A linear time process algebra

ALAN JEFFREY

ABSTRACT. This paper presents a variant of Milner's Calculus for Communicating Systems enriched with a notion of time. Time here is considered to be a totally ordered monoid rather than a particular numerical domain. A set of laws for the algebra are presented, as well as a transition system semantics. The laws are then shown to be consistent and complete.

1 Introduction

Milner's *Calculus for Communicating Systems* (CCS) [Mil80, Mil89] is a well developed theory of untimed concurrency. This has recently been extended to include a notion of time by Wang [Wan90, Wan91], Hennessy and Regan [HR90], and Moller and Tofts [MT90]. In addition, Bergstra and Klop's ACP [BW90] has been given timed variants by Baeten and Bergstra [BB91] and Nicollin and Sifakis [NS90]; and Hoare's CSP [Hoa85] has a timed model given by Reed and Roscoe [RR86].

Despite these models being developed independently, they have many features in common, one being that they are all *real time* models—the notion of time is assumed to be either $\mathbb{N}$ or $[0, \infty)$. In this paper we develop a generalization of Wang's Timed CCS, with three new concepts.

- Time is considered to be an *abstract* notion. We do not specify what the time domain should be, as long as it is a totally ordered monoid, with a few extra conditions. This means that Wang's calculus and Hennessy and Regan's calculus can be seen as examples of the model presented here. If we take the trivial time domain $\{0\}$ with only one time, we have a model isomorphic to Milner's untimed CCS.
- We can produce a *complete axiomatization*. Wang has a complete axiomatization for regular agents, that is ones built without parallelism, restriction, or alphabet transformation. However, due to the lack of an expansion theorem, he was unable to provide a complete axiomatization for parallelism. He has suggested [Wan91] a different prefixing operator $\alpha@t.P_t$ to alleviate this problem. Here, we suggest another prefix, $\alpha{:}P$, and show an expansion theorem using it.
- We explicitly allow *time-stop* processes. In many other calculi, time-stops are implicitly allowed through constructs like $\mu x.x$. Here, we are allowing them because of the interaction of *maximal progress* and *unbounded sum*. If we assume that $\tau{:}P$ will not allow time to pass, but insists on performing its τ action, then the process $\sum\{\varepsilon t{:}\tau{:}P \mid t \neq 0\}$ will not allow time to pass, but

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Also available as Programming Methodology Group Technical Report 61.
Author's address: Department of Computer Sciences, S-412 55 Göteborg, Sweden.
E-mail: jeffrey@cs.chalmers.se.

cannot perform a τ action. Hence it is a time-stop. It might be possible to restrict ourselves to a language where such processes were not possible, but it turns out to be algebraically much simpler just to allow time-stops. We can regard these as the 'complex numbers' of this calculus—they do not correspond to anything we might have a computational intuition for, but they simplify the algebra.

Unfortunately, this paper is by no means complete.

- The language considered here only allows finite processes, and has no primitive for recursion. This should not be too difficult to rectify, although finding a complete set of laws to include recursion might be slightly tricky.
- The calculus includes an uncountable sum operator $\sum$. This is the operator that gives our calculus its expressive power, but with an obvious loss—equivalence is no longer decidable. It is an interesting question as to whether we can restrict ourselves to some notion of decidable sum. We still want to be able to make uncountable summations (such as in the definition of $\mu@t.P$) but we may be able to restrict ourselves so as to recover decidability. For example, if our time domain has a topological structure, we might be able to restrict ourselves to summation over closed sets.

This model is a *transition system* model, in the tradition of Milner [Mil80] and Plotkin [Plo81] so there is an implicit assumption that all history can be made linear. This is the reason we are limited to totally ordered time domains. In [Jef91], the author showed how timed process algebra could be applied to partially ordered time domains.

2 Assumptions

We are going to produce a timed variant of Milner's CCS, and so we will need the same notion of *name*.

ASSUMPTION 1. *There is a set $\mathcal{A}$ of names, ranged over by a, b and c.*

A *label* is either a name a or its complement $\overline{a}$.

