Cours/Lecture Series

1985 – 1986 ACADEMIC TRAINING PROGRAMME

LECTURER	: J. FITCH / University of Bath
TITLE	: Symbolic and algebra[redacted]
DATES	: 6, 7 and 8 Novembe[redacted]
TIME	: 11.00 to 12.00 hrs
PLACE	: Auditorium

ABSTRACT

Computers are an ideal tool for increasing the accuracy of algebraic calculations, and taking the tedium out of pen and paper manipulations. The current generation of algebra systems are drawing on recent advances in mathematics to extend their capabilities beyond human ability. These lectures will describe the methods used to implement algebra on a computer, some of the underlying mathematical theory and the techniques avalable to users to make the most of the algebraic facilities in the solution of physical problems. The lectures will be illustrated by references to standard available algebra systems.

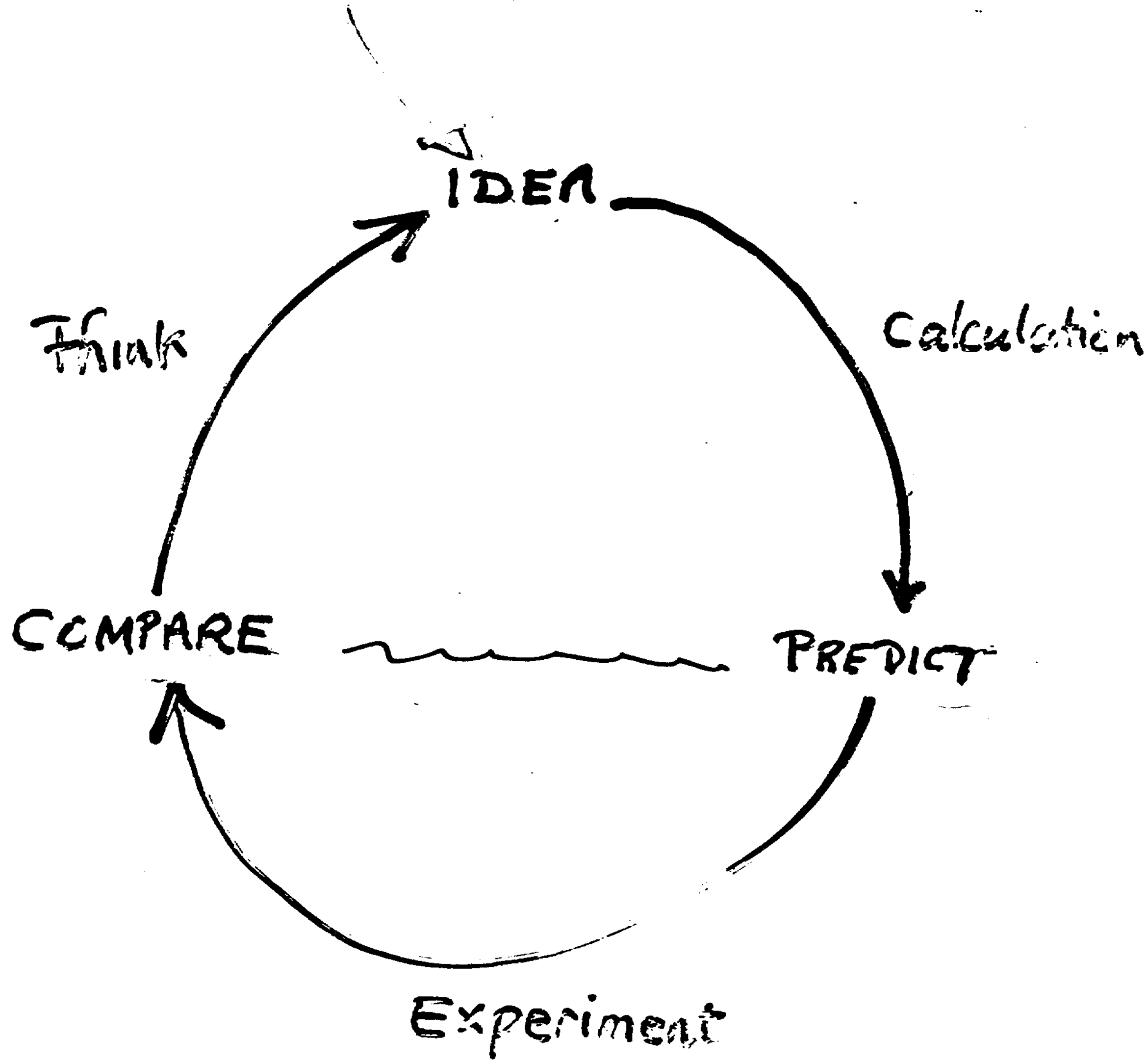
LECTURE NOTES

DOC/PU/ED
Distr. int. + ext.

