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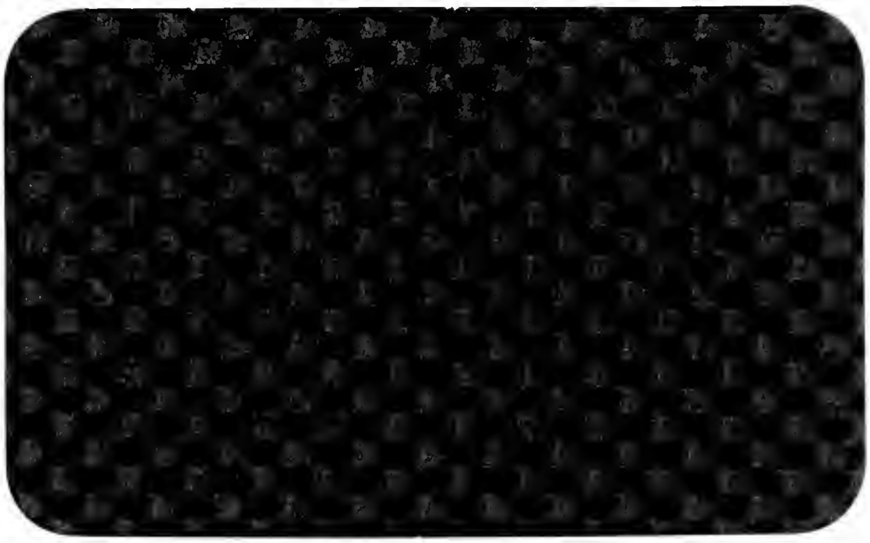

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DECISION PROCEDURES FOR ELEMENTARY SUBLANGUAGES OF SET THEORY.
XIV. THREE LANGUAGES INVOLVING RANK RELATED CONSTRUCTS

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1. INTRODUCTION.

In this paper we present three decidability results for some quantifier-free and quantified theories of sets involving rank related constructs.

For the unquantified case, we will show that the theories in the language $\emptyset$ (empty set), $=$ (equality), $\in$ (membership), $\cup$ (union), $\setminus$ (set difference) plus rank comparison and singleton (MLSSR), or plus the operator $\text{pred}_<$ (set-of-predecessors) (see [Vau]) defined as

$$\text{pred}_<(x) = \{z : rk(z) < rk(z)\} \ (\text{MLSPR}_<),$$

have a solvable satisfiability problem.

As for the quantified case, we will prove that the propositional closure of simple prenex formulas in the language $\emptyset$, $=$, $\in$, rk (*rank* operator) has a solvable *finite* satisfiability problem.

The notion of *trapped places* and *trapped variables* previously introduced in [CFS] is here generalized in two ways and plays an important rôle.

Other results concerning *rank* constructs are contained in [CFMS] where the theory MLS (cf. [FOS]) extended by the *rank* operator or by the *rank comparison* predicate are shown to be decidable.

[BFOS] solves the ordinary satisfiability problem for some elementary quantified theories.

We use techniques and ideas developed in [CFMS], [CFS] and [BFOS]. For all the definitions and basic properties in set theory we refer to [Jec] and [Vau].

2. PRELIMINARIES.

In [FOS], the theory MLS, which is the set of formulas built using the boolean connectives (conjunction, disjunction, implication and negation) from set theoretic atoms of the following types:

$$\begin{aligned} &x = y \cup z, \quad x = y \setminus z \\ &x \in y, \quad x = \emptyset \end{aligned} \tag{2.1}$$

is shown to be decidable.

Here we summarize briefly the basic concepts and results.

It can be shown that the decision problem for the theory MLS is equivalent to giving an algorithm for deciding satisfiability of any conjunction P of literals of type:

$$
\begin{array}{lll}
 & (=) & x = y \cup z, \quad x = y \setminus z \\
(2.2) & (\in) & x \in y \\
 & (\notin) & x \notin y
\end{array}
$$

The following definitions play a central rôle in subsequent sections.

Definition 2.1. A *place* π of P is a 0/1-valued function on the set of all variables in P such that

$$\pi(x) = \pi(y) \vee \pi(z) \text{ if } x = y \cup z \text{ is in } P$$

and

$$\pi(x) = \pi(y) \wedge \neg\pi(z) \text{ if } x = y \setminus z \text{ is in } P.$$

Definition 2.2. Given a variable x of P, a place π is said to be a *place of* P *at* x if:

$$\pi(y) = 1 \text{ if } x \in y \text{ is in } P$$

and

$$\pi(y) = 0 \text{ if } x \notin y \text{ is in } P.$$

In the next sections we will also make use of the following notions.

Definition 2.3. An *injective model* of a formula ϕ is any model of ϕ which maps distinct variables into distinct sets.

Definition 2.4. ϕ is *injectively satisfiable* if it has an injective model.

Clearly the following holds:

Theorem 2.1. *ϕ is satisfiable if and only if it is injectively satisfiable.* •

The theorem in [FOS] can then be rewritten

Theorem 2.2. *Let* P *be a normalized conjunction of literals of type* (2.2). *Let* $V = \{y_1, \ldots, y_m\}$ *be the set of variables occurring in* P. *Then* P *is injectively satisfiable if and only if there exist*

(i) *a set* $\Pi = \{\pi_1, \ldots, \pi_n\}$ *of places of* P;

(ii) *a mapping* $x \mapsto \pi^x$ *from* V *into* Π;

(iii) *a linear ordering of* Π

such that:

(a) *no two distinct variables in* P *are* Π*-equivalent;*

(b) *for each* x *in* V *and* π *in* Π, *if* $\pi(x) = 1$ *then* $\pi < \pi^x$. •

3. MLS EXTENDED BY RANK COMPARISON AND SINGLETON.

Let MLSSR be the unquantified theory which extends MLS by adding to the atoms of (2.1) the following

$$\begin{aligned} x &\le y \text{ which means } rank(x) \le rank(y) \\ x &< y \text{ which means } rank(x) < rank(y) \\ x &= \{y\}, \text{ where } \{\cdot\} \text{ is the singleton operator.} \end{aligned} \tag{3.1}$$

In [FOS] and [CFMS] the extensions of MLS with each of these constructs were shown to be decidable. Here we will show that both extensions can be handled simultaneously, thus obtaining the decidability of MLSSR. Arguing as in the preceding section, in order to prove the decidability of MLSSR it is sufficient to give an algorithm for detecting injective satisfiability of a conjunction P of literals of type (2.2) and (3.1). We can assume without loss of generality that P contains the literals:

$$\begin{aligned} y_0 &= \emptyset \\ y_1 &= \{y_0\} \end{aligned} \tag{3.2}$$

Let $\Pi = \{\pi_0, \dots, \pi_n\}$ be a set of places of P and let $y_0, \dots, y_m$ be the variables in P. Put

$$\Delta_i = \{\pi_j : \pi_j(y_i) = 1\}$$

Notice that $\Delta_0 = \emptyset$.

Definition 3.1. Let Δ_i, Δ_j be such that $\Delta_i \neq \Delta_j$. We write $\Delta_i \longrightarrow \Delta_j$ if and only if either $y_i = \{y_j\}$, or $y_i \le y_j$, or $y_i < y_j$ is in P .

Definition 3.2. A set Δ_i is said to be *bounded* if and only if either $\Delta_i = \emptyset$ or $\Delta_i \overset{*}{\longrightarrow} \emptyset$, where $\overset{*}{\longrightarrow}$ is the transitive closure of the relation $\longrightarrow$ defined above.

Definition 3.3. A place $\pi \in \Pi$ is called *trapped* if and only if $\pi \in \Delta_i$ for some bounded Δ_i. A variable y_i is *trapped* if and only if every $\pi \in \Delta_i$ is trapped.

