Concept Sublattices

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Abstract. We consider the following problem: Given a "universe" of primitive and composed entities, where non-primitive entities may contain other ones. How should we represent these entities, such that their containment relation is decidable? As an answer to this problem we propose a representation based on a Galois connection. An application of this idea in modelling human memory is given as well.

1 Introduction

We consider conceptual knowledge and its representation by (formal) concept lattices which is based on a Galois connection. We use the lattice theoretical notions of a *context* and the corresponding *concept lattice*, introduced by R. Wille ([4]).

Our aim is to generalize these notions, as follows. Basically, a concept lattice is meant to be used to determine a lattice element which corresponds to a given set of input objects *or*, alternatively, of attributes (where *or* denotes exclusive or). We generalize the input and allow both objects *and* attributes to be given. It turns out that in this case the input can be represented by a Galois connection as well. Eventually we obtain the result that our generalization allows a *concept lattice* to be an input. The task is then to determine which sublattice of the concept lattice corresponds to the given input.

We are motivated for this generalization by practical problems in which the input is the yield of some complex, e.g. visual observation process. Such an input contains objects and attributes which belong to some "observed" part of the context. Another motivation is due to a new model of human memory, which is briefly described in Sect. 4.

2 Concept lattices

2.1 Basic definitions

Concepts are defined to exist in a context or "universe". Formally, a *context* is a triple (G, M, I), where G is a set of objects, M is a set of attributes, I is a binary relation between G and M, i.e. for $g \in G$, $m \in M$, $(g, m) \in I$ iff object g has the attribute m. In our definition of a context the sets G and M are finite.

Definition 1. *For a context the following mappings are defined: $A' = \{m \in M|$ gIm for all $g \in A\}$ for $A \subseteq G$, and $B' = \{g \in G|$ gIm for all $m \in B\}$ for $B \subseteq M$.*

A *concept* of a context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, which satisfies the conditions (i) $A' = B$ and (ii) $A = B'$.

For any concepts (A_1, A_1') and (A_2, A_2') of the context (G, M, I), the *hierarchy of concepts* is captured by the definition: $(A_1, A_1') \leq (A_2, A_2')$ iff $A_1 \subseteq A_2$ (or equivalently, iff $A_1' \supseteq A_2'$). The set of all concepts of (G, M, I) with this order relation is called the *concept lattice*.

The mappings of Definition 1 define a Galois connection between the power-sets of G and M.

2.2 Extensions of the basic model

In our model the containment relation on lattices plays a central rôle. This is defined as follows.

Definition 2. *A concept sublattice of a concept lattice L_2 is a complete sublattice L_1 of L_2, such that there exists a lattice homomorphism h from L_2 to L_1, and a pair of functions (g, f) such that the following diagram commutes:*

$$
\begin{array}{ccc}
L_2 & \xleftarrow{\;\;\gamma_2\;\;} & (G_2, M_2, I_2) \\
\downarrow {\scriptstyle h} & & \downarrow {\scriptstyle (g,f)} \\
L_1 & \xleftarrow{\;\;\gamma_1\;\;} & (G_1, M_1, I_1)
\end{array}
$$

where γ_2 and γ_1 are mappings from contexts to concept lattices; $g :: G_2 \to G_1$ and $f :: M_2 \to M_1$ are surjective functions; and the mapping of I_2 to I_1 is such that $(x, y) \in I_2$ implies $(gx, fy) \in I_1$.

We say that an entity has an *occurence* in another entity if it is contained in the latter one. Formal concepts and concept sublattices can be seen as some kind of canonical representations. When another, arbitrary representation is used, it may happen that an entity is differently represented in its occurrences. This is because different subsets of objects and attributes of the entity are considered important in the different occurrences. Therefore it can be very difficult to find out whether an entity has an occurrence elsewhere, as part of another one. A simple example can clarify this. Assume that the geometrical entities, angle, triangle and quadrangle are defined. If these definitions are separately given, and unless special care has been taken, it is very unlikely that one can derive that the latter two geometrical entities "contain" the first one.

In Sect. 3 we give examples which demonstrate how the Galois connection based representation overcomes this problem. We must point out however, that this representation may not be called canonical, simply because a philosophical concept or entity can have different mathematizations, which may be incomparable within a formal system.