Secretariat : tel. 2844 - 3364

SIS/H/346

The Science Loop



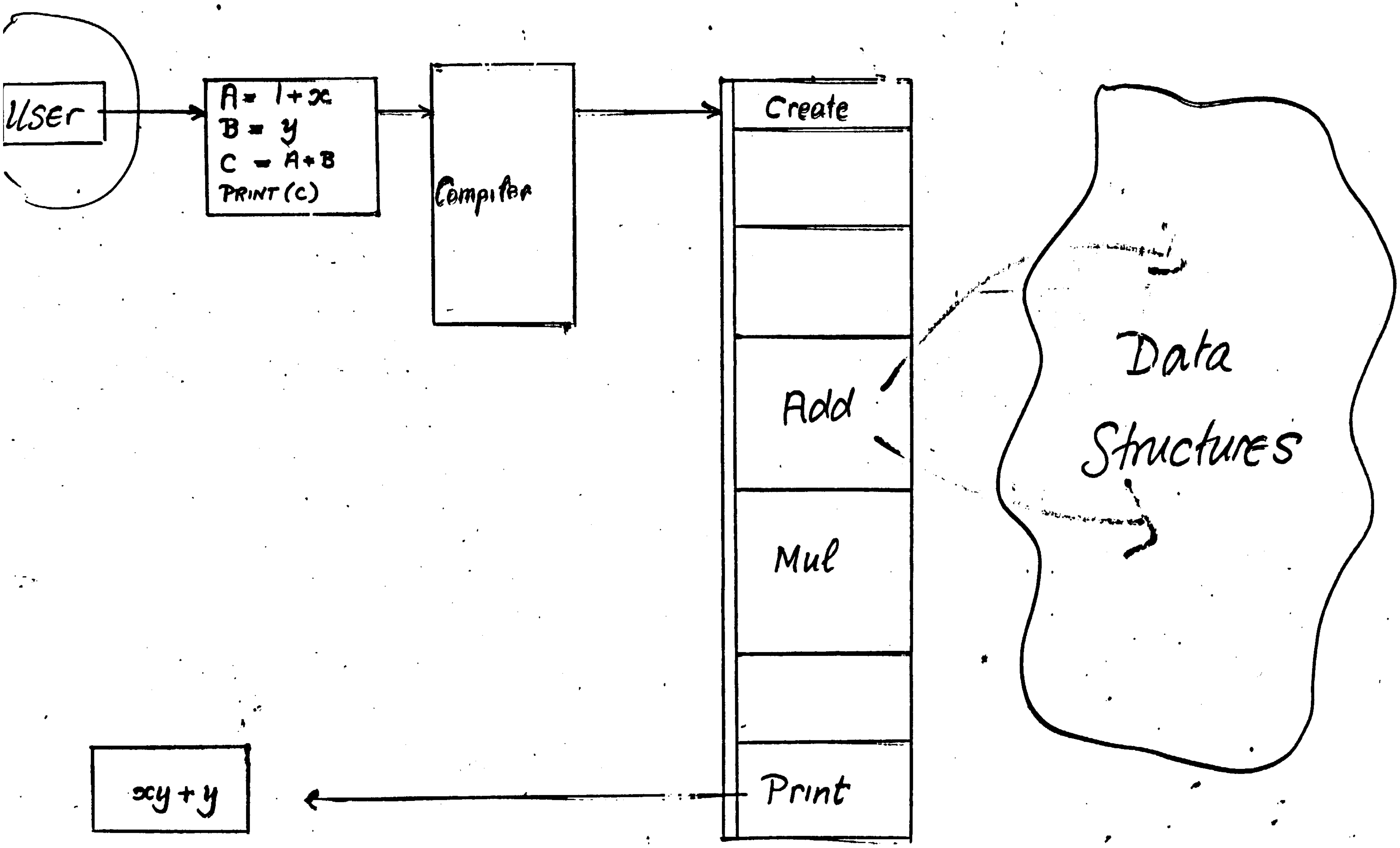
```
:MACSYMA

THIS IS MACSYMA 255

FIX 255 DSK MACSYM BEING LOADED
LOADING DONE


(C1) DIFF((X↑2-1)↑5,X,5)/(2↑5*FACT(5));
                          2      2             5            3    2
                7200 X (X   - 1)   + 3840 X   + 19200 X   (X   - 1)
(D1)            ----------------------------------------------------
                                    3840

(C2) EXPAND(%);
                                  5        3
                              63 X     35 X     15 X
(D2)                          ----- - ----- + ----
                                8       4       8

(C3)
```

$$\frac{\frac{d^5}{dx^5}\left((x^2-1)^5\right)}{2^5\ \underline{|5}}$$

$$P_n(x) = \frac{1}{2^n \underline{|n}} \frac{d^n}{dx^n}(x^2-1)^n$$

```
A[10]
FOR I=1:1:10
     A[I]=(hh-1).I
     FOR J=1:1:I
          A[I]=(1/(2J))dA[I]/dh
     REPEAT
     PRINT[A[I]]
     PRINT[TIME]; PRINT[STORE]
REPEAT
STOP
```

```
A[1] =      h

TIME=0.01 SECS
Store in Use=11, Ceiling=71 out of 14839 words
A[2] =  -( (1/2) - (3/2) h↑2 )

TIME=0.02 SECS
Store in Use=31, Ceiling=95 out of 14839 words
A[3] =  -( (3/2) h - (5/2) h↑3 )

TIME=0.03 SECS
Store in Use=51, Ceiling=135 out of 14839 words
A[4] =    ( (3/8) - (15/4) h↑2 + (35/8) h↑4 )

TIME=0.05 SECS
Store in Use=79, Ceiling=179 out of 14839 words
A[5] =    ( (15/8) h - (35/4) h↑3 + (63/8) h↑5 )

TIME=0.07 SECS
Store in Use=107, Ceiling=231 out of 14839 words
A[6] =  -( (5/16) - (105/16) h↑2 + (315/16) h↑4 - (231/16) h↑6 )

TIME=0.10 SECS
Store in Use=143, Ceiling=283 out of 14839 words
A[7] =  -( (35/16) h - (315/16) h↑3 + (693/16) h↑5 - (429/16) h↑7 )

TIME=0.14 SECS
Store in Use=179, Ceiling=343 out of 14839 words
A[8] =    ( (35/128) - (315/32) h↑2 + (3465/64) h↑4 - (3003/32) h↑6 + (6435/128)
```


User
A = 1 + x
B = y
C = A + B
PRINT (C)
Compiler
Create
Add
Mul
Print
Data Structures
xy + y

Types of Algebra System

- Big but "straightforward" problems

  (REDUCE) CAMAL SHEEP TRIGMAN etc

- Making harder things "straightforward"