Notice that π_0 and y_0 are both trapped.

Decidability of MLSSR is an immediate consequence of the following theorem.

Theorem 3.1. *Let P be a normalized conjunction of* MLSSR. *Let $V = \{y_0, \dots, y_m\}$ be the set of variables occurring in* P. *Then* P *is injectively satisfiable if and only if there exist:*

(i) *a set $\Pi = \{\pi_0, \dots, \pi_n\}$ of places of* P; (*without loss of generality we can suppose that there exist $0 < k \le n$ and $0 < h \le m$ such that: only $\pi_0 \dots, \pi_k$ are trapped, π_0 is a place at $\emptyset$ and only $y_0, \dots, y_h$ are trapped*);

(ii) *nonempty pairwise disjoint hereditarily finite sets $\overline{\pi_j}$, $0 \le j \le k$, of rank lower than $h+1$ such that the assignment $My_i = \bigcup_{\pi_j(y_i)=1} \overline{\pi_j}$ is an injective model for the subset of* P *involving only trapped variables*;

(iii) *a mapping $x \mapsto \pi^x$ from* V *into* Π; (*for simplicity we define a function $F : \{0, \dots, m\} \to \{0, \dots, n\}$ such that $F(i) = j$ if $\pi^{y_i} = \pi_j$*)

(iv) *a sequence of integers:* $r_0 = 0 < r_1 < \ldots < r_e = n-k$ *and a function* $R : \{k+1, \ldots, n\} \to \{0, 1, \ldots, e\}$ *such that:*

(a) *no two variables in* P *are* Π-equivalent;

(b) $\pi^{y_i}(= \pi_{F(i)})$ *is a place at* y_i *for all variables in* P;

(c) *if* y_i *and* π_j *are trapped and* $My_i \in \overline{\pi_j}$ *then* $\pi^{y_i} = \pi_j$;

(d) *if* $j > k$ *(i.e. if* π_j *is nontrapped) then* $r_{R(j)-1} < j - k \leq r_{R(j)}$;

(e) *if* $i > h$, $j > k$ *(i.e. if* y_i *and* π_j *are not trapped) and* $\pi_j(y_i) = 1$ *then* $r_{R(j)} < r_{R(F(i))}$

.

For all $i \in \{0, \ldots, m\}$ *such that* y_i *is nontrapped we put*

$$i^* = max\{R(t) : \pi_t(y_i) = 1\}.$$

Then we have

(f) *if* $y_{i_1} \leq y_{i_2}$ *is in* P *and* y_{i_1} *is nontrapped then* $i_1^* \leq i_2^*$;

(g) *if* $y_{i_1} < y_{i_2}$ *is in* P *and* y_{i_1} *is nontrapped then* $i_1^* < i_2^*$;

(h) *if* $y_{i_1} = \{y_{i_2}\}$ *and* y_{i_2} *is nontrapped then*

(h_1) $\pi^{y_{i_2}}(y_{i_1}) = 1$;

(h_2) *if* $\pi_j \neq \pi^{y_{i_2}}$ *then* $\pi_j(y_{i_1}) = 0$, $j \in \{0, \ldots, n\}$;

(h_3) *if* $F(i) = F(i_2)$ *then* $i = i_2$, *for all* $i \in \{0, \ldots, m\}$ *(i.e.,* $\pi^{y_{i_2}}$ *is a place only at the variable* y_{i_2}*);*

(h_4) $R(F(i_2)) = i_2^* + 1$.

Proof: ($\Rightarrow$) Assume that P has an injective model M. Let $\sigma_0, \ldots, \sigma_n$ be the nonempty, disjoint parts of the Venn diagram defined by $My_0, \ldots, My_m$ in the universe $My_0 \cup \ldots \cup My_m \cup \{My_0, \ldots, My_m\}$.

Let

$$\pi_j(x) = \begin{cases} 1 & \text{if } \sigma_j \subseteq Mx \\ 0 & \text{if } \sigma_j \cap Mx = \emptyset \end{cases}, \quad \text{for all } j \in \{0, \ldots, n\}.$$

Let $\Pi = \{\pi_0, \ldots, \pi_n\}$ and put $\pi^{y_i} = \pi_j$ if and only if $My_i \in \sigma_j$, i.e. $F(i) = j$ if and only if $My_i \in \sigma_j$.

Assume that $\pi_0, \ldots, \pi_k$ are the trapped places, $y_0, \ldots, y_h$ are the trapped variables and that π_0 is the place at $\emptyset$. Suppose also that $j_1 < j_2$ implies $rank(\sigma_{j_1}) \leq rank(\sigma_{j_2})$ and $i_1 < i_2$ $rank(My_{i_1}) \leq rank(My_{i_2})$.

Lemma 3.1. *For* $0 \leq j \leq k$, σ_j *is hereditarily finite and* $rank(\sigma_j) < h + 1$.

Proof: Since π_j is trapped by Definition 3.3, $\pi_j \in \Delta_i$ for some bounded Δ_i . Clearly $\Delta_i \neq \emptyset$. Hence by Definition 3.2, $\Delta_i \xrightarrow{*} \emptyset$. This means that there is a chain

$$\Delta_i = \Delta_{i_t} \to \Delta_{i_{t-1}} \to \ldots \to \Delta_{i_0} = \Delta_0 = \emptyset \tag{3.3}$$

with $t \geq 1$ (see Definition 3.1).

Claim. For every $1 \leq l \leq t$ and for every $\pi_j \in \Delta_{i_l}$

$$rank(\sigma_j) < l + 1 \tag{3.4}$$

Proof of the Claim: We proceed by induction on l. If $l = 1$ and $\pi_j \in \Delta_{i_1}$ we have

$$\Delta_{i_1} \to \emptyset \text{ and } \Delta_{i_1} \neq \emptyset.$$

By Definition 3.1, it follows that the literal $y_{i_1} = \{y_0\}$ is in P. Since $My_{i_1} = \{My_0\} = \{\emptyset\}$, it follows that $\sigma_0 = \{\emptyset\}$ so that $rank(\sigma_0) = 1 < 2$.

Assume now that the claim is true for every $1 \leq l' < l$ and let $\pi_j \in \Delta_{i_l} \rightarrow \Delta_{i_{l-1}}$. We distinguish the following subcases.

Case 1). $y_{i_l} = \{y_{i_{l-1}}\}$ is in P. Since $My_{i_l} = \{My_{i_{l-1}}\}$ then $\sigma_j = My_{i_l}$. By induction hypothesis:

$$rank(My_{i_{l-1}}) = \max_{\pi_s(y_{i_{l-1}})=1} rank(\sigma_s) < l - 1 + 1 = l\ .$$

Hence $rank(\sigma_j) = rank(My_{i_{l-1}}) + 1 < l + 1$.

Case 2). Either $y_{i_l} \leq y_{i_{l-1}}$ or $y_{i_l} < y_{i_{l-1}}$ is in P. It follows by induction hypothesis

$$rank(\sigma_j) \leq rank(My_{i_l}) \leq rank(My_{i_{l-1}}) < l < l + 1.$$

This completes the proof of the claim.

Without loss of generality we can assume that in (3.3) all the Δ_{i_l} are pairwise distinct (since any cycle can be skipped). Therefore each reduction $\rightarrow$ introduces a new trapped variable. This means that the length t of (3.3) is at most h. This together with (3.4) proves Lemma 3.1. •

We choose $r_1, \ldots, r_e$ in such a way that for $k < j_1, j_2 \leq n$:

$$r_{a-1} < j_1 - k, \quad j_2 - k \leq r_a \leftrightarrow rank(\sigma_{j_1}) = rank(\sigma_{j_2})$$

and we put $R(j_1) = R(j_2) = a$.

Trivially (b), (d) and (e) are true.

