A brief overview of possibilistic logic

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A BRIEF OVERVIEW OF POSSIBILISTIC LOGIC

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The intended purpose of this research[1] is i) to show the specificity of possibilistic logic with respect to other logics with weighted statements, in particular its ability to deal both with uncertainty and vagueness, ii) to develop proof methods for possibilistic logic, and iii) to relate its ability to cope with partial inconsistency to belief revision and non-monotonic reasoning issues. See Léa Sombé (1990) for an introduction to possibilistic logic among other non-classical logics.

Several approaches have been proposed for dealing with uncertainty and/or vagueness in theorem proving ; see Dubois et al. (1991a) for an overview. However a large part of them are based on *fuzzy* logic, which completely departs from *possibilistic* logic. Fuzzy logic deals with propositions involving vague predicates (or properties whose satisfaction can be a matter of degree) and manipulates truth degrees which are truth-functional with respect to each connective, whereas possibilistic logic involves certainty and possibility degrees which are not compositional for all connectives and which are attached to classical formulae, i.e. containing only non-vague propositions or predicates (in the simplest case). The lack of complete certainty about the truth of a considered formula is to be understood as a consequence of a lack of complete information.

A possibilistic logic formula is a first order logic formula with a numerical weight between 0 and 1 which is a lower bound on a possibility measure Π or on a necessity measure N. Thus this lower bound should obey the characteristic axioms governing these measures, i.e. $\forall p, \forall q, N(p \wedge q) = \min(N(p),N(q))$ and $\Pi(p \vee q) = \max(\Pi(p),\Pi(q))$ respectively for necessity and possibility measures (Zadeh, 1978 ; Dubois and Prade, 1988), with the duality relation $N(p) = 1 - \Pi(\neg p)$. However we only have $N(p \vee q) \geq \max(N(p),N(q))$ and $\Pi(p \wedge q) \leq \min(\Pi(p),\Pi(q))$. Moreover we have the usual limit conditions $\Pi(\bot) = N(\bot) = 0$, $\Pi(\top) = N(\top) = 1$, where $\bot$ and $\top$ stand for the

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contradiction and the tautology respectively. The weight attached to a formula represents to what extent it is possible or it is certain that the formula holds for true given the available information. A semantics has been proposed first when only lower bounds on a necessity measure are used (Dubois et al., 1989) and then extended to the general case where lower bounds of both possibility and necessity are allowed (Lang et al., 1991). For the sake of brevity let us only indicate the semantics attached to a set $\mathcal{K}$ of (classical) formulas p_i, $i = 1,n$ weighted by lower bounds α_i of the necessity type, i.e. $\forall i$, $N(p_i) \geq \alpha_i$. The fuzzy set $M(p_i,\alpha_i)$ of interpretations of (p_i,α_i) is defined by the characteristic function

$$\forall \omega,\ \mu_{M(p_i,\alpha_i)}(\omega) = \max(\mu_{M(p_i)}(\omega),\ 1 - \alpha_i)$$

where $\mu_{M(p_i)}(\omega) = 1$ if ω is a model of p_i and $\mu_{M(p_i)}(\omega) = 0$ if ω is not a model of p_i. The lack of certainty in p_i, estimated by $1 - \alpha_i$, is committed to the interpretations which are not models of p_i. By performing the conjunction of the $M(p_i,\alpha_i)$'s, we associate each interpretation ω with a weight equal to $\pi(\omega) = \min_{i=1,n} \mu_{M(p_i,\alpha_i)}(\omega)$. Thus the weights attached to formulas in the knowledge base $\mathcal{K} = \{(p_i,\alpha_i)\}$ induce an ordering among the interpretations (according to their level of possibility $\pi(\omega)$). It is very similar to Shoham (1988)'s preferential model semantics ; see (Dubois and Prade, 1991b) on this point. It can be checked that $\forall i$, $N(p_i) = 1 - \Pi(\neg p_i) \geq \alpha_i$, with $\Pi(p_i) = \sup\{\pi(\omega), \omega \in M(p_i)\}$, which is the definition (Zadeh, 1978) of a possibility measure Π from a possibility distribution π, in this setting.