  REDUCE, MACSYMA

- Casual, Day to Day, calculations

  MACSYMA SCRATCHPAD

  μMATH

  REDUCE

MAJOR ALGEBRA SYSTEMS

MACSYMA

Multics, Symbolics, VAX

Large; very broad

REDUCE

VAX, HP, DEC20, IBM, 68000 based
CDC CRAY .... HLH Orion ...... IBM PC

Not so complete; still developing

MAPLE

VAX

Fast growing; innovative

SCRATCHPAD

IBM only

Advanced; unavailable

Equations in General Relativity

Covariant metric

$$ds^2 = g_{ij}\, dx^i dx^j \qquad \begin{array}{l} i = 0, \dots 3 \\ j = 0, \dots 3 \end{array}$$

indice répété $\Rightarrow$ summation

Contravariant metric

$$g_{ip}\, g^{jp} = \delta_i^j$$

inverse de la matrice

Christoffel Symbols

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

$$\left\{ {k \atop ij} \right\} = g^{kp}\, [ij,p]$$

Riemann Tensor

$$R_{ijkl} = \frac{\partial}{\partial x^k}[jl,i] - \frac{\partial}{\partial x^l}[jk,i] + [il,p]\left\{ {p \atop jk} \right\} - [ik,p]\left\{ {p \atop jl} \right\}$$

Ricci Tensor

$$R_{ij} = g^{pq} R_{ipjq}$$

Ricci Scalar

$$R = g^{pq} R_{pq}$$

La métrique de Bondi

Bondi's Metric

$$g_{ij} = \begin{bmatrix} -r^2 U e^{2\gamma} + V e^{2\beta}/r & e^{2\beta} & r^2 U e^{2\gamma} & 0 \\ e^{2\beta} & 0 & 0 & 0 \\ r^2 U e^{2\gamma} & 0 & -r^2 e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta\, e^{2} \end{bmatrix}$$

U, V, β, γ functions of u, r and θ

La métrique de Sachs est impossible (???)

General Relativity

- Calculation of Riemann, Ricci, Einstein & Weyl tensors
- Calculation of Petrov types
- Solution of the field equations by approximation
- Establishing a consistent set of equations for a type of universe
- Kaltek & Aman's thing.

The Equivalence Problem in G.R.

Does there exist a real transformation

$$g_{ij} \longrightarrow g^{*}_{ij} \qquad ?$$

Are they the same space ?

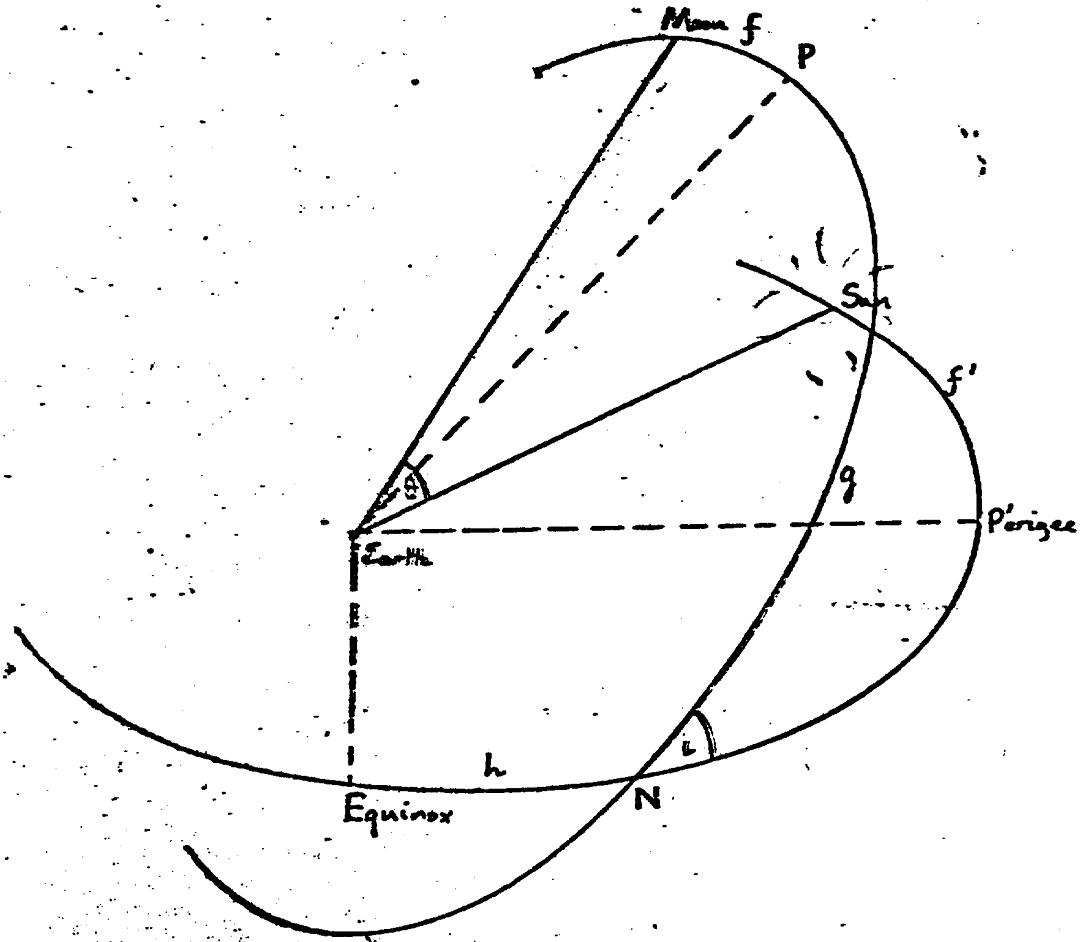
Classical Answer:

Tenth Deriv. R_{ijke} ; 14 rank tensor

$\sim 20 \times 4^{10}$ expressions

Karlhede & Åman answer:

Up to 4^{th} deriv R_{ijke} , incremental



$a, e, \gamma = \sin i/2$

$\xi = \cos\theta$, arc $ENP' = g' + h'$

arc $EN = h$, arc $NP = g$, arc $PM = f$, arc $P'S = f'$

Figure 22: Definition of the coordinate system used to calculate the disturbing function

Method of the Lunar Theory

Determine total energy,

Newtonian energy + disturbing energy

Use Hamilton's equations;

Delaunay made ~300 contact transformations by hand in 20 yrs

Deprit's method ~ Lie transform – ~24 [illegible] cpu

This was done ~1970 : but the technique is very important

A Basic Algebraic Technique – Repeated Approximation

Simple Example:

Solve $y^2 = 1 - e$ for small e

as a series $y = f(e)$

We will ignore the binomial theory

Step 1: Assume e is zero

$$y_0 = \pm 1$$

Step 2: Assume $y = y_n + \eta + O(n+2)$

with y_n correct to order n in e, and η is of order $n+1$

Substituting
$$y_n^2 + 2y_n\eta + \eta^2 = 1 - e + O(n+2)$$

$$y_n^2 + 2y_0\eta = 1 - e - \eta^2 + O(n+2)$$

$$\Longrightarrow \quad \eta = \left[\frac{1 - e - y_n^2}{2y_0}\right]_{n+1}$$

This technique is very general

$E = u + e\sin E$ (Kepler's Equation)

Picard's Method

$\ddot{y} + y = \varepsilon y^3$ Duffing's Equation

etc

Repeated Approximation

To solve $y^2 = 1 - e$

where e is small.