Let us prove that also (f) holds. If $y_{i_1} \leq y_{i_2}$ then $rank(My_{i_1}) \leq rank(My_{i_2})$. Now $My_{i_1} = \bigcup_{\pi_j(y_{i_1})=1} \sigma_j$ and $My_{i_2} = \bigcup_{\pi_j(y_{i_2})=1} \sigma_j$. If a is the maximum index of elements of $\Delta_{i_2} = \{\pi_j : \pi_j(y_{i_2}) = 1\}$ then

$$rank(My_{i_2}) = rank(\sigma_a).$$

For each j such that $\pi_j(y_{i_1}) = 1$,

$$rank(\sigma_j) \leq rank(My_{i_1}) \leq rank(\sigma_a).$$

It follows that for every $\pi_j \in \Delta_{i_1}$, $R(j) \leq R(a)$ and this completes the proof of (f).

Similarly we can show that (g) holds.

Finally, it is trivial to see that (i) also holds, completing the proof of the theorem in one direction.

($\Leftarrow$) Conversely, if there exist $\Pi, \overline{\pi_0}, \ldots, \overline{\pi_k}, x \mapsto \pi^x, r_0, \ldots, r_e$ such that conditions (b)-(i) hold, we build a model for P in the following way: let γ be an integer such that

$$\gamma > \sum_{\pi_j \text{ trapped}} (|\overline{\pi_j}|) + n + m.$$

For $k < j \leq n$ let

$$I_j = \{0, 1, \ldots, n, \ldots, \gamma + R(j)\} \setminus \{j\}.$$

So

$$|I_j| = \gamma + R(j) \text{ and } rank(I_j) = \gamma + R(j) + 1,$$

and for each j we have $rank(I_j) = rank(I_{R(j)})$.

For $k < j \leq n$ put

$$(3.5) \qquad \sigma_j = \begin{cases} \{My_i : F(i) = j\} & \text{if there is } y_{i'} = \{y_i\} \text{ in P and } F(i) = j \\ \{I_j\} \cup \{My_i : F(i) = j\} & \text{otherwise.} \end{cases}$$

Notice that, by condition (h_3), (3.5) is independent of the literal $y_{i'} = \{y_i\}$.

The following lemma can be proved much in the same way of Lemma 3.1.

Lemma 3.2. For $0 \leq j \leq k$, σ_j is hereditarily finite and $rank(\sigma_j) < h + 1$.

Lemma 3.3. $rank(\sigma_j) = rank(I_{r_{R(j)}}) + 1$, $k < j \leq n$.

Proof: We proceed by induction on j. If $j = k + 1$ then since π_{k+1} is nontrapped it cannot be the case that $\{\pi_j\} = \Delta_i$ for some literal $y_i = \{y_{i'}\}$ in P. Hence

$$\sigma_{k+1} = \{I_{k+1}\} \cup \{My_i : F(i) = k + 1\}.$$

Now, if $My_i \in \sigma_{k+1}$ it follows by Lemma 3.2 that $rank(My_i) < h + 1 < \gamma + 1 = rank(I_{k+1})$. Therefore $rank(\sigma_{k+1}) = rank(I_{k+1}) + 1 = rank(I_{R(k+1)})$.

Inductive step: case a) $\sigma_j = \{My_i\}$. By condition (h_4) $rank(\sigma_j) = rank(My_i) + 1 = rank(\sigma_t) + 1$ for some t such that $\pi_t(y_i) = 1$ and $i^* = R(t)$, and

$$R(j) = R(t) + 1 = i^* + 1$$

Since y_i is not trapped then π_j is not trapped and by induction hypothesis $rank(\sigma_j) = rank(\sigma_t) + 1 = rank(I_{r_{R(t)}}) + 1 + 1 = rank(I_{r_{R(t)+1}}) + 1 = rank(I_{r_{R(j)}}) + 1$.

Case b). $\sigma_j = \{I_j\} \cup \{My_i : F(i) = j\}$. Now if $My_i \in \sigma_j$ and $\sigma_{j'} \subseteq My_i$ then by (e)

$$R(j') < R(F(i)) = R(j)$$

So $rank(My_i) < rank(I_{r_{R(j)}})$ thus $rank(\sigma_j) = rank(I_j) + 1 = rank(I_{r_{R(j)}}) + 1$ completing the proof of Lemma 3.3. •

Lemma 3.4. *If $s \leq k$ and $s < j$ then $\sigma_s \cap \sigma_j = \emptyset$.*

Proof: If $j \leq k$ then $\sigma_s \cap \sigma_j = \emptyset$ by condition (ii) of Theorem 3.1. if $j > k$ then $\sigma_j = \{I_j\} \cup \{My_i : F(i) = j\}$. $I_j \notin \sigma_s$ by Lemma 3.2. Similarly if y_i is not trapped then $My_i \notin \sigma_s$ by Lemmas 3.2 and 3.3. Moreover if y_i is trapped and $F(i) = j$ then by condition (b) $My_i \notin \sigma_s$ since $s \neq j = F(t)$. Consequently, $\sigma_s \cap \sigma_j = \emptyset$. Lemma 3.4 is thus proved. •

Lemma 3.5. *If $k < s < j then \sigma_s \cap \sigma_j = \emptyset$.*

Proof: We proceed by induction on s.

Base Case: If $s = k + 1$ then $\sigma_{k+1} = \{I_{k+1}\} \cup \{My_i : F(i) = k + 1\}$ where by condition (e) all the y_i such that $F(i) = k + 1$ are trapped. So if $\sigma_j = \{My_{i'}\}$ with $F(i') = j$ then $My_{i'} \neq My_i$ since

$My_{i'}$ is not trapped. Moreover $My_{i'} \neq I_{k+1}$ because $|I_{k+1}| = \gamma + R(k+1)$ whereas $|My_{i'}| < \gamma$. Thus if $\sigma_j = \{My_{i'}\}$ then $\sigma_{k+1} \cap \sigma_j = \emptyset$. If $\sigma_j = \{I_j\} \cup \{My_{i'} : F(i') = j\}$, obviously $I_j \neq I_{k+1}$. If $F(i) = s = k+1$ and $F(i') = j$ we know that y_i must be trapped. If $y_{i'}$ is also trapped then by condition (ii) of the theorem $My_i \neq My_{i'}$. On the other hand if $y_{i'}$ is not trapped then by Lemmas 3.2 and 3.3 $My_i \neq My_{i'}$. This shows that $\sigma_{k+1} \cap \sigma_j = \emptyset$ for every $j > k+1$.

Inductive step. Assume that the assertion is true for every $k < s_0 < s$ and let $j > s$. Since

$$\begin{aligned} \sigma_s &= \{I_s\} \cup \{My_i : F(i) = s\} \\ \sigma_j &= \{I_j\} \cup \{My_{i'} : F(i') = j\} \end{aligned} \tag{3.6}$$

it is sufficient to show that the right-hand side members in (3.6) are disjoint. Indeed, $I_s \neq I_j$ and $I_s \neq My_i$ by Lemmas 3.2 and 3.3. Finally if $F(i) = s, F(i') = j$ then clearly if $s \neq j$, $My_i \neq My_{i'}$. In fact we have the following two cases.

Case a). If there there exists π_b such that $\pi_b(y_i) = 1$ and $\pi_b(y_{i'}) = 0$ then $\sigma_b \subseteq My_i$ whereas by induction hypothesis and by Lemma 3.4 $\sigma_b \cap My_{i'} = \emptyset$ yielding $My_i \neq My_{i'}$.