DEFINITION 2. *The set $\mathcal{L}$ of labels is $\mathcal{A} \cup \overline{\mathcal{A}}$, where $\overline{\mathcal{A}} = \{\overline{a} \mid a \in \mathcal{A}\}$. $\mathcal{L}$ is ranged over by l. Let $\overline{\overline{a}} = a$.*

An *action* is either a label, the special silent action τ.

DEFINITION 3. *The set Act of actions is $\mathcal{L} \cup \{\tau\}$, ranged over by α and β.*

So far, we have followed Milner's untimed calculus. If we are going to produce a timed calculus, though, we will need some notion of time. In fact, we are never interested in *absolute* time (such as '9.26am on 24 January 1990') just *relative* time (such as 'five hours from now'). We can assume there is a *zero* time ('now') and any two times can be added together ('five hours from now' plus 'three hours from now' is 'eight hours from now'). So we assume time is a totally ordered monoid.

ASSUMPTION 4. *There is a monoid $(\mathcal{T}, +, 0)$ of times, ranged over by t, u and v, such that if $t + u = t + v$ then $u = v$.*

DEFINITION 5. $t \leq v$ *iff* $\exists u \,.\, t + u = v$

Assumption 6. *$\leq$ is a total order, with bottom* 0*, and every non-empty set T has a greatest lower bound* $\inf T$.

For example:

- $(\mathbb{N}, +, 0)$ gives a discrete timed model of CCS, similar to [HR90].
- $([0, \infty), +, 0)$ gives a continuous timed model, similar to [Wan91].
- $(\{0\}, +, 0)$ gives the untimed model of CCS.

The *events* we can use in our language are labels, τ actions or times.

Definition 7. *The set Σ of events is $\mathcal{L} \cup \{\tau\} \cup \{\varepsilon t \mid t \in \mathcal{T}\}$, ranged over by σ and ρ.*

A *prefix* is an action, or the special time-stop prefix δ. We are using a prefix to represent time-stop in the same way as ACP [BW90] uses δ to represent deadlock.

Definition 8. *The set Pre of prefixes is Act $\cup \{\delta\}$, ranged over by μ and ν.*

Finally, for technical reasons, we need an upper bound on the nondeterminism our language allows.

Assumption 9. *There is a regular cardinal $\lambda > |\mathcal{T}|$.*

3 Syntax

Let us consider *Linear Timed CCS*, based on Wang's Timed CCS, in turn based on Milner's CCS.

Definition 10. *LTCCS is defined by*

$$P ::= \mu{:}P \mid \varepsilon t{:}P \mid \textstyle\sum \mathcal{P} \mid P \mid P$$

where P, Q and R range over LTCCS and $\mathcal{P}$ and $\mathcal{Q}$ are subsets of LTCCS strictly smaller than λ.

This is the same as Milner's CCS, except that:

- $l{:}P$ can only do an l at time 0, otherwise it deadlocks.
- $\tau{:}P$ will not let time pass, but insists that the τ happens immediately. This is called *maximal progress* by Wang.
- $\delta{:}P$ will not let time pass at all.
- $\varepsilon t{:}P$ delays P by time t.

From these primitives, we can build the operators in Wang's calculus:

Definition 11.

$$\begin{aligned} \mathbf{0} &= \textstyle\sum \emptyset \\ P + Q &= \textstyle\sum \{P, Q\} \\ \alpha @ t.P_t &= \textstyle\sum \{\varepsilon t{:}\alpha{:}P_t \mid t \in \mathcal{T}\} \end{aligned}$$

We now give a set of laws for this language, and write $\vdash P = Q$ if we can prove P and Q are equal using them. To begin with, we inherit three properties from Wang's Timed CCS:

Law 1 (Time continuity). $\vdash \varepsilon t{:}\varepsilon u{:}P = \varepsilon(t + u){:}P$

LAW 2 (TIME DETERMINACY). $\vdash \varepsilon t{:}\sum\mathcal{P} = \sum\{\varepsilon t{:}P \mid P \in \mathcal{P}\}$

LAW 3 (ZERO DELAY). $\vdash \varepsilon 0{:}P = P$

and three new properties of $\delta{:}P$:

LAW 4 (MAXIMAL PROGRESS). $\vdash \tau{:}P = \tau{:}P + \delta{:}Q$

LAW 5 (TIME-STOP). *If* $t \neq 0$ *then* $\vdash \delta{:}P = \delta{:}P + \varepsilon t{:}Q$