The following deduction rules (Dubois and Prade, 1987, 1990a) have been proved sound and complete for the above-mentioned semantics, see (Dubois et al., 1989, Lang et al., 1991)

$$\frac{N(p) \geq \alpha\ ,\ N(q) \geq \beta}{N(\mathrm{Res}(p,q)) \geq \min(\alpha,\beta)} \qquad\qquad \frac{N(p) \geq \alpha\ ,\ \Pi(q) \geq \beta}{\Pi(\mathrm{Res}(p,q)) \geq \begin{cases}\beta \text{ if } \alpha + \beta > 1\\ 0 \text{ otherwise.}\end{cases}}$$

where $\mathrm{Res}(p,q)$ is the resolvent of p and q. If we want to compute the certainty degree which can be attached to a formula, we add to the knowledge base the negation of the formula to evaluate with a necessity degree equal to 1. Then it can be shown that any lower bound obtained on $\perp$, by resolution, is a lower bound of the necessity of the formula to evaluate. First order logic automatic deduction methods can be extended to possibilistic logic. Various strategies for applying the above extended resolution principles, which make use of ordered search methods (Dubois et al., 1987), as well as, the generalization of semantic evaluation techniques like the Davis and Putnam' procedure (Lang, 1990) have been carried out. Also preliminary results on possibilistic logic programming have been obtained (Dubois et al., 1991b).

The introduced semantics enables us to define the degree *of partial inconsistency* of a knowledge base $\mathcal{K}$ which is equal, in the case of necessity-weighted formulas, to $\mathrm{Inc}(\mathcal{K}) = 1 - \sup_\omega \min_{i=1,n} \mu_{M(p_i,\alpha_i)}(\omega)$. Then it can be shown that this degree estimates to what extent the lower bounds in the knowledge base violate the characteristic axiom of necessity measures and to what extent the fuzzy set of models of the knowledge base is empty. It has been also shown that is it possible to reason with such partially inconsistent knowledge bases, still preserving the above-mentioned soundness and completeness results (Lang et al., 1991). An important point when reasoning with a partially inconsistent possibilistic knowledge base is that the conclusions which can be deduced with a degree strictly greater than the degree of inconsistency are still valid.

Possibilistic logic implements a non-monotonic reasoning in case of partial inconsistency. Indeed, it has been shown (Dubois and Prade, 1991b) that the preferential entailment (in the sense of Shoham (1988)) $\models_\pi$, defined by

$$p \models_\pi q \Leftrightarrow N(q \mid p) > 0$$

$$\text{with } N(q \mid p) = 1 - \Pi(\neg q \mid p) \text{ and } \Pi(q \mid p) = \begin{cases} 1 \text{ if } \Pi(p) = \Pi(p \wedge q) \\ \Pi(p \wedge q) \text{ if } \Pi(p) > \Pi(p \wedge q) \end{cases}$$

(where π is the possibility distribution, associated with the semantics of the knowledge base $\mathcal{K}$, underlying Π), is in complete agreement with non-monotonic consequence relations obeying the axiomatics of system P proposed by Kraus et al. (1990). See also Gärdenfors (1991) on the link between non-monotonicity issues and necessity-like measures called "expectations".