Case b). If $\pi_b(y_i) = 1 \rightarrow \pi_b(y_{i'}) = 1$ then $My_i \subseteq My_{i'}$. On the other hand there must exist b' such that $\pi_{b'}(y_{i'}) = 1$ and $\pi_b(y_i) = 0$ otherwise $i = i'$ and so $F(i) = F(i')$. Hence by induction hypothesis $\sigma_b \cap \sigma_{b'} = \emptyset$ for every b such that $\pi_b(y_i) = 1$ showing $My_i \neq My_{i'}$ since $\sigma_{b'} \subseteq My_{i'} \setminus My_i$. This completes the proof of Lemma 3.5.

By Theorem 2.1 we can affirm that M is also an injective model for the literals of type (2.2) with occurrences of nontrapped variables. Also $y_i \leq y_{i'}$ is in P with y_i trapped and $y_{i'}$ nontrapped, then

$$My_i \leq My_{i'} \text{ because } rank(My_i) \leq h \text{ and } rank(My_{i'}) > \gamma.$$

If $y_i, y_{i'}$ are both nontrapped then $rank(My_i) \leq rank(My_{i'})$ by condition (f). Therefore M is a model of all the literals of type $\leq$. Literals of type $y_i < y_{i'}$ are handled in a similar way by making use of condition (g). Finally if $y_i = \{y_{i'}\}$ is in P and $y_{i'}$ is not trapped then, by (i), $My_{i'} = \sigma_{F(i')} = \{My_i\}$, proving that M is indeed a model of P and in turn concluding the proof of the theorem. •

4. MLS EXTENDED BY THE SET OF PREDECESSOR OPERATOR.

Consider the theory MLSPR$_<$ which extends MLS by adding to the atoms of type (2.1) the following:

$$x = pred_<(y). \tag{4.1}$$

where $pred_<(y) = \{z : rank(z) < rank(y)\}$.

As in [FOS] decidability of MLSPR$_<$ is equivalent to checking injective satisfiability of any conjunction P of literals of type (2.2) and (4.1). The following theorem establishes the decidability of MLSPR$_<$.

Theorem 4.1. *Let* P *be a conjunction of literals of type* (2.2) *and* (4.1) *and let* V *be the set of all variables in* P. *Then* P *is satisfiable if and only if there exist:*

(i) *a set* $\Pi = \{\pi_1, \dots, \pi_n\}$ *of pairwise distinct places. Let* $V = \{y_1, \dots, y_m\}$ *be the set of all variables in* P.

(ii) *a mapping* $y_i \mapsto \pi^{y_i}$.
For simplicity we introduce the function

$$F : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\},$$

$$\pi^{y_i} = \pi_j \text{ if and only if } F(i) = j.$$

(iii) *a sequence of integers,* $0 = r_0 < r_1 < \dots < r_k = n,$ *and a function* $R : \{1, 2, \dots, n\} \to \{1, 2, \dots, k\}$ *such that:*

(*a*) $r_{R(j)-1} < j \leq r_{R(j)}, \quad 1 \leq j \leq n;$

(*b*) $\pi_{F(i)} = \pi^{y_i}$ *is a place at* y_i, $1 \leq i \leq m;$

(*c*) *if* $\pi_j(y_i) = 1$ *then* $r_{R(j)} < r_{R(F(i))}$;

(*d*) *if* $y_{i_1} = \mathrm{pred}_<(y_{i_2})$ *is in* P *then if we put* $i_1^* = \max\{R(j) : \pi_j(y_{i_1}) = 1\}$ *for every* $1 \leq i_1 \leq m$, *then*

(d_1) $\pi_j(y_{i_1}) = 1$ *if and only if* $j \leq r_{i_1^*}$

(d_2) *if* y_s *is such that* $s^* < i_1^*$ *then* $F(s) \leq r_{i_1^*}$, *i.e. if* $\pi^{y_s} = \pi_j$ *then* $j \leq r_{i_1^*}$.

Proof: ($\Rightarrow$) Assume that P has an injective model M. Let $\{y_1, \dots, y_m\}$ be the set of all variables in P. Let $\sigma_1, \dots, \sigma_n$ be the nonempty, disjoint parts of the Venn diagram defined by $My_1, \dots, My_m$ in the universe

$$My_1 \cup \dots \cup My_m \cup \{My_1, \dots, My_m\}.$$

Let

$$\pi_j(x) = \begin{cases} 1 & \text{if } \sigma_j \subseteq Mx \\ 0 & \text{if } \sigma_j \cap Mx = \emptyset \end{cases},$$

$\Pi = \{\pi_1, \dots, \pi_n\}$ and $F(i) = j \leftrightarrow My_i \in \sigma_j$. Then, by [FOS], Π is a set of places of P satisfying $Mx = \bigcup_{\pi_j(x)=1} \sigma_j$. Without loss of generality we can suppose that if $j_1 < j_2$ then $rank(\sigma_{j_1}) \leq rank(\sigma_{j_2})$. So we choose $r_1, \dots, r_k$ in such a way that: for all $j_1, j_2, r_{h-1} < j_1, j_2 \leq r_h \leftrightarrow rank(\sigma_{j_1}) = rank(\sigma_{j_2})$ and in this case we put: $R(j_1) = R(j_2) = h$. Trivially (a), (b) and (c) are true. To see that (d) also holds assume that $y_{i_1} = pred_<(y_{i_2})$ is in P. Then, since $My_{i_1} = pred_<(My_{i_2})$ we have

$$\pi_j(y_{i_1}) = 1 \leftrightarrow \sigma_j \subseteq My_{i_1} \leftrightarrow rank(\sigma_j) \leq rank(My_{i_2}) \leftrightarrow j \leq r_{i_2^*}.$$

Moreover if y_s is such that $s^* < i_2^*$ then

$$rank(My_s) < rank(My_{i_2}) \to My_s \in My_{i_1} \to \sigma_{F(s)} \subseteq My_{i_1} \to F(s) \leq r_{i_2^*}.$$

($\Leftarrow$) Conversely, assume that Π, F, $r_1, \dots, r_k$ and R exist in such a way that (a)-(d) are verified. Let $I_j = \{2r_{R(j)}, j\}$. So we have $rank(I_j) = 2r_{R(j)} + 1$. Following the increasing order of indices put

$$\sigma_j = \{I_j\} \cup \{My_t : F(t) = j\}, \quad j \neq r_h, \quad 1 \leq h \leq k$$

$$\sigma_{r_h} = pred_<(\{I_{r_h}\}) \setminus \left(\left(\bigcup_{t < r_h} \sigma_t \right) \cup \{My_s : s^* < h\} \right) \tag{4.2}$$

and $My_s = \bigcup_{\pi_j(y_s)=1} \sigma_j$.

Lemma 4.1. $rank(\sigma_j) = rank(I_j) + 1$, $1 \le j \le n$.

Proof: We proceed by induction on j.

Base case: $\sigma_1 = \{I_1\} \cup \{My_s : F(s) = 1\}$. By (c) if $F(s) = 1$ i.e. $\pi^{y_s} = \pi_1$ then $\pi_j(y_s) = 0$ for all j. Thus

$$\sigma_1 = \begin{cases} \{I_1\} \cup \{\emptyset\} & \text{if} \quad \mathrm{F(s)} = 1 \quad \text{for some s} \\ \{I_1\} & \text{otherwise} \end{cases}$$

In any case $rank(\sigma_1) = rank(I_1) + 1$.

Induction step: Suppose that the assertion is true for any σ'_j with $j' < j$ and let us show that it holds for σ_j.

Case a) $j = r_h$ for some h. So

$$\sigma_j = pred(\{I_j\}) \setminus ((\bigcup_{t<j} \sigma_t) \cup \{My_s : s^* < h\})$$

First we show that:

$$I_j \notin \bigcup_{t<j} \sigma_t. \tag{4.3}$$

Indeed let $t < j$ and consider two cases.

