^*$ such that $[t_1]+[h^*_1]=[t_2]+[h^*_2]$ iff there exist $h'_1,h'_2 \in H'$ such that $t_1+h^*_1+h'_1=t_2+h^*_2+h'_2$. Hence, according to the hypothesis over T, the sum $T+H$ is direct iff the sum $[T]+[H^*]$ is direct. Now, if $T \oplus H=G$, for every $[g]$ in G/H' we have $[g]=[t+h^*+h']=[t]+[h^*]$ for some t,h^*,h', hence, $[T] \oplus [H^*]=G/H'$. Finally, if $[T] \oplus [H^*]=G/H'$ and $[g]=[t]+[h^*]$, for each element $\bar{g}$ in $[g]$ there exist an unique h' in H' such that $\bar{g}=t+h^*+h'$, so $T \oplus H^* \oplus H'=T \oplus H=G$. ∎

3. SKEWING SCHEMES

Let A be a set of data and $M=\{1,...,m\}$ a set of $m \leq |A|$ memory modules. By a skewing scheme it is meant any rule to store the data of A into the m memory units. In our formulation we identify each element of A with an element of a group G with the same cardinality. A more precise definition follows.

3.1. Definition: A *skewing scheme* s is any surjective map $s:G \longrightarrow M$.

In this definition it must be understood that any element g of G is stored by the skewing scheme in the bank $s(g)$. Our goal is to characterize those schemes that are suitable for some prescribed subsets of data, that is, skewing schemes that store every element of such subsets into different memory banks. To this end, the following concepts are needed.

3.2. Definition: A *data template* T of size q is any subset of G with q elements, $q \leq m$.

3.3. Definition: The skewing scheme s is *valid* for a data template $T=\{t_1,...,t_q\}$, iff for any g in G the restriction of s to $g+T$ is injective. In particular, when this restriction is bijective, $q=|T|=m$, the skewing scheme s is said to be *proper* for T.

In other words, s is a valid skewing scheme for T when no two elements in $g+T$ are stored in the same memory bank for every g in G. The skewing scheme s is proper for T when it is valid for T and each memory unit contains one element of $g+T$. From Definition 3.3, it is clear that s is valid for T iff it is valid for $g+T$ for any g in G. So, there is no loss of generality in assuming $0 \in T$.

Any skewing scheme s induces an equivalence relation $\approx$ in G defined as $g \approx g'$ iff $s(g)=s(g')$. Let $H_i=s^{-1}(i)$, $i=1,\ldots,m$, denote the equivalence classes of $\approx$, that is, the set of data stored in the ith memory module. Notice that a skewing scheme s is valid for T iff $|(g+T)\cap H_i|\leq 1$ for every g in G and each $i=1,\ldots,m$. Similarly, s is proper for T iff $|(g+T)\cap H_i|=1$ for every g in G and each $i=1,\ldots,m$. This fact leads to the following result.

3.4. Theorem: *A skewing scheme is valid (proper) for the data template T iff, for any i in M, T pseudotessellates (tessellates) G by H_i.*

Proof: Suppose that s is valid for T. For each $i=1,\ldots,m$ and every element g in G, we have $|(g+T)\cap H_i|\leq 1$. If $|(g+T)\cap H_i|=1$, there exist an unique t in T and an unique h in H_i such that $g+t=h$, or $g=h-t$, and if $|(g+T)\cap H_i|=0$, $g\notin H_i+(-T)$, so $-T$ pseudotessellates G by H_i for every i. By Lemma 2.2, T pseudotessellates G by H_i for every i. Finally, if s is a proper skewing scheme for T, for every g in G we have $|(g+T)\cap H_i|=1$, $i=1,\ldots,m$. Hence, $g=h-t$ for some $h\in H_i$ and $t\in T$, and $(-T)\oplus H_i=G$ for each $i=1,\ldots,m$. By Lemma 2.2, T tessellates G by H_i, $i=1,\ldots,m$. These arguments can be reversed to obtain the converse statement. ■

This theorem is a simpler statement and a generalization of the main result obtained in [4]. It is clear that, if T tessellates G by H, every subset T' of T pseudotessellates G by H. So, a proper skewing scheme for T is valid for any of its subsets. In what follows we will concentrate on proper schemes, although all statements and proofs can be easily rewritten for valid schemes and pseudotessellations.