LAW 6 (TIME-STOP CONTINUITY). *If* $I \neq \emptyset$ *then:*

$$\vdash \sum\{\varepsilon t_i{:}\delta{:}P_i \mid i \in I\} = \varepsilon(\inf\{t_i \mid i \in I\}){:}\delta{:}Q$$

as well as the standard rules for $\sum$ from CCS:

LAW 7 (SUM UNIT). $\vdash \sum\{P\} = P$

LAW 8 (SUM HOMOMORPHISM). $\vdash \sum\{\sum\mathcal{P}_i \mid i \in I\} = \sum\bigcup\{\mathcal{P}_i \mid i \in I\}$

and a continuity condition on summation:

LAW 9 (SUM CONTINUITY). *If* $\forall Q \in \mathcal{Q}. \vdash P = P + Q$ *then* $\vdash P = P + \sum\mathcal{Q}$.

Finally, we have a variant of the expansion theorem:

LAW 10 (EXPANSION THEOREM). *If:*

$$\begin{aligned} P &= \sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\} \\ Q &= \sum\{\varepsilon u_j{:}\nu_j{:}Q_j \mid j \in J\} \end{aligned}$$

then:

$$\begin{aligned} \vdash P \mid Q \;=\; & \sum\{\varepsilon t_i{:}\mu_i{:}(P_i \mid Q_{t_i}) \mid i \in I\} \\ & + \sum\{\varepsilon u_j{:}\nu_j{:}(P_{u_j} \mid Q_j) \mid j \in J\} \\ & + \sum\{\varepsilon t_i{:}\tau{:}(P_i \mid Q_j) \mid i \in I \wedge j \in J \wedge t_i = u_j \wedge \overline{\mu_i} = \nu_j\} \end{aligned}$$

where:

$$\begin{aligned} P_t &= \sum\{\varepsilon u{:}\mu_i{:}P_i \mid i \in I \wedge t + u = t_i\} \\ Q_t &= \sum\{\varepsilon u{:}\nu_j{:}Q_j \mid j \in J \wedge t + u = u_j\} \end{aligned}$$

This is just a variant on the standard expansion theorem, which says that when P is placed in parallel with Q, one of three things can happen:

- P delays by t_i, performs μ_i, and becomes P_i. In the mean time, Q must move on to time t_i and become Q_{t_i}.
- Q delays by u_j, performs ν_j, and becomes Q_j. In the mean time, P must move on to time u_j and become P_{u_j}.
- Both P and Q delay by time t_i, then P performs μ_i, Q performs $\overline{\mu_i}$, and the resulting system performs a τ action.

It turns out that these laws are the only ones we shall need to prove bisimulation of any agents.

4 Semantics

As with Wang's calculus, we shall give our syntax a *transition system* semantics, with arrows labeled by Σ.

- $P \xrightarrow{l} Q$ means P performs an l action, and becomes Q. This transition takes no time.
- $P \xrightarrow{\tau} Q$ means P performs a silent move, and becomes Q. This transition takes no time.
- $P \xrightarrow{\varepsilon t} Q$ means that P can idle for time t and become Q.

Furthermore, we can place some restrictions on which transition systems we will consider, taken from [Wan91].

AXIOM 1 (TIME CONTINUITY). *If* $P \xrightarrow{\varepsilon(t+u)} R$ *then* $\exists Q \,.\, P \xrightarrow{\varepsilon t} Q \xrightarrow{\varepsilon u} R$.

AXIOM 2 (MAXIMAL PROGRESS). *If* $P \xrightarrow{\tau} Q$ *and* $P \xrightarrow{\varepsilon t} R$ *then* $t = 0$.

AXIOM 3 (TIME DETERMINACY). *If* $P \xrightarrow{\varepsilon t} Q$ *and* $P \xrightarrow{\varepsilon t} R$ *then* $Q \equiv R$.

Note, however, that we do *not* have Wang's 'time persistency', because our prefixing primitive $l{:}P$ will offer an l at time 0, but not at any later time. We are allowing $P \xrightarrow{\varepsilon 0} Q$ as a transition, though, so we need an extra axiom.