Case a_1. $R(t) < h$. In this case by induction hypothesis

$$rank(\sigma_t) = rank(I_t) + 1 = 2r_{R(t)} + 2 < 2r_h + 1.$$

In fact, $r_{R(t)} + 1 \le r_h$, so $r_{R(t)} + 2 \le 2r_h < 2r_h + 1$. But $rank(I_j) = 2r_h + 1$, thus $rank(\sigma_t) < rank(I_j)$. So $I_j \notin \sigma_t$.

Case a_2. $R(t) = h$. In this case t is not of type r_b, $1 \le b \le h$. So

$$\sigma_t = \{I_t\} \cup \{My_s : F(s) = t\}$$

Since $t < j$, $I_j \ne I_t$. Moreover $rank\{My_s\} = rank\left(\bigcup_{\pi_a(y_s)=1} \sigma_a\right)$. Furthermore

$$\pi_a(y_s) = 1 \rightarrow a < F(s) = t < j.$$

By (c) it follows that $r_{R(a)} < r_{R(t)} = j$. By induction hypothesis

$$rank(\sigma_a) = rank(I_a) + 1 = 2r_{R(a)} + 2 < 2r_h + 1.$$

It follows hence

$$rank(My_s) < 2r_h + 1, \quad \text{whereas} \quad rank(I_{r_h}) = 2r_h + 1.$$

Therefore $I_j \ne My_t$ and $I_j \notin \sigma_t$. This completes the proof of (4.3).

Let us now prove

$$I_j \notin \{My_s : s^* < h\}. \tag{4.4}$$

By the argument of (Case a_2) and by induction hypothesis

$$rank(My_s) = rank(\sigma_{r_{s^*}}) < rank(\sigma_{r_{s^*}}) + 1 < rank(\sigma_{r_h}) + 1.$$

Therefore if $s^* < h$, $My_t \neq I_j$. (4.3) and (4.4) show that

$$I_j = I_{r_h} \notin \left(\bigcup_{t < r_h} \sigma_t \right) \cup \{My_t : t^* < h\}.$$

So we can conclude

$$rank(\sigma_j) = rank(I_j) + 1$$

Case b). $j \neq r_h$, $1 \leq h \leq k$. In this case $\sigma_j = \{I_j\} \cup \{My_s : F(s) = j\}$. On the other hand, if $F(s) = j$ then by (c) $\pi_b(y_s) = 1 \rightarrow r_{R(b)} < r_{R(j)}$, yielding $2r_{R(b)} + 2 < 2r_{R(j)} + 1$. Therefore if $F(s) = j$ and $\pi_b(y_s) = 1$, $rank(\pi_b) = 2r_{R(b)} + 2 < rank(I_i)$ and consequently $rank(My_s) < rank(I_j)$ which shows that even in this case $rank(\sigma_j) = rank(I_j) + 1$. Thus, the proof of Lemma 4.1 is completed. •

Lemma 4.2. $\sigma_{j_1} \cap \sigma_{j_2} = \emptyset$ whenever $j_1 < j_2$.

Proof: If $j_2 = r_h$ for some $1 \leq h \leq n$, then by (4.2) the lemma holds. So we can assume that $j_2 \neq r_h$, $1 \leq h \leq k$ and

$$\sigma_{j_2} = \{I_{j_2}\} \cup \{My_s : F(s) = j_2\}. \tag{4.5}$$

We proceed by induction on j_1.

Base case: Let us show that $\sigma_1 \cap \sigma_{j_2} = \emptyset$ if $j_2 > 1$. By the argument used in the proof of the preceding lemma

$$\sigma_1 = \begin{cases} \{I_1\} \cup \{\emptyset\} & \text{if } F(s) = 1 \text{ for some } 1 \leq s \leq m \\ \{I_1\} & \text{otherwise .} \end{cases}$$

Now $I_1 \neq I_{j_2}$ and $I_1 \neq My_s$ since I_1 has odd rank, whereas, by the preceding lemma, My_s has even rank. It follows by (4.5) that $I_1 \notin \sigma_{j_2}$. On the other hand if $F(s) = 1$ then $\emptyset \notin \sigma_{j_2}$. Consequently $\sigma_1 \cap \sigma_{j_2} = \emptyset$.

Inductive step): Assume that the lemma holds for $1 \leq j' < j_1$ and let us show that $\sigma_{j_1} \cap \sigma_{j_2} = \emptyset$ whenever $j_1 < j_2$.

Case 1). $j_1 \neq r_h$, $1 \leq h \leq k$. Then

$$\sigma_{j_1} = \{I_{j_1}\} \cup \{My_s : F(s) = j_1\}$$
$$\sigma_{j_2} = \{I_{j_2}\} \cup \{My_{s'} : F(s') = j_2\}$$

We have: $I_{j_1} \neq I_{j_2}$; $I_{j_1}, I_{j_2} \neq My_s, My_{s'}$ for every s, s', since the I_j's have odd ranks whereas My_s's have even rank for every s. Therefore to show disjointness it is sufficient to prove that

$$My_s \neq My_{s'} \quad \text{if} \quad F(s) = j_1 \quad \text{and} \quad F(s') = j_2.$$

Indeed by the induction hypothesis and by (c)

$$\sigma_{t_1} \cap \sigma_{t_2} = \emptyset$$

for every t_1 such that $\pi_{t_1}(y_s) = 1$, $t_2 \neq t_1$. It follows that:

$$My_s = \bigcup_{\pi_t(y_s)=1} \sigma_t \neq My_{s'} = \bigcup_{\pi_{t'}(y_{s'})=1} \sigma_{t'}.$$

Since $F(s) \neq F(s')$, then $s \neq s'$ and $y_s \neq y_{s'}$, implying $My_s \neq My_{s'}$. This completes the proof of disjointness in case 1).

Case 2) $j_1 = r_h$ for some $1 \leq h \leq k$.

$$\sigma_{j_1} = \sigma_{r_h} = \mathrm{pred}_<(\{I_{r_h}\}) \setminus \left(\bigcup_{t<r_h} \sigma_t \cup \{My_s : s^* < h\} \right)$$

$$\sigma_{j_2} = \{I_{j_2}\} \cup \{My_{s'} : F(s') = j_2\}.$$

Since $rank(I_{j_2}) > rank(I_{r_h})$ then $I_{j_2} \notin \sigma_{r_h}$. Moreover $My_{s'} \in \sigma_{j_2}$ implies that either $rank(My_{s'}) > rank(I_{r_h})$ and so $My_{s'} \notin \sigma_{r_h}$ or $rank(My_{s'}) < rank(I_{r_h})$ and so $s'^* < h$ which by (3.2) implies $My_{s'} \notin \sigma_{r_h}$. Lemma 2 is then completely proved. •

From Theorem 2.1 it follows that M is a model for all the terms of type (2.2) in P. Moreover assume that $y_{i_1} = \mathrm{pred}_<(y_{i_2})$ is in P. Let $z \in \mathrm{pred}_<(My_{i_2})$ then $rank(z) < rank(My_{i_2}) = rank(\sigma_{r_{i_2^*}}) = rank(I_{r_{i_2^*}}) + 1$. It follows that $z \in \mathrm{pred}_<(\{I_{r_{i_2^*}}\})$. By condition (d_1) we have $\pi_{r_{i_2^*}}(y_{i_1}) = 1$ and then

$$\mathrm{pred}_<(\{I_{r_{i_2^*}}\}) \setminus (\bigcup_{t<r_{i_2^*}} \sigma_t \cup \{My_s : s^* < i_2^*\}) \subseteq My_{i_1}.$$

This implies that if $z \notin (\bigcup_{t<r_{i_2^*}} \sigma_t \cup \{My_s : s^* < i_2^*\})$ then $z \in My_{i_1}$. On the other hand if $z \in \bigcup_{t<r_{i_2^*}} \sigma_t$ then $z \in \sigma_{t'}$ for some $t' < r_{i_2^*}$. It follows that by condition (d_1) we get

$$\pi_{t'}(y_{i_1}) = 1 \text{ giving } \sigma_{t'} \subseteq My_{i_1} \text{ and } z \in My_{i_1}.$$