Moreover it has been established in (Dubois and Prade, 1991b) that we have $N(q \mid p) > 0$ if and only if it is possible to deduce (q,β) from $\mathcal{K} \cup \{(p,1)\}$ with $\beta > \mathrm{Inc}(\mathcal{K} \cup \{(p,1)\})$, where N is the necessity measure defined from π associated with $\mathcal{K}$. Since $N(q \mid p) > 0$ behaves like a non-monotonic consequence relation $p \mid\!\sim q$, it illustrates the close relation that there exists between non-monotonic reasoning and belief revision (Makinson and Gärdenfors, 1991), in the possibilistic framework. The links between possibility theory and the theory of revision of symbolic knowledge bases developed by Gärdenfors indicate that there is a deep coherence between the reasoning methods in possibilistic logic and recent developments in purely symbolic approaches to reasoning with incomplete or contradictory knowledge. More specifically it has been shown that Gärdenfors (1988)' epistemic entrenchment relations are equivalent to the qualitative counterpart of necessity measures (Dubois and Prade, 1990b). This explains that the ability of possibilistic logic to deal with partial inconsistency is related to a belief revision mechanism in agreement with Gärdenfors' epistemic entrenchment relation. Moreover the lack of known updating rules in possibility theory has led us to investigate counterparts of updating rules existing in probability theory ; a possibilistic Jeffrey-like rule for updating a possibility distribution on the basis of another possibility distribution has been proposed (Dubois and Prade, 1990d). The reader is referred to (Dubois and Prade, 1991c) for a detailed analysis of belief revision in possibility theory. Besides, the problem of recovering consistency in a partially inconsistent knowledge base $\mathcal{K}$ by building maximal consistent sub-bases (obtained by deleting suitable pieces of knowledge in $\mathcal{K}$) is discussed in (Dubois et al., 1991c). The problem of reasoning with paraconsistent pieces of knowledge which violate the requirement $\min(N(p),N(\neg p)) = 0$ (a consequence of the axiomatics of necessity measures) has been recently discussed (Dubois et al., 1991e).

As pointed out in (Dubois et al., 1989), the weighted clause $(\neg p \vee q, \alpha)$, understood as $N(\neg p \vee q) \geq \alpha$ is semantically equivalent to the weighted clause $(q, \min(\alpha, v(p)))$ where $v(p)$ is the truth value of p, i.e. $v(p) = 1$ if p is true and $v(p) = 0$ if p is false. This remark is very useful for hypothetical reasoning, since by "transferring" a sub-formula from a clause to the weight part of the formula we are introducing explicit assumptions. Indeed changing $(\neg p \vee q, \alpha)$ into $(q, \min(v(p), \alpha))$ leads to state the piece of knowledge under the form "q is certain at the degree α, *provided that* p is true". More generally, the weight or label can be a function of logical (universally quantified) variables involved in the clause. The weight is no more just a degree but in fact a label which expresses the context in which the piece of knowledge is more or less certain. This is to be related to "possibilistic Assumption-based Truth Maintenance Systems" (with weighted justifications and/or hypotheses, which have been defined (Dubois et al., 1990, 1991d) and exemplified on a diagnosis problem. The approach contrasts with other uncertainty handling ATMS in the sense that the symbolic processing and the calculus of uncertainty are no longer separated here. Besides, applications to discrete optimization and to the handling of prioritized constraints are presented in (Lang, 1991).

Moreover the presence of logical variables in the weight also enables the expression of some graduality attached to *vague predicates* (as in the rule "the younger the person, the more certain he/she is single", where "young" is a vague predicate) in a simple way, as $N(\text{single}(x)) \geq \mu_{\text{young}}(\text{age}(x))$ in our example. It would then allow for a flexible interface between the symbolic knowledge base and numerical inputs. Vague predicates can thus be handled by introducing their characteristic functions in the weights. This remark together with theoretical results (Dubois and Prade, 1990a) on the extension of the resolution rules in possibilistic logic in presence of vague predicates enables us to accommodate vague predicates ; see also (Dubois et al., 1991e).

Lastly, deduction in possibilistic logic has been shown in perfect agreement (see Dubois et al., 1991a, e) with Zadeh (1979)'s approach to approximate reasoning which is based on the combination and the projection of possibility distributions. Paralleling existing results about network inference techniques for reasoning with probability measures or belief functions, some preliminary work (Dubois and Prade, 1990c) has been done indicating that the framework of possibility theory is also liable of inference methods based on hypergraphs (Shafer and Shenoy, 1990).

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