AXIOM 4 (ZERO DELAY). $P \xrightarrow{\varepsilon 0} P$

This is a matter of style, and is used because it makes the transition rules simpler.

We can now give the transition rules for each of the operators. These semantics are the same as Wang's, with the exception of prefixing, and some technicalities to do with transitions of the form $P \xrightarrow{\varepsilon 0} P$.

To begin with, a prefix $\alpha{:}P$ can either do a α action immediately, or wait for no time. Also, the process $l{:}P$ can wait and become $\mathbf{0}$. Note that the only transition $\delta{:}P$ has is $\delta{:}P \xrightarrow{\varepsilon 0} \delta{:}P$.

$$\frac{}{\alpha{:}P \xrightarrow{\alpha} P} \qquad \frac{}{\mu{:}P \xrightarrow{\varepsilon 0} \mu{:}P} \qquad \frac{}{l{:}P \xrightarrow{\varepsilon t} \mathbf{0}}\,[t \neq 0]$$

Note that we insist $l{:}P \xrightarrow{\varepsilon t} \mathbf{0}$ rather than Wang's $l.P \xrightarrow{\varepsilon t} l.P$. It is this crucial difference that allows us to find a complete axiomatization for our language. The rules for delay are, however, similar to Wang's.

$$\frac{P \xrightarrow{\alpha} P'}{\varepsilon 0{:}P \xrightarrow{\alpha} P'} \qquad \frac{}{\varepsilon(t+u){:}P \xrightarrow{\varepsilon t} \varepsilon u{:}P} \qquad \frac{P \xrightarrow{\varepsilon u} P'}{\varepsilon t{:}P \xrightarrow{\varepsilon(t+u)} P'}$$

As is the rule for summation.

$$\frac{P \xrightarrow{\alpha} P'}{\sum \mathcal{P} \xrightarrow{\alpha} P'}\,[P \in \mathcal{P}] \qquad \frac{\mathcal{P} \xrightarrow{\varepsilon t} \mathcal{Q}}{\sum \mathcal{P} \xrightarrow{\varepsilon t} \sum \mathcal{Q}}$$

Here $\mathcal{P} \xrightarrow{\varepsilon t} \mathcal{Q}$ iff $\forall P \in \mathcal{P} \,.\, \exists Q \in \mathcal{Q} \,.\, P \xrightarrow{\varepsilon t} Q$ and $\forall Q \in \mathcal{Q} \,.\, \exists P \in \mathcal{P} \,.\, P \xrightarrow{\varepsilon t} Q$.

The only real problem, as in [Wan91] is how to deal with parallel composition. It is easy to give the rules for when each side can perform an action, as these are just Milner's rules from [Mil89].

$$\frac{P \xrightarrow{\alpha} P'}{P \mid Q \xrightarrow{\alpha} P' \mid Q} \qquad \frac{Q \xrightarrow{\alpha} Q'}{P \mid Q \xrightarrow{\alpha} P \mid Q'} \qquad \frac{P \xrightarrow{l} P' \quad Q \xrightarrow{\bar{l}} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'}$$

The problem comes with delay. It is *not* true that if $P \xrightarrow{\varepsilon t} P'$ and $Q \xrightarrow{\varepsilon t} Q'$ then $P \mid Q \xrightarrow{\varepsilon t} P' \mid Q'$. For example, $a{:}P \mid \bar{a}{:}Q$ can do a τ action, so by the assumption of maximal progress, it cannot do a time transition.

The solution, as in [Wan91] is to look at the *initial actions* that a process can do before time t. If no communication is possible, then we can allow a time t transition to take place.

Define *inits* P to be the initial actions of P together with the times they are available. If $(t, \alpha) \in$ *inits* P and P can wait for time t then P can do an α at time t.

DEFINITION 12.

$$\begin{aligned}
\mathit{inits}\ \alpha{:}P &= \{(0, \alpha)\} \\
\mathit{inits}\ \delta{:}P &= \emptyset \\
\mathit{inits}\ \varepsilon t{:}P &= \{(t + u, \alpha) \mid (u, \alpha) \in \mathit{inits}\ P\} \\
\mathit{inits} \textstyle\sum \mathcal{P} &= \bigcup\{\mathit{inits}\ P \mid P \in \mathcal{P}\} \\
\mathit{inits}\ P \mid Q &= \mathit{inits}\ P \cup \mathit{inits}\ Q \cup \{(t, \tau) \mid (t, l) \in \mathit{inits}\ P \wedge (t, \bar{l}) \in \mathit{inits}\ Q\}
\end{aligned}$$

Then we can say a process is *stable* until t if no τ action can happen before t.