Finally, if $z \in \{My_s : s^* < i_2^*\}$ then $z = My_s$ for some s such that $s^* < i_2^*$. By (d_2) we have $F(s) \leq r_{i_2^*}$ and by (d_1)

$$z = My_s \in \sigma_{F(s)} \subseteq My_{i_1}.$$

Thus we have showed that $\mathrm{pred}_<(My_{i_2}) \subseteq My_{i_1}$. Conversely, let $z \in My_{i_1}$. Then $z \in \sigma_t$ for some $i \leq t \leq n$ and $\pi_t(y_{i_1}) = 1$. By (d_1) $t \leq r_{i_2^*}$, yielding $rank(z) < rank(\sigma_t) \leq rank(\{I_{r_{i_2^*}}\})$ $= rank(My_{i_2})$. Therefore $My_{i_1} \subseteq \mathrm{pred}_<(My_{i_2})$. This shows that $My_{i_1} = \mathrm{pred}_<(My_{i_2})$ and the proof of the theorem affirming the decidability of $\mathrm{MLSPR}_<$ is complete.

5. FINITE SATISFIABILITY OF FORMULAS INVOLVING RESTRICTED QUANTIFIERS AND THE RANK OPERATOR.

A prenex formula $Q_1 Q_2 \dots Q_n p$ is called *simple* if for $i = 1, 2, \dots, n$ either every Q_i is $(\exists y_i \in z_i)$ or every Q_i is $(\forall y_i \in z_i)$, and no z_j is a y_i for any $i, j = 1, 2, \dots, n$ (cf. [BFOS]). Let T be the quantifier-free theory in the language $\underline{\emptyset}$, $=$, $\in$, rk (where rk is a function symbol which maps sets into their rank). The following theorem contains an implicit algorithm for deciding finite satisfiability of the propositional closure of the class of simple formulas over matrices belonging to the theory T.

Theorem 5.1. *Let* P *be a conjunction of simple prenex formulas of the theory* T, *and let* $V = \{y_1, \dots, y_m\}$ *be the set of free variables occurring in* P. *Without loss of generality we can assume that existential quantifiers are not present in P since they can be eliminated by introducing a new variable for each existentially quantified variable. Let* U *be a set of variables disjoint from* V *and such that:*

$$|U| \leq m^2 + 6m + |\mathcal{V}_{m+1}|,$$

where $\mathcal{V}_m$ *is the collection of all sets having rank less than* m. *Put* $V_0 = V \cup \{\underline{\emptyset}\}$ *and let* P′ *be the formula resulting from* P *by replacing each formula* $(\forall x \in z)p$ *by the set of formulas*

$$\{(x \in z \rightarrow p)^x_w : w \in U \cup V_0\}\dagger$$

until all the univeral quantifiers are eliminated. Then P *is injectively satisfiable if and only if there exist*

(1) *a function* $' : V \rightarrow U \cup V_0$ *(predecessor)*;
(2) *a function* $^- : U \cup V_0 \rightarrow U \cup V_0$ *(rank)*;
(3) *a set* Q *of membership relations such that for all* x *and* y *in* $U \cup V_0$ *either* $x \in y$ *or* $x \notin y$ *occurs in* Q;
(4) *a disjunct* P″ *of a disjunctive normal form of* P′ *such that:*
  (*a*) $\overline{P''} \wedge Q$ *does not contain any explicit contradiction of the form* $\mathcal{A} \wedge \neg\mathcal{A}$, *where* $\overline{P''}$ *denotes the formula obtained by recursively substituting each term* $rk(x)$ *by* $\overline{x}$, *until all terms* $rk(x)$ *are eliminated;*
  (*b*) Q *does not contain any cycle of memberships* $x_0 \in x_1 \in \dots \in x_0$;
  (*c*) *if* $x \in y$ *is in* Q *then* $\overline{x} \in \overline{y}$ *is in* Q;
  (*d*) *if* $x \in \overline{y}$ *is in* Q *then* $\overline{x} \equiv x$;
  (*e*) *if* y *is in* V *then* $y' \in y$ *is in* Q. *Moreover if* $x \in \overline{y}$ *is in* Q *then either* $x \in \overline{y'}$ *or* $x \equiv \overline{y'}$;
  (*f*) $x \notin \underline{\emptyset}$ *is in* Q *for all* x *in* $U \cup V_0$;
  (*g*) $\overline{\underline{\emptyset}} = \underline{\emptyset}$ *is in* Q;
  (*h*) *for all* x, y *in* $U \cup V_0$ *such that* $\overline{x} \not\equiv \overline{y}$, *either* $\overline{x} \in \overline{y}$ *or* $\overline{y} \in \overline{x}$ *is in* Q.

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† By $\phi^{x_1, \dots, x_n}_{w_1, \dots, w_n}$ we denote the result of simultaneously substituting in ϕ all free occurrences of $x_1, \dots, x_n$ with the terms $w_1, \dots, w_n$

Given x in $U \cup V_0$, we say that x is trapped *if and only if either $x \equiv \underline{\emptyset}$ or $x \equiv \overline{z}$ for some trapped z, or $\overline{x}$ is trapped, or x is in V and x' is trapped. Then*

(i_1) *$|\{x \text{ in } U \cup V_0 : x \text{ is nontrapped}\}| \leq m^2 + 6m$;*

(i_2) *if we define a partial assignment M^* over the trapped variables, by recursively putting $M^*\underline{\emptyset} = \emptyset$ and*

$$M^*x = \{M^*y : (y \text{ is in } U \cup V_0) \wedge (y \text{ is trapped}) \wedge (y \in x \text{ is in } Q)\}$$

then

$$M^*\overline{x} = rk(M^*x) \quad \text{for all trapped } x;$$

(i_3) *for every pair x, y in the set $S = V \cup \{x : x \text{ is trapped}\} \cup \{\overline{x} : x \text{ is in } U \cup V_0\}$, if x, y are distinct then there exists z in $U \cup V_0$ such that exactly one of the two literals $z \in x$, $z \in y$ is in Q.*

Proof: Assume first that P is finitely satisfiable and let M be a model of P. Since Mx is finite for all x in V, we can define the map $'$ as follows: let x be in V and let s_x be any element of Mx such that $rank(s_x) + 1 = rank(Mx)$. Then, if $s_x = My$ for some y for which M is defined, we put $x' \equiv y$ otherwise we pick up a new variable z_x and put $x' \equiv z_x$ and $Mz_x = s_x$. Let U_1 be the set of the new variables z introduced in the preceding step. Clearly $|U_1| \leq m$.

Next we partition the variables in $V_0 \cup U_1$ according to the rank of their model. For each class C of variables in the partition we do the following: let x be any variable in C; then if $rank(Mx) = My$ for some y for which M is defined, we put $\overline{x} \equiv y$ and also $\overline{z} \equiv y$ for all $z \in C$, otherwise we introduce a new variable z_x and put $Mz_x \equiv rank(Mx)$, $\overline{z_x} \equiv z_x$, $\overline{z} = z_x$ for all z in C. Let U_2 be the set of variables introduced during the preceding step. Trivially $|U_2| \leq 2m$. We also put

$$Q_1 = \{(x \in y) : x, y \text{ are in } V_0 \cup U_1 \cup U_2 \text{ and } Mx \in My\} \cup$$
$$\{(x \notin y) : x, y \text{ are in } V_0 \cup U_1 \cup U_2 \text{ and } Mx \notin My\}.$$

Using much the same definition given before condition (i) of the theorem (but with respect to $V_0 \cup U_1 \cup U_2$ in place of $V_0 \cup U$ and the set of membership relations Q_1 in place of Q), we can define the notion of trapped variables. Let ℓ_0 be the maximum length of any chain of membership relation in Q_1 of trapped variables. Then

$$\ell_0 \leq m. \tag{5.1}$$

Indeed, by inducting on the length of the derivations needed to prove the trappedness of variables, it is easy to see that for each trapped variable x there is a variable z_x in V_0 such that $\overline{x} \equiv \overline{z_x}$. Therefore, if $x_0 \in x_1 \in \ldots \in x_r$ is any chain of memberships of trapped variables, then there must exist $y_{i_0}, y_{i_1}, \ldots, y_{i_r}$ in V_0 such that $rank(My_{i_j}) \in rank(My_{i_{j+1}})$ for all $j = 0, 1, \ldots, r-1$. Hence $r \leq m$, which proves (5.1).