The following result provides necessary and sufficient conditions for the existence of a proper skewing scheme for a given data template T.

3.5. Theorem: *There exists a proper skewing scheme s for a given data template $T=\{t_1,\ldots,t_q\}$ iff T tessellates G by some subset H.*

Proof: The direct sense of the statement is clear from Theorem 3.4, since T tessellates G by H_1, for example. For the converse, define s as $s(g)=i$ iff $g\in H+t_i$, $i=1,\ldots,q$. Now, $H_i=H+t_i$ and, by Lemma 2.2, T tessellates G by H_i for each i. Therefore s is proper for T. ■

As a particular case of Theorem 3.5, since any subgroup of G tessellates G, there always exists a proper skewing skewing scheme for T when T is a subgroup of G or a transversal to some subgroup in G.

In the context of the conflict-free access problem, there are usually several data templates that should be fetched without conflict. For instance, rows, columns and diagonals in a matrix. Theorem 3.5 can be easily generalized for this purpose.

3.6. Theorem: *There exists a proper skewing scheme for all the data templates $T_1,\dots,T_r$ (all of them with the same size) iff there exists a subset H of G such that each T_α tessellates G by H, $\alpha=1,\dots,r$.*

Proof: Suppose that T_α tessellates G by some subset H for each $\alpha=1,\dots,r$. Let $T_1=\{t_1,\dots,t_q\}$. As we have seen in the proof of Theorem 3.5, the skewing scheme s defined as $s(g)=i$ iff $g\in H+t_i$, $i=1,\dots,q$, is proper for T_1. Now, by Lemma 2.2, for every $\alpha=2,\dots,r$, T_α tessellates G by $H_i=H+t_i$, $i=1,\dots,m$. Therefore, by Theorem 3.4, s is also proper for $T_2,\dots,T_r$. Reciprocally, if s is proper for all T_α, each one tessellates G by H_1, for example. ∎

4. PERIODIC SKEWING SCHEMES

From a practical point of view, skewing schemes should be easily computed. This fact leads to the study of periodic schemes, the definition of which follows.

4.1. Definition: A skewing scheme s is said to be *periodic* when there exists a subgroup $H'<G$ such that $[g]=[g']$ implies $s(g)=s(g')$. The subgroup H' is the *period* of s, and the index of H' in G is called the *index* of the scheme.

In other words, the skewing scheme s has period H' iff it is constant over each of the cosets of H' in G. In [4], Shapiro considered periodic skewing schemes over $\mathbb{Z}^2$ with period $H'=m\mathbb{Z}\times m\mathbb{Z}$. In [2], and independently in [7], the situation was generalized to periodic skewing schemes defined over $\mathbb{Z}^d$ with period $H'=\mathbf{M}\mathbb{Z}^d$, the lattice generated by the column vectors of the $d\times d$ integral matrix $\mathbf{M}$.

Notice that, if s is a periodic skewing scheme with period H' and index k, there exists an induced map s' such that the following diagram commutes,

$$\begin{tikzcd} G \arrow[r, "s"] \arrow[d, "P"'] & M \\ G/H' \arrow[ur, "s'"'] & \end{tikzcd}$$

where P is the canonical projection of G over G/H'. Therefore, s can be deduced from s' (that is defined over only k values). This fact is on the basis of the use of periodic schemes, and leads to the following main results. As said before, it is always supposed that no two elements in T belong to the same class modulo H'.