Suppose e is zero; then zero approximation is

$$y_0 = 1$$

Let y_n be correct to order n in e

Let $y_n + \eta$ be true solution

$$y_n^2 + 2y_n\eta + \eta^2 = 1 + e$$

$$2y_0\eta = 1 + e - y_n^2 + O(n+2)$$

$$\eta = \frac{1}{2}\left(1 + e - y_n^2\right)_{n+1} + O(n+2)$$

Lindstedt – Poincaré

$$\ddot{y} + A^2 y = e f(y, \dot{y}) + g(t)$$

Initial approximation

$$y_0 = a \cos(\omega t)$$
$$\omega_0 = A$$

Changing variables to $\tau = \omega t$

$$y'' + y = \frac{1}{A}\left\{ \underbrace{e f(y, y') - (\omega - A) y''}_{\text{small}} \right\}$$

$$= \sum K_j \cos(j\tau) + M_j \sin j\tau$$

So

$$y_n = \sum_{j \neq A} \frac{K_j \cos j\tau}{1^2 - j^2} + \sum_{j \neq A} \frac{M_j \sin j\tau}{1 - j^2}$$
$$+ c \cos \tau + d \sin \tau$$

$$\omega_n = A \cdots\cdots + f e^{n+1}$$

The Perturbed Harmonic Oscillator

$$\ddot{y} + y = ey^3$$

with solution having no secular terms.

Stretch time

Phase constraint $y'(0) = 0$

Initial solution $a \cos[t]$

Transform to $t = cT$

$c_0 = 1$

$$y'' + y = \underbrace{ey^3 + (c-1)^2 y'' + 2(c-1)y''}_{\text{small}}$$

Write this as

$$(D^2+1)y = G(y, c-1)$$

- Extend c by $k e^n$
- Calculate the right hand side
- Remove any resonance terms by solving the coefficients of $\sin[t]$ and $\cos[t]$ for any introduced constant
- Calculate $y = \frac{1}{D^2+1}(\text{rhs})$
- Apply phase constraint

$c_0 = 1 \qquad y_0 = a\cos[t]$

$\text{rhs} = \left(-2eak + \frac{3}{4}ea^3\right)\cos[t] + \frac{1}{4}ea^3\cos[3t]$

to remove resonance $\quad k = \frac{3}{8}a^2$

$c_1 = 1 + \frac{3}{8}a^2 e \qquad y_1 = a\cos[t] - \frac{1}{32}ea^3\cos[3t]$

$\text{rhs} = -\left(2e^2ak + \frac{21}{128}e^2a^5\right)\cos[t] + \left(\frac{1}{4}ea^3 + \frac{21}{128}e^2a^5\right)\cos[3t]$
$\qquad - \frac{3}{128}e^2a^5\cos[5t]$

to remove resonance $\quad k = -\frac{21}{256}a^4$

$c_2 = 1 + \frac{3}{8}a^2 e - \frac{21}{256}a^4e^2 \qquad y_2 = a\cos[t] - \left(\frac{1}{32}ea^3 + \frac{21}{1024}e^2a^5\right)\cos[3t]$

Celestial Mechanics

- Expansion of Lunar disturbing function
- Performing Delaunay Operations
- Discovering approximate contact transformations
- Production of literal form of the variational orbit
- Solution of the main problem of the Lunar theory by Hill's technique

Theoretical Physics

- Study of the breaking effect on stars caused by a gas cloud
- Study of the temperature distribution inside a planet
- Perturbation theory in Quantum mechanics
- Elastic Waves in a sheared medium

Engineering

- Solution of boundary value problems for the elastic stress in a pipe junction
- Design and simulation of control systems
- Design of non-linear filters

etc

- Application of operators to wave equations
- Transformation of pictures
- Solution of differential equations near a singularity
- Solution of stiff differential equations
- Process flow in a computer system
- Number theory
- Spread of epidemics
- Random walks
- Flow of fluid through a membrane

Trouver la solution de [JWKB]

$$\frac{d^2f}{dx^2} + k^2(x) f = 0$$

avec $f = \frac{1}{\sqrt{q}} \exp\left[i \int q \, dx \right]$

et $q = k(x) \sum_{n=0} Y_{2n}(x)$

C'est suffisand trouver Y_{2n}

Laisse $\theta = \sum Y_{2n}$ et $Y_0 = 1$

Define $\dot{A} \equiv \frac{1}{k} \frac{dA}{dx}$

En substituant dans l'Équation d'origine, on obtient

$$\theta^2 \varepsilon_0 + \theta^2 - \theta^4 - \frac{\theta \ddot{\theta}}{2} + \frac{3 \dot{\theta}^2}{4} = 0$$

$$\varepsilon_0 = \frac{1}{4k^2} \left\{ 3 \left(\frac{1}{k} \frac{dk}{dx} \right)^2 - \frac{2}{k} \frac{d^2k}{dx^2} \right\}$$