Let TRANS be the transitive closure of the set $\{Mx : x \text{ is in } V_0 \cup U_1 \cup U_2 \text{ and x is trapped}\}$. Notice that if $s \in$ TRANS then $rank(s) \in$ TRANS. Notice also that since $\ell_0 \leq m$ then TRANS $\subseteq \mathcal{V}_{m+1}$ and consequently $|\text{TRANS}| \leq |\mathcal{V}_{m+1}|$. Now, for each set $s \in \text{TRANS} \setminus \{Mx : x \in V \cup U_1 \cup U_2\}$

introduce a new variable z_s and extend M by putting $Mz_s = s$. Let U_3 be the set of all new variables introduced at this step; clearly $|U_3| \leq |\mathcal{V}_{m+1}|$. In addition, we extend the map $^-$ to U_3 by putting $\overline{x} = z$ where $Mz = rank(Mx)$.

For each pair of distinct x, y in V_0 such that the set $(Mx \setminus My) \cup (My \setminus Mx)$ does not contain any element of type Mz, we choose an element $s_{x,y}$ in $(Mx \setminus My) \cup (My \setminus Mx)$, introduce a new variable $z_{x,y}$ and define $Mz_{x,y} = s_{x,y}$. Also, if $rank(s_{x,y}) = rank(Mz)$, for some z, we put $\overline{z_{x,y}} \equiv \overline{z}$ otherwise we introduce a new variable z_r and put $\overline{z_{x,y}} \equiv \overline{z_r} \equiv z_r$ Let U_4 be the set of the new variables introduced. Trivially $|U_4| \leq 2\binom{m+1}{2} = m^2 + m$.

Finally, for each variable x in V such that Mx is not an ordinal, we distinguish the following two cases according to whether Mx contains nonordinal elements or not. In the first case, we pick a nonordinal element of Mx, say s. We introduce a new variable z_x and put $Mz_x = s$. In addition, if $rank(Mz_x)$ is not already present, we introduce another new variable z_r and put $Mz_r = rank(Mz_x)$ and extend $^-$ by putting $\overline{z_r} = \overline{z_x} = z_r$, otherwise we put $\overline{z_x} = z$, where $rank(Mz_x) = Mz$.

In the second case, i.e. if Mx is a set of ordinals, Mx cannot be transitive for it would be an ordinal itself, so we can pick two sets s_1 and s_2 such that $s_2 \in s_1 \in Mx$ and $s_2 \notin Mx$. Again, if it is the case we introduce new variables for s_1, s_2 and their ranks extending the map $^-$ accordingly. Let U_5 be the set of new variables introduced in the above step. Clearly $|U_5| \leq 2m$.

Finally we put $U = \bigcup_{i=1}^{5} U_i$. We plainly have

$$|U| = |\bigcup_{i=1}^{5} U_i| \leq m^2 + 6m + |\mathcal{V}_{m+1}|.$$

Now define

$$Q = \{(x \in y) : x, y \in U \cup V_0, Mx \in My\} \cup \{(x \notin y) : x, y \in U \cup V_0, Mx \notin My\}$$

Clearly condition (3) is satisfied.

Let P$'$ be the formula resulting from P after eliminating quantifiers from it in the way described in the statement of the theorem. Obviously M is also a model of P$'$. So let P$''$ be a disjunct of a disjunctive normal form of P$'$ which is satisfied by M.

The way in which the original model M has been extended assures that conditions (a)-(i) are all satisfied, thus establishing the theorem in one direction.

Conversely, assume that the set U, the functions $'$,$^-$, the set Q and a conjunction P$''$ can be found as in (1)-(4) and such that all conditions (a)-(i) are satisfied. We can also assume, without loss of generality, that there are nontrapped variables. Indeed, if all variables were trapped, then by (i_2) M^* would be a model of P. So, let w be an $\in$-minimal nontrapped variable such that $\overline{w} = w$. Let $\emptyset \in x_1 \in \ldots \in x_k$ be a longest chain of trapped variables. Observe that $k \leq m$. Indeed, by reasoning as in the proof of Lemma 1, for each trapped variable x there exists a variable z_x in V_0 such that $\overline{x} \equiv \overline{z_x}$. Thus, in correspondence of $x_1, \ldots, x_k$ we can find $y_{i_1}, \ldots, y_{i_k}$ in V_0 such that $\overline{y_{i_j}} \equiv \overline{x_j}$, for all $j = 1, \ldots, k$. But since $x_j \in x_{j+1}$, then $\overline{x_j} \in \overline{x_{j+1}}$, i.e., $\overline{y_j} \in \overline{y_{j+1}}$, $j = 1, \ldots, k-1$. Therefore from (b) we deduce that the variables y_{i_j} must be pairwise distinct, thus showing that $k \leq m$.

Let w be an $\in$-minimal nontrapped variable such that $\overline{w} = w$ and let $z_1, z_2, \ldots, z_{m+6-k}$ be newly introduced variables. Add to Q the sets of relations:

$$\bigcup_{i=1}^{m+6-k} Q^w_{z_i}$$

$$\{z_i \in z_j \,:\, i < j,\ i,j = 1,\ldots,m+6-k\}$$

$$\{z_i \notin z_j \,:\, i \geq j,\ i,j = 1,\ldots,m+6-k\}$$

Also extend $^-$ to $z_1, z_2, \ldots, z_{m+6-k}$ by putting $\overline{z}_i \equiv z_i$ for all $i = 1, \ldots, m+6-k$. Let $W = U \cup V_0 \cup \{z_1, z_2, \ldots, z_{m+6-k}\}$. It is immediate to verify that after the insertions of variables z and the consequent update of $^-$ and Q, conditions (a)-(i) of the theorem still hold.

Definition 5.1. A variable x in W is said to be an *ordinal variable* if $\overline{x} \equiv x$.
Given an ordinal variable x, we denote by $height(x)$ the length of a longest chain of memberships $\emptyset \in x_1 \in x_2 \in \cdots \in x_s \equiv x$.
For each variable z in W, we put

$$prk(z) = height(\overline{z})\ (pseudorank).$$

Let $s_1, s_2, \ldots, s_{m^2+6m}$ be pairwise distinct elements of $\mathcal{V}_{m+2} \setminus \{\mathcal{V}_{m+1}\}$. For each $h \geq m+7$ and $j = 1, \ldots, m^2+6m$ we put

$$i_{h,j} = \{\mathcal{V}_{h-2}\} \cup (\mathcal{V}_{h-2} \setminus \{s_j\}),$$

and call the sets $i_{h,j}$ *individuals.* Clearly, $\text{rank}(i_{h,j}) = h-1$.