DEFINITION 13. *$P \downarrow t$ iff $\forall u < t \,.\, (u, \tau) \notin$ inits P.*

So we can give a side-condition on the rule for parallelism to make sure that we are not breaking maximal progress.

$$\frac{P \xrightarrow{\varepsilon t} P' \quad Q \xrightarrow{\varepsilon t} Q'}{P \mid Q \xrightarrow{\varepsilon t} P' \mid Q'} [P \mid Q \downarrow t]$$

We have now defined the transition system $(LTCCS, \Sigma, \longrightarrow)$. All we have to do now is ensure that it respects our axioms for timed transition systems.

LEMMA 14. *The transition system $(LTCCS, \Sigma, \longrightarrow)$ satisfies Axioms 1–4.*

PROOF. A variant of [Wan91], except for Axiom 4, which is an induction. □

5 Bisimulation

Following [Mil89] and [Wan91], we can define a *strong bisimulation* (from now on just *bisimulation*), which we shall use as our equivalence on $LTCCS$.

DEFINITION 15. *A relation $\mathcal{R}$ is a bisimulation iff, for every $P \mathcal{R} Q$:*

- *if $P \xrightarrow{\sigma} P'$ then $\exists Q' \,.\, Q \xrightarrow{\sigma} Q'$ and $P' \mathcal{R} Q'$, and*
- *if $Q \xrightarrow{\sigma} Q'$ then $\exists P' \,.\, P \xrightarrow{\sigma} P'$ and $P' \mathcal{R} Q'$.*

We shall then say P and Q are *bisimilar* iff there is a bisimulation which identifies them.

DEFINITION 16. *$P \sim Q$ iff there is a bisimulation $\mathcal{R}$ such that $P \mathcal{R} Q$.*

We then have to show that $\sim$ is a congruence.

LEMMA 17. *$\sim$ is a congruence.*

PROOF. A variant of the proof in [Wan91]. □

6 Consistency

We now have two notions of equivalence on *LTCCS*—the laws which prove $\vdash P = Q$, and the bisimulation equivalence $P \sim Q$. We would like to show that these are in fact the same thing, i.e. that they are *consistent* and *complete*. To begin with, we can see that our laws are consistent.

THEOREM 18. *If $\vdash P = Q$ then $P \sim Q$.*

PROOF. This is a matter of showing that all of our laws are sound. In each case we construct a relation $\mathcal{R}$ containing our law, and show that it must be a bisimulation. Again, most of our laws are contained in either [Mil89] or [Wan91], but we shall prove some of the more interesting ones here.

For sum continuity, assume that for every $Q \in \mathcal{Q}$ there is a bisimulation $\mathcal{R}_Q$ containing $(P, P + Q)$. Then define P_t and Q_t such that $P \xrightarrow{\varepsilon t} P_t$ and $Q \xrightarrow{\varepsilon t} Q_t$, and $\mathcal{Q}_t$ as $\{Q_t \mid Q \in \mathcal{Q}\}$. Then define $\mathcal{R}$ as:

$$\mathcal{R} \;=\; \bigcup\{\mathcal{R}_Q \mid Q \in \mathcal{Q}\} \cup \{(P_t, P_t + \sum \mathcal{Q}_t) \mid t \in \mathcal{T}\}$$

By a simple case analysis, $\mathcal{R}$ is a bisimulation, and so since by zero delay and time determinacy, $P_0 \equiv P$ and $Q_0 \equiv Q$, we have shown sum continuity to be sound.