4.2. Theorem: *Let s be a periodic skewing scheme with period H'. Then s is proper for T iff $s':G/H' \longrightarrow M$ is proper for $[T]$.*

Proof: Suppose that s is a proper periodic scheme for T with period H'. Since s is constant over the elements of each coset modulo H', $H_i=s^{-1}(i)$ can be written as $H_i=H'\oplus H^*_i$ for some subset H^*_i of G and each i. By Theorem 3.4, T tessellates G by H_i , so T tessellates G periodically with period H' Hence, by Lemma 2.4, $[T]$ tessellates G/H' by $[H^*_i]$ for each i. From Theorem 3.4 again, since $s'^{-1}(i)=[H^*_i]$, s' is proper for $[T]$. This argument can be reversed to complete the proof. ■

In the proof of the above lemma, we have also proved the following result.

4.3. Corollary: *Let s be a periodic skewing scheme with period $H'<G$. The following statements are equivalent.*

(i) *s is proper for T;*

(ii) *T tessellates G by H_i periodically with period H' for every $i=1,...,m$;*

(iii) *$[T]$ tessellates G/H' by $[H^*_i]=s'^{-1}(i)$ for every $i=1,...,m$.* ■

The following theorem, analogous to Theorem 3.6, generalizes the main result in [4].

4.4. Theorem: *There exists a periodic skewing scheme s with period H' proper for the family $T_1,...,T_r$ of data templates (of size m) iff there exists a subset H^* such that every T_i tessellates G by $H=H'\oplus H^*$ with period H'.*

Proof: By Corollary 4.3, if s is proper for every $T_1,\ldots,T_r$, each T_i tessellates G periodically by H_1, for example, with period H'. Reciprocally, if there exists H^* such that every T_i tessellates G by $H=H'\oplus H^*$ with period H', then each $[T_i]$ tessellates G/H' by $[H^*]$. By Theorem 3.6, there exists a skewing scheme s' (defined over G/H') proper for each $[T_i]$. Then the skewing scheme s defined by $s(g)=s'([g])$, $g\in G$, is proper for every T_i and clearly it has period H'. ∎

From a practical point of view, Theorem 4.4 provides necessary and sufficient conditions for the existence of a periodic skewing scheme, proper for a given family of data templates, that can be easily tested with an appropriate choice of the subgroup H' —namely, when G/H' is a small group. Moreover, it supplies a simple way to define s through s'.

5. CONCLUSIONS

In this paper we have reformulated the theory introduced by Shapiro in [4] about the problem of conflict-free access in parallel memories using the concept of algebraic tessellations of a group. In this context, the basic equivalence between tessellations of the plane and existence of proper skewing schemes has been generalized in a simple and concise way. On the other hand, the study of the useful class of periodic schemes has also been set up in this new framework.

In the literature, the main concern in relation to the conflict-free access problem in parallel memories has been the matrix storage. When the set of data A is a $p_1\times\ldots\times p_d$ d-dimensional array, every element is labelled with a d-tuple $(i_1,\ldots,i_d)$, $0\le i_j\le p_j-1$. In this case, we can use the group $\mathbb{Z}^d$, and periodic skewing schemes with period a subgroup H' such that $\mathbb{Z}^d/H'\cong\mathbb{Z}_{p_1}\oplus\ldots\oplus\mathbb{Z}_{p_d}$. Then the identification between the set of data and this last group is quite natural. Moreover, it can be useful to use periodic skewing schemes in $\mathbb{Z}_{p_1}\oplus\ldots\oplus\mathbb{Z}_{p_d}$. The study of such schemes can be related to the integral matrix $\mathbf{M}$ such that $H''=\mathbf{M}\mathbb{Z}^d$, $\mathbb{Z}^d/H''<\mathbb{Z}^d/H'$. In this context, the structure of $\mathbb{Z}^d/H''$ can be derived from the Smith normal form of $\mathbf{M}$, see [3]. Its application to the characterization of the so-called linear schemes (an interesting subclass of periodic schemes in which $\mathbb{Z}^d/H''$ is cyclic) is considered in [7].

The theory developed in this work, can also be applied to other geometrical or combinatorial structures in which the set of data to be stored is embedded. In each case, the problem consists on finding, when possible, an appropriate identification between the set of data and an Abelian group.

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