From (a) and (b), we can define the model M by induction on the pseudorank of the variables. We put $M\underline{\emptyset} = \emptyset$. Next, assume that M has been defined for all variables y such that $prk(y) < k$. Let $u_1, u_2, \ldots, u_{\ell_k}$ be all variables having pseudorank equal to k. If $k < m+7$, we put

(5.2.1) $$Mu_j = \{My : y \in u_j \text{ is in } Q\},\quad j = 1,2,\ldots,\ell_k$$

On the other hand, if $k \geq m+7$, we can assume without loss of generality that $\overline{u}_1 = u_1$ and that $u_2, \ldots, u_{r_k}$ are in V, whereas $u_{r_k+1}, \ldots, u_{\ell_k}$ are not in V. Then we put

(5.2.2) $$Mu_j = \begin{cases} \{My : y \in u_j \text{ is in } Q\} & \text{if } i = 1,2,\ldots,r_k \\ \{My : y \in u_j \text{ is in } Q\} \cup \{i_{k,j}\} & \text{if } i = r_{k+1},\ldots,\ell_k \end{cases}$$

We will prove that M is an injective model of P by showing that

- M is injective;
- M is a model for $\overline{P''}$;
- M is a model for P'';
- M is a model for P'.

We have the following elementary lemma.

Lemma 5.1. For all variables x in W and individuals $i_{h,j}$, $|Mx| < |i_{h,j}|$.
Proof. Indeed

$$|Mx| \leq |W| + 1 \leq |\mathcal{V}_{m+1}| + m^2 + 7m + 7 < |\mathcal{V}_{m+2}| \leq |\mathcal{V}_{h-2}| = |i_{h,j}| .$$

•

The preceding lemma implies easily the injectivity of M.
Lemma 5.2. M is injective.
Proof. Assume by contradiction that M is not injective. Let x_1 be a variable in W of lowest pseudorank such that $Mx_1 = Mx_2$ for some x_2 distinct from x_1. In view of (5.2), we can write

$$Mx_1 = \{My : y \in x_1 \text{ is in } Q\} \cup I_{x_1}$$
$$Mx_2 = \{My : y \in x_2 \text{ is in } Q\} \cup I_{x_2},$$

where I_{x_1} (resp. I_{x_2}) is empty or is the singleton of the individual associated with x_1 (resp. x_2). Since $x_1 \not\equiv x_2$, then $I_{x_1} \cap I_{x_2} = \emptyset$. Thus by the preceding lemma $I_{x_1} = I_{x_2} = \emptyset$. This implies that x_1 and x_2 are in $V \cup \{x : x \text{ is trapped}\} \cup \{\overline{x} : x \text{ is in } W\}$. Therefore, by (i_3) there exists z in W such that $z \in x_1$ is in Q if and only if $z \in x_2$ is not in Q. Assume without loss of generality that $z \in x_1$ is in Q and that $z \in x_2$ is not in Q. Since $Mz \in Mx_1 = Mx_2$, it follows that there exists a variable $z' \not\equiv z$ such that $z' \in x_2$ is in Q and $Mz' = Mz$. But this contradicts the minimality of the pseudorank of x_1, thus proving the lemma. •

Lemma 5.3. M is a model for $\overline{P''}$.
Proof: If the literal $x \in y$ occurs in $\overline{P''}$, then by (a) and (3) $x \in y$ is also in Q. Thus by (5.2) $Mx \in My$. On the other hand, if $x \notin y$ is in $\overline{P''}$, by reasoning as in the preceding case it follows that $x \notin y$ is in Q. Therefore the preceding two lemmas imply that $Mx \notin My$. This proves that all conjuncts in $\overline{P''}$ are correctly modeled by M and in turn that M is a model of $\overline{P''}$. •

Lemma 5.4. M is a model for P''.
Proof: It is enough to prove that for all x in W

$$M\overline{x} = rank(Mx).$$

Notice that if x is trapped, then $Mx \equiv M^*x$. Thus by (i_2) $M\overline{x} = rank(Mx)$. So we can assume that x is nontrapped. Suppose first that x is an ordinal variable. We will show that in this case

$$Mx = height(x). \tag{5.3}$$

We proceed by induction on $height(x)$. If $height(x) = 0$ then $x \equiv \emptyset$ and by (f) and (5.2) we have $Mx = \emptyset = height(x)$. Suppose that (5.3) holds for all ordinal variables y such that $height(y) < height(x)$. Observe that by definition $Mx = \{My : y \in x \text{ is in } Q\}$. If $y \in x$ is in Q, then by (d) y is an ordinal variable. Clearly $height(y) < height(x)$. Thus $Mx \subseteq height(x)$. Conversely, assume that $s \in height(x)$. Then there exists an ordinal variable y such that there is a chain in Q of membership relations leading from y into x and such that $height(y) = s$. Thus by (h)

and (b) the literal $y \in x$ is in Q and therefore $s = height(y) = My \in Mx$. Hence $height(x) \subseteq Mx$ which together with the previously proved set inclusion yields $Mx = height(x)$. Observe that in the case in which x is an ordinal variable, (5.3) clearly implies $M\overline{x} = rank(Mx)$.

Next suppose that x is not an ordinal variable. We distinguish two cases according to whether x is in V or not. Assume first that x is in V. Let $s \in M\overline{x}$. Thus by (5.2) $s = My$ for some variable y for which $y \in \overline{x}$ is in Q. From (e) it follows that either $y \in \overline{x'}$ or $y \equiv \overline{x'}$. In any case $My \leq M\overline{x'}$. Thus, again by (e), $s = My \leq rank(Mx') < rank(Mx)$ and in turn $M\overline{x} \subseteq rank(Mx)$. Conversely, let $s \in rank(Mx)$. Then $s = rank(My)$ for some y for which $y \in x$ is in Q. Clearly $prk(y) < prk(x)$. Thus by induction $s = M\overline{y}$. But $\overline{y} \in \overline{x}$; therefore $s \in M\overline{x}$ which implies $rank(Mx) \subseteq M\overline{x}$. In conclusion we proved that $M\overline{x} = rank(Mx)$ in the case in which x is in V too. It only remains to verify that the same equality holds even if x is not in V.

So, suppose that x is a nontrapped, nonordinal variable which is not in V. By (5.2), $Mx = \{My : y \in x \text{ is in } Q\} \cup \{i_x\}$, where $rank(i_x) = prk(x) - 1$. If $y \in x$ is in Q, then by (c) $\overline{y} \in \overline{x}$ is also in Q. Thus $M\overline{y} \in M\overline{x} = prk(x)$ which implies $M\overline{y} \leq prk(x) - 1$. Hence, $rank(Mx) = prk(x) = M\overline{x}$.

Summing up, we have proved that $M\overline{x} = rank(Mx)$ for all x in W. Therefore M is a model for P''. •

Lemma 5.5. M is a model for P'.

Proof: Since P'' is a disjunct of P', it follows immediately that M is also a model of P'. •

We are now ready to prove that M is a model of P. We do this by showing that every conjunct C of P is satisfied by M. So let C be any conjunct of P. We can assume that C has the form

$$(\forall x_1 \in y_1) \dots (\forall x_n \in y_n) p,$$

since all unquantified conjuncts of P are contained in P'. Let $s_1 \in My_1, \dots, s_n \in My_n$. Then $s_i = Mz_i$ for some z_i such that the literal $z_i \in y_i$ is in Q, $i = 1, \dots, n$. Thus,

$$(x_1 \in y_1 \rightarrow (x_2 \in y_2 \rightarrow \dots \rightarrow (x_n \in y_n \rightarrow p) \cdots))^{x_1, \dots, x_n}_{z_1, \dots, z_n}$$

is in P' and therefore it is satisfied by M. In particular, since $Mz_i \in My_i$, $i = 1, \dots, n$, it follows that $(p^{x_1, \dots, x_n}_{z_1, \dots, z_n})^M = \mathbf{true}$, i.e $p^{M[x_1/s_1] \cdots [x_n/s_n]} = \mathbf{true}$. Hence M satisfies C. This proves that M is a model for P and concludes the proof of the theorem. •

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