For the expansion law, assume:

$$\begin{aligned} P &= \sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\} \\ Q &= \sum\{\varepsilon u_j{:}\nu_j{:}Q_j \mid j \in J\} \end{aligned}$$

and define:

$$\begin{aligned} P_t &= \sum\{\varepsilon u{:}\mu_i{:}P_i \mid i \in I \wedge t + u = t_i\} \\ Q_t &= \sum\{\varepsilon u{:}\nu_j{:}Q_j \mid j \in J \wedge t + u = u_j\} \\ R_t &= \sum\{\varepsilon u{:}\mu_i{:}(P_i \mid Q_{t_i}) \mid i \in I \wedge t + u = t_i\} \\ &\quad + \sum\{\varepsilon u{:}\nu_j{:}(P_{u_j} \mid Q_j) \mid j \in J \wedge t + u = u_j\} \\ &\quad + \sum\{\varepsilon u{:}\tau{:}(P_i \mid Q_j) \mid i \in I \wedge j \in J \wedge t_i = u_j \wedge \overline{\mu_i} = \nu_j\} \end{aligned}$$

It is a straightforward case analysis to show that $I \cup \{(P_t \mid Q_t, R_t) \mid t \in \mathcal{T}\}$ is a bisimulation, and therefore $P_0 \mid Q_0 \sim R_0$. However, $P_0 \equiv P$, $Q_0 \equiv Q$, and R_0 is the rhs of the expansion law. □

7 Completeness

We can now turn to the meat of this paper—the proof that Laws 1–10 are *complete*, so if $P \sim Q$ then $\vdash P = Q$. As usual, we shall find a *normal form* which we can transform all of our agents into, and then show that any equivalent normal agents must be identical.

DEFINITION 19. *P is in summand form if it is of the form:*

$$P \equiv \sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\}$$

where each of the P_i are in summand form.

This is *not* a normal form, since $\sum\{\varepsilon 0{:}\tau{:}P\}$ and $\sum\{\varepsilon 0{:}\tau{:}P, \varepsilon 0{:}\delta{:}0\}$ are bisimilar, and both are in summand form. However, it is a step towards a normal form, and every agent can be transformed to one in summand form.

LEMMA 20. *If P and Q are in summand form, then there is an R in summand form such that $\vdash P \mid Q = R$.*

PROOF (BY INDUCTION ON P AND Q). Assume:

$$\begin{aligned} P &\equiv \sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\} \\ Q &\equiv \sum\{\varepsilon u_j{:}\nu_j{:}Q_j \mid j \in J\} \end{aligned}$$

and define:

$$\begin{aligned} P_t &\equiv \sum\{\varepsilon u{:}\mu_i{:}P_i \mid i \in I \wedge t + u = t_i\} \\ Q_t &\equiv \sum\{\varepsilon u{:}\nu_j{:}Q_j \mid j \in J \wedge t + u = u_j\} \end{aligned}$$

then by the expansion law:

$$\begin{aligned} \vdash P \mid Q = {} & \sum\{\varepsilon t_i{:}\mu_i{:}(P_i \mid Q_{t_i}) \mid i \in I\} \\ & + \sum\{\varepsilon u_j{:}\nu_j{:}(P_{u_j} \mid Q_j) \mid j \in J\} \\ & + \sum\{\varepsilon t_i{:}\tau{:}(P_i \mid Q_j) \mid i \in I \wedge j \in J \wedge t_i = u_j \wedge \overline{\mu_i} = \nu_j\} \end{aligned}$$

By induction, we can find summand forms for $P_i \mid Q_{t_i}$, $P_{u_j} \mid Q_j$ and $P_i \mid Q_j$, and so we are finished. □

LEMMA 21. *For any P, there is a Q in summand form such that $\vdash P = Q$.*

PROOF (BY INDUCTION ON P).

$P \equiv \mu{:}P'$ By induction, there is an Q' in summand form such that $\vdash P' = Q'$. By sum unit and zero delay, $\vdash P = \sum \varepsilon 0{:}\mu{:}Q'$, which is in summand form.

$P \equiv \varepsilon t{:}P'$ By induction, we can show $\vdash P' = \sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\}$ and so by time continuity and time determinacy, $\vdash P = \sum\{\varepsilon(t + ti){:}\mu_i{:}P_i \mid i \in I\}$.

$P \equiv \sum\{P_i \mid i \in I\}$ By induction, for every $i \in I$ we can find a $\sum \mathcal{Q}_i$ in summand form such that $\vdash P_i = \sum \mathcal{Q}_i$. Then by sum homomorphism, $\vdash P = \sum \bigcup\{\mathcal{Q}_i \mid i \in I\}$ which is in summand form.