Ecrit $\quad \mathcal{E}_r = \dot{\mathcal{E}}_{r-1}$

$$\text{Ordre}(\mathcal{E}_0) = 2$$

$$\text{Ordre}(\mathcal{E}_r) = r+2$$

Alors, $\quad \text{Ordre}(Y_{2n}) = 2n$

$$\theta_{2n} = \bar{\theta} + Y_{2n} \qquad \text{ordre}(\bar{\theta}) = 2n-2$$

$$\begin{aligned}
&\bar{\theta}^2 \mathcal{E}_0 + \bar{\theta}^2 + \underline{2\bar{\theta} Y_{2n}} + Y_{2n}^2 + 2\bar{\theta} Y_{2n} \mathcal{E}_0 + Y_{2n}^2 \mathcal{E}_0 \\
&- \bar{\theta}^4 - \underline{4\bar{\theta}^3 Y_{2n}} + 6\bar{\theta}^2 Y_{2n}^2 + 6\bar{\theta} Y_{2n}^3 + Y_{2n}^4 \\
&+ \frac{1}{2}(\bar{\theta} + Y_{2n})\ddot{\bar{\theta}} + \ldots \\
&+ \frac{3}{4}\dot{\bar{\theta}}^2 + \ldots \qquad = 0
\end{aligned}$$

$$\Longrightarrow \qquad Y_{2n} = \frac{1}{2}\left[\mathcal{E}_0 \bar{\theta}^2 + \bar{\theta}^2 - \bar{\theta}^4 + \frac{1}{2}\bar{\theta}\ddot{\bar{\theta}} + \frac{3}{4}\dot{\bar{\theta}}^2\right]_{2n}$$

```
OPERATOR E;

FORALL N LET DF(E(N),X)= E(N+1);

FOR N=0:5 DO ORDER (E(N)):= N+1;

ARRAY Y(10);

Y(0) := 1;        TH:= 1;

FOR N=1:5 DO <<

      WEIGHT (2*N);

      Y(2*N) := ((E(0)+1)*TH**2 - TH**4

                 +DF(TH,X,2)* TH/2

                 + 3* DF(TH,X)**2 /4 ) /2;

    TH := TH + Y(2*N)      >>;
```

Une méthode très importante

(voir Fitch, Norman et Moore 1981)

Cowling Stellar Model

$$2x^2 \frac{d}{dx}(yz) = 5wz$$

$$\frac{dw}{dx} = x^2 z$$

$$\begin{cases} z = y^{3/2} & x < x_0 \\ x^2 \dfrac{dy}{dx} = -Qz^2 y^{-13/2} & x > x_0 \end{cases}$$

$x = 0$ $\quad w = 0 \quad$ y & z finite

$x = x_0$ $\quad y, z\, w\, y' \; z' \; w'$ continuous

$x = x_s$ $\quad y = z = 0$

x is radius

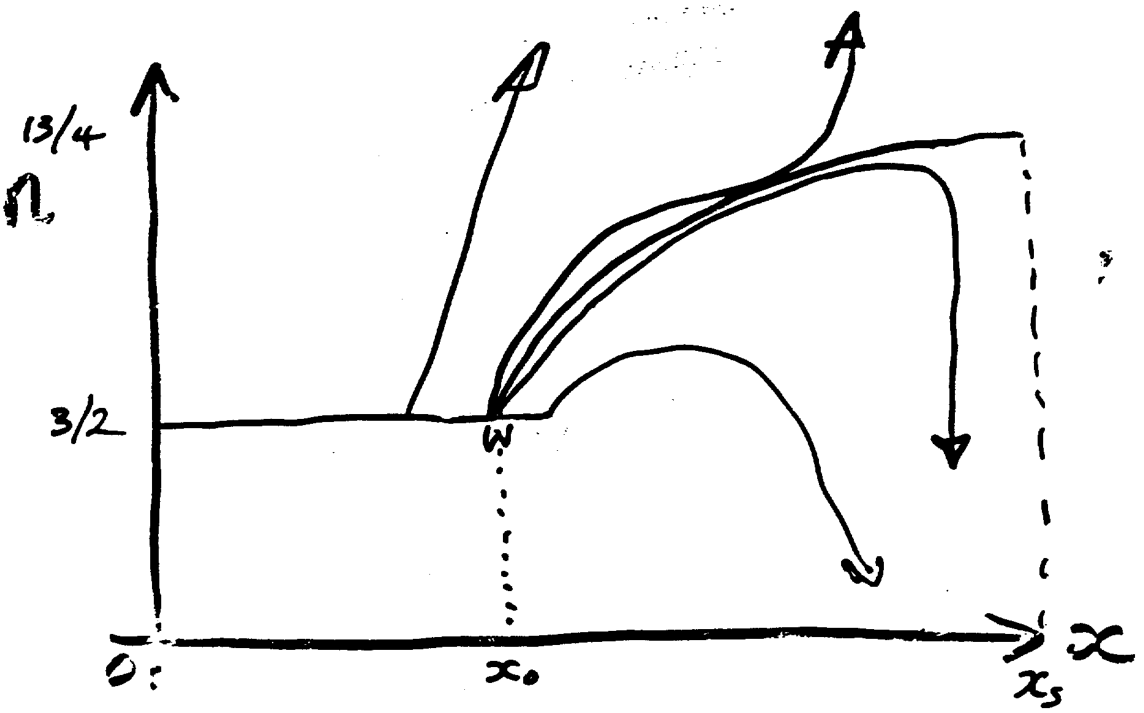
w is mass

y, z pressure & luminosity

Numerical calculation:

The inner zone is Emden equation $n = 3/2$

At the surface approaches Emden $n = 13/4$



Algebraic Calculation

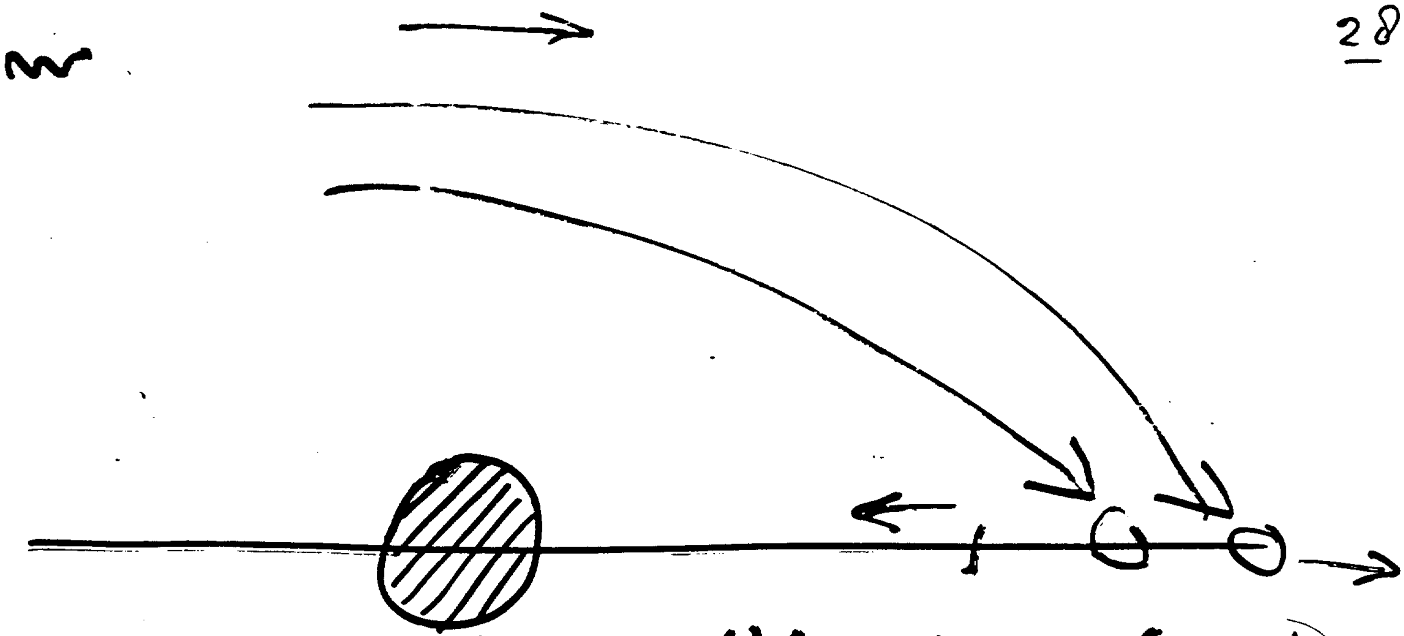
Develop the solution around the surface x_s unknown

The solution has the form

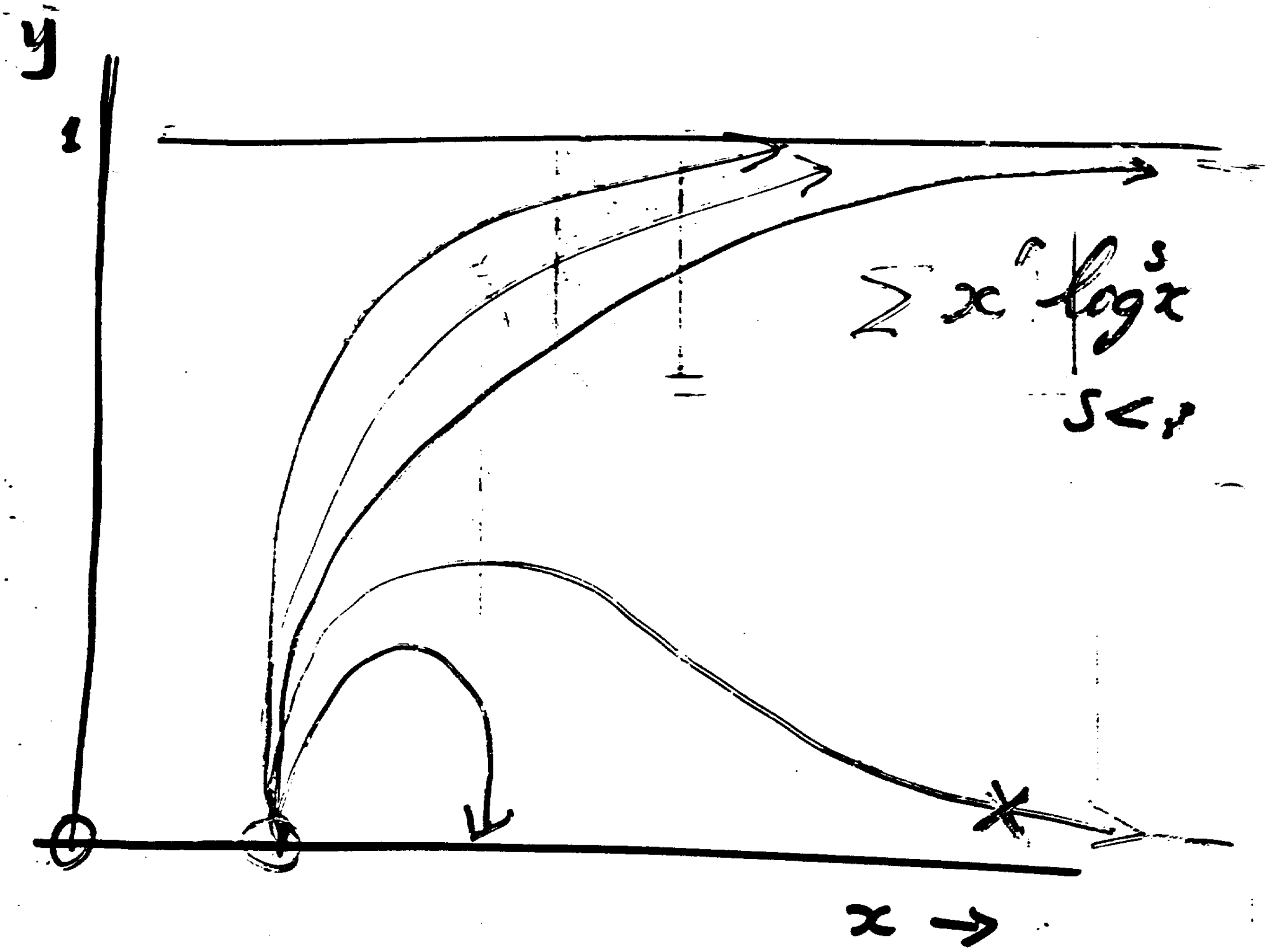
$$y = \sum_{i=1} y_i \, (x_s - x)^i$$

$$z = (x_s - x)^{9/4} \sum_{i=1} z_i \, (x_s - x)^i$$

$$w = \mu - (x_s - x)^{13/4} \sum_{i=1} w_i \, (x_s - x)^i$$



$$x(1-yy')(1-kx) = y(1-y)$$



ome References on Applications

General:

Barton & Fitch
Reports on Progress in Physics
35 1972

Brown & Hearn
Comp. Phy. Comm 1980

Fitch
Lecture Notes in Computer Sciens
72 1979

Relativity:

Cohen, Leringe & Sundblat
GRG 7 1976

Fitch & Cohen
GRG 1980

Optics:

Hawkes
Optik 48 29 1977

Goto & Soma
Optik 48 255 1977

Fluids:

Graham & Moore
MNRAS 183 617 1978

REDUCE Facilities

Interactive ALGOL (PASCAL) like language

FORTRAN Output

General system

LISP based

includes

Simplification

Indefinite Integration

Factorization

SOLVE

Big integers & big floats

MACSYMA Facilities

Interactive language – BASIC like ?

FORTRAN Output

Very General

LISP Based

includes

User controlled simplification

Definite & Indefinite integration

Factorization

Limits

Power Series

Solve

Big integers and floats

Plotting

Factorization over finite fields

The Construction of Algebra Systems

Language:

LISP is commonest

MACSYMA, REDUCE, SCRATCHPAD, SHEEP....

C

MAPLE, SMP

BCPL

CAMAL, (MAPLE)

FORTRAN

TRIGMAN, ALDES

etc

Assembler : many

Representation of Algebraic Expressions

More important than language

Three main groups

- Strings, Parse Trees

  Recursive Polynomial Structure

- Tables

Parse Trees.

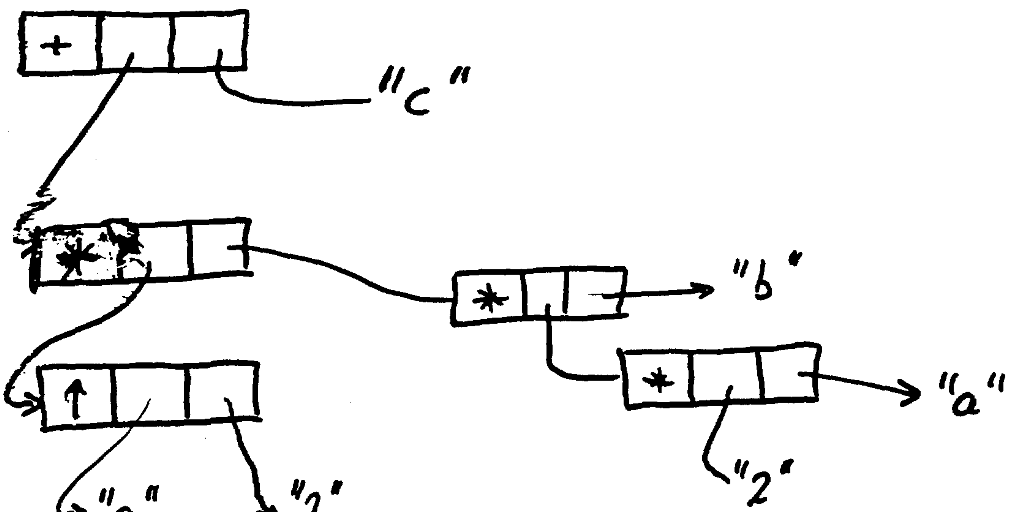
No serious system has ever used strings

but starting point is Polish notation

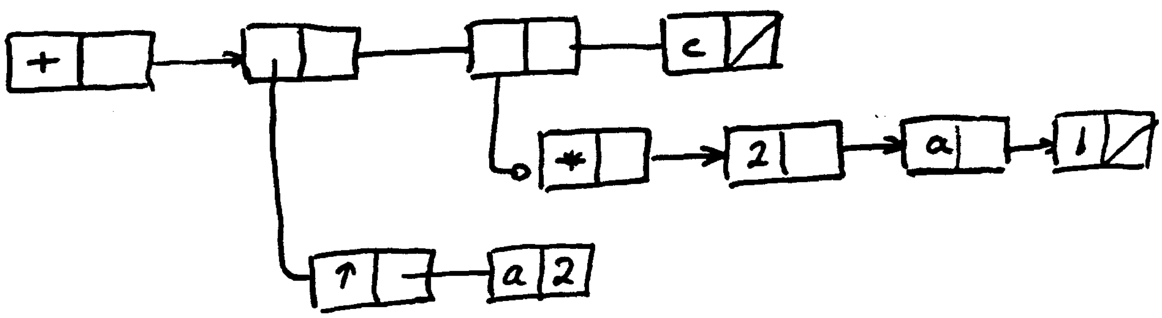
$$a^2 + 2ab + c \longrightarrow + + \uparrow a\, 2 * 2 * a\, b\, c$$

No ambiguity, but inconvenient

$\Longrightarrow$ A tree



or variant



The Recursive Data Structure

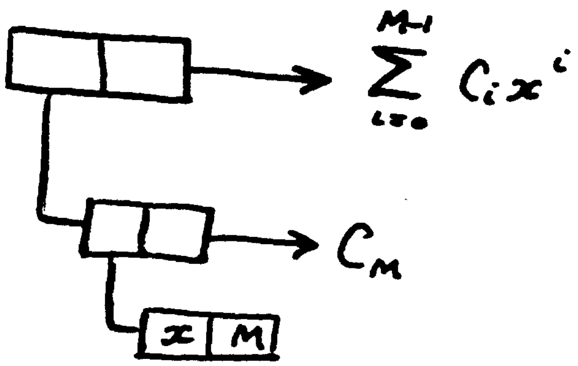
Consider a polynomial in n variables

$$\sum C_{ijk\ldots} \, x^i y^j \ldots z^k \qquad C_{ij\ldots k} \text{ number}$$

or

$$\sum C_i \, x^i \qquad \text{with } C_i \text{ polynomial in } n-1$$

This second form gives the representation



Variants: Can $M=0$?