$P \equiv P_1 \mid P_2$ By induction, P_1 and P_2 can be transformed into summand form, and so by Lemma 20, P can be transformed into summand form. □

We can now define the normal form we've been looking for.

DEFINITION 22. *P is in normal form if it is of the form:*

$$P \equiv \sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\}$$

where:

- *each of the P_i are in normal form,*

- *if* $\mu_i = \tau$ *then* $\exists j \,.\, t_i = t_j \wedge a_j = \delta$, *and*
- *if* $\mu_i = \delta$ *then* $\forall j \,.\, t_j \leq t_i$ *and* $P_i \equiv \mathbf{0}$.

This is the same as summand form, except we insist on maximal progress, remove any actions which happen after a time-stop, and insist that all time-stops are of the form $\delta{:}\mathbf{0}$. For example $\sum\{\varepsilon 0{:}\tau{:}P\}$ is not in normal form, but $\sum\{\varepsilon 0{:}\tau{:}P, \varepsilon 0{:}\delta{:}\mathbf{0}\}$ is. We can now show that any agent in summand form can be converted to normal form.

LEMMA 23. *If P is in summand form, then there is a Q in normal form such that* $\vdash P = Q$.

PROOF. Assume:

$$P \;\equiv\; \textstyle\sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\}$$

then by induction we can find Q_i in normal form such that $\vdash P_i = Q_i$. Define:

$$I_M \;=\; \{i \in I \mid \mu_i \in M\}$$

If $I_{\{\tau,\delta\}}$ is empty, then $P = \sum\{\varepsilon t_i{:}\mu_i{:}Q_i\}$, which is in normal form. Otherwise:

$$\begin{aligned} t_\delta &= \inf\{t_i \mid i \in I_{\{\tau,\delta\}}\} \\ J &= \{j \in I_{\mathcal{L}\cup\{\tau\}} \mid t_j \leq t_\delta\} \\ Q &\equiv \textstyle\sum(\{\varepsilon t_j{:}\mu_j{:}Q_j \mid j \in J\} \cup \{\varepsilon t_\delta{:}\delta{:}\mathbf{0}\}) \end{aligned}$$

Then for any $i \in I \setminus J$, there are two possibilities. If $t_i \leq t_\delta$, then from the definition of J this is only possible if $t_i = t_\delta$ and $\mu_i = \delta$. Then:

$$\begin{aligned} \vdash Q &= Q + \textstyle\sum\{\varepsilon t_\delta{:}\delta{:}\mathbf{0}\} && \text{(sum laws)} \\ &= Q + \varepsilon(\inf\{t_\delta\}){:}\delta{:}Q_i && \text{(time-stop continuity)} \\ &= Q + \varepsilon t_\delta{:}\delta{:}Q_i && \text{(definition of inf)} \\ &= Q + \varepsilon t_i{:}\mu_i{:}Q_i && \text{(above)} \end{aligned}$$

Otherwise, if $t_\delta < t_i$, there is some $t \neq 0$ such that $t_\delta + t = t_i$. Then:

$$\begin{aligned} \vdash Q &= Q + \varepsilon t_\delta{:}\delta{:}0 && \text{(sum laws)} \\ &= Q + \varepsilon t_\delta{:}(\delta{:}0 + \varepsilon t{:}\mu_i{:}Q_i) && \text{(time-stop)} \\ &= Q + \varepsilon t_\delta{:}\delta{:}0 + \varepsilon t{:}_\delta \varepsilon t{:}\mu_i{:}Q_i && \text{(time determinacy)} \\ &= Q + \varepsilon t_\delta{:}\delta{:}0 + \varepsilon t{:}_i\mu_i{:}Q_i && \text{(time continuity)} \\ &= Q + \varepsilon t{:}_i\mu_i{:}Q_i && \text{(sum laws)} \end{aligned}$$

So:

$$\begin{aligned} \vdash Q &= Q + \textstyle\sum\{\varepsilon t_i{:}\mu_i{:}Q_i \mid i \in I \setminus J\} && \text{(sum continuity)} \\ &= \textstyle\sum\{\varepsilon t_i{:}\mu_i{:}Q_i \mid i \in I\} + \varepsilon t_\delta{:}\delta{:}0 && \text{(sum laws)} \\ &= \textstyle\sum\{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\} + \varepsilon t_\delta{:}\delta{:}0 && \text{(definition of } Q_i\text{)} \\ &= P + \varepsilon t_\delta{:}\delta{:}0 && \text{(definition of } P\text{)} \\ &= P + \varepsilon(\inf\{t_i \mid i \in I_{\{\tau,\delta\}}\}){:}\delta{:}0 && \text{(definition of } t_\delta\text{)} \\ &= P + \textstyle\sum\{\varepsilon t_i{:}\delta{:}P_i \mid i \in I_{\{\tau,\delta\}}\} && \text{(time-stop continuity)} \end{aligned}$$

$$\begin{array}{rll} = & P + \sum\{\varepsilon t_i{:}\delta{:}P_i \mid i \in I_{\{\delta\}}\} + \sum\{\varepsilon t_i{:}\delta{:}P_i \mid i \in I_{\{\tau\}}\} & \text{(sum laws)} \\ = & P + \sum\{\varepsilon t_i{:}\delta{:}P_i \mid i \in I_{\{\tau\}}\} & \text{(definition of } I_{\{\delta\}}) \\ = & P + \sum\{\varepsilon t_i{:}\tau{:}P_i \mid i \in I_{\{\tau\}}\} + \sum\{\varepsilon t_i{:}\delta{:}P_i \mid i \in I_{\{\tau\}}\} & \text{(definition of } I_{\{\tau\}}) \\ = & P + \sum\{\varepsilon t_i{:}\tau{:}P_i \mid i \in I_{\{\tau\}}\} & \text{(maximal progress)} \\ = & P & \text{(definition of } I_{\{\tau\}}) \end{array}$$

So $\vdash P = Q$, and Q is in normal form. □

COROLLARY 24. *For any P, there is a Q in normal form such that $\vdash P = Q$.*

Finally, all we have to do is show that our normal form is indeed normalizing.

LEMMA 25. *If $P \sim Q$ are in normal form, then $P \equiv Q$.*

PROOF (BY INDUCTION ON P AND Q). Assume:

$$\begin{array}{rcl} P & \equiv & \sum \mathcal{P} \\ \mathcal{P} & \equiv & \{\varepsilon t_i{:}\mu_i{:}P_i \mid i \in I\} \\ Q & \equiv & \sum \mathcal{Q} \\ \mathcal{Q} & \equiv & \{\varepsilon u_j{:}\nu_j{:}Q_j \mid j \in J\} \end{array}$$

If $\varepsilon t{:}\delta{:}0 \in \mathcal{P}$, then $P \xrightarrow{\varepsilon u}$ iff $u \leq t$, so $Q \xrightarrow{\varepsilon u}$ iff $u \leq t$, so $\varepsilon t{:}\delta{:}0 \in \mathcal{Q}$. If $\varepsilon t{:}\tau{:}P_i \in \mathcal{P}$, then $P \xrightarrow{\varepsilon t}\xrightarrow{\tau} P_i$, so $Q \xrightarrow{\varepsilon t}\xrightarrow{\tau} Q_j$ and $P_i \sim Q_j$, so by induction $P_i \equiv Q_j$, so $\varepsilon t{:}\tau{:}P_i \in \mathcal{Q}$. If $\varepsilon t{:}l{:}P_i \in \mathcal{P}$, then $P \xrightarrow{\varepsilon t}\xrightarrow{l} P_i$, so $Q \xrightarrow{\varepsilon t}\xrightarrow{l} Q_j$ and $P_i \sim Q_j$, so by induction $P_i \equiv Q_j$, so $\varepsilon t{:}P{:}_i \in \mathcal{Q}$. So $\mathcal{P} \subseteq \mathcal{Q}$, and similarly $\mathcal{Q} \subseteq \mathcal{P}$, so $P \equiv Q$. □

This means we can now show the main result of this paper.

THEOREM 26. *If $P \sim Q$ then $\vdash P = Q$.*

PROOF. By Corollary 24, we can show $\vdash P = P'$ and $\vdash Q = Q'$ where P' and Q' are in normal form. Then by Lemma 25, $P' \equiv Q'$, so $\vdash P = Q$. □

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