Is mention of x necessary?

Tabular Form

ALTRAN representation : 3 tables

variables
coefficients
exponents

x	y	z	t

3
-2
5
15
1

0	0	1	0
1	0	0	0
1	2	0	0
0	1	1	6
2	0	0	1

Represents:

$$3 + z + (-2)x + 5xy^2 + 15yzt^5 + x^2t$$

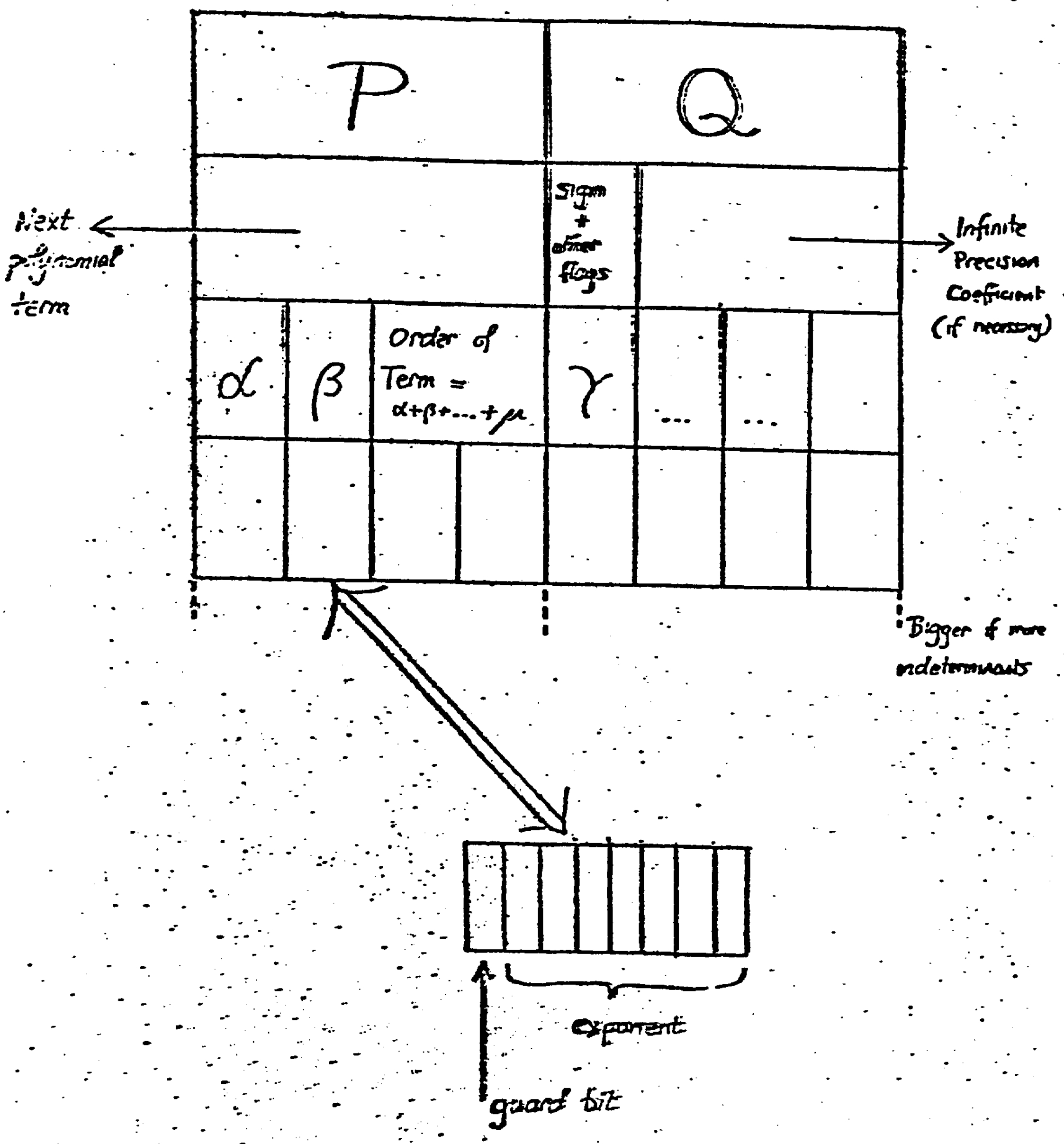


Figure 3 Representation of $\frac{P}{Q}$ $a^{\alpha}b^{\beta}\ldots m^{\mu}$

Ordering

The entries in any of the structures need to have an order to avoid unnecessary expressions

$$x + y + x + 2y + 3x - 3y - 4x$$

In recursive structure, order variables & then increasing or decreasing powers

REDUCE uses decreasing powers & "random" order of variables

Perturbation systems tend to use increasing powers

Basic Algorithms

ADD

SUBTRACT

MULTIPLE

DIFFERENTIATE

Harder Algorithms

DIVISION

FACTORIZATION

INTEGRATION

ADDITION: REDUCE Style

$$X \equiv C_i x^i + \sum C_j x^j \quad \text{or} \quad \text{number}$$

$$Y \equiv C_k y^k + \sum C_l y^l \quad \text{or} \quad \text{number}$$

```
PROCEDURE ADD(X,Y);

  IF NUMBERP(X) THEN
     IF NUMBERP(Y) THEN   X+Y
     ELSE                 LT Y  .+ ADD(X,RED Y)
  ELSE IF NUMBERP(X) THEN LT X  .+ ADD(RED X, Y)
  ELSE
   IF LVAR X = LVAR Y  THEN
     IF LPOW X = LPOW Y   THEN
               ADD(LC X, LC Y) .* (LVAR X . LPOW X)
                  .+ ADD (RED X , RED Y)
      ELSE IF LPOW X > LPOW Y  THEN
               LT X  .+ ADD( RED X, Y)
          ELSE LT Y  .+ ADD ( X, RED Y)
   ELSE IF ORDER(X,Y) THEN
               LT X  .+ ADD (RED X , Y)
       ELSE    LT Y  .+ ADD (X , RED Y)
```