3   Examples

We demonstrate the construction of a concept lattice by examples taken from plane geometry. Alignment of a point on a line x is denoted by the attribute on_x.

The context of the angle (G_a, M_a, I_a) is described by three distinct points defining a pair of bisecting straight lines, that is $G_a = \{P_0, P_1, P_2\}$, $M_a = \{on_e, on_f\}$ and $I_a = \{(P_0, on_e), (P_0, on_f), (P_1, on_e), (P_2, on_f)\}$. The members of the concept lattice are: $C_0 = (\{P_0\}, \{on_e, on_f\})$, $C_1 = (\{P_0, P_1, P_2\}, \{\ \})$, $C_2 = (\{P_0, P_1\}, \{on_e\})$, $C_3 = (\{P_0, P_2\}, \{on_f\})$.

The context of the triangle (G_t, M_t, I_t) is similarly defined. We have that $G_t = \{P_0, P_1, P_2\}$, $M_t = \{on_e, on_f, on_g\}$ and $I_t = \{(P_0, on_e), (P_0, on_f), (P_1, on_f), (P_1, on_g), (P_2, on_e), (P_2, on_f)\}$. The concept lattice of the triangle has members: $C_0 = (\{\ \}, \{on_e, on_f, on_g\})$, $C_1 = (\{P_0, P_1, P_2\}, \{\ \})$, $C_2 = (\{P_0\}, \{on_e, on_f\})$, $C_3 = (\{P_1\}, \{on_f, on_g\})$, $C_4 = (\{P_2\}, \{on_e, on_f\})$, $C_5 = (\{P_0, P_1\}, \{on_f\})$, $C_6 = (\{P_0, P_2\}, \{on_e\})$, $C_7 = (\{P_1, P_2\}, \{on_g\})$.

We can observe that the concept lattice of the angle is contained in the concept lattice of the triangle (as sublattices $\{C_1, C_2, C_5, C_6\}$, $\{C_1, C_3, C_5, C_7\}$ and $\{C_1, C_4, C_6, C_7\}$).

4   Modelling human memory

In modelling human reasoning two main approaches can be identified: the models of logical inference of cognitive psychology (e.g. [1]), and the functional models of experimental neurosurgery (e.g. [3]).

The proposed simple functional model of human memory, depicted in Fig. 1, tries to benefit from both of the mentioned models.
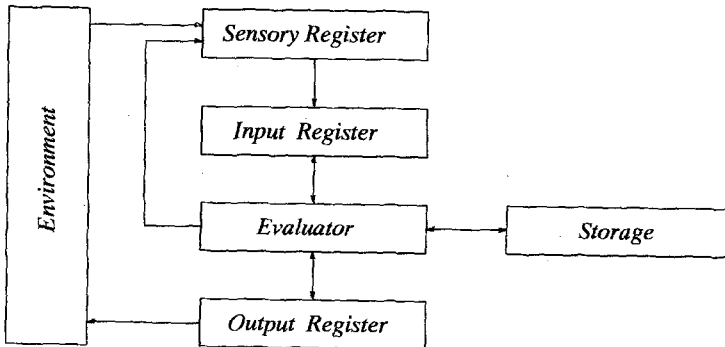

Fig. 1. A functional model of human memory.

In Fig. 1 *Environment* denotes the outside world. The *Sensory Register* is an

input unit which can recognize and identify object-attribute pairs. The input is stored and filtered in the *Input Register*. The *Evaluator* is the unit where the input concept lattice is generated; the feedback path is used, when incomplete data has been input. In that case the input, or certain part of it, is re-sampled at greater precisety. The *Storage* is the memory in the classical sense, which can match the input lattice with a part of the stored concept lattice. Finally, the *Output Register* is the unit of answer generation.

For sake of completeness we will simply assume that the concepts of the concept lattice of the *Storage* possess some information about the answer to be generated. This is a primary answer, which, depending on the distance between the input lattice and the sublattice found by the matching process, can be blocked in the *Output Register*. Answer generation can be interrrupted when due to some change in the *Environment*, the answer under processing is not needed anymore.

The adequacy of the above model can be demonstrated by the functional description of the mechanism of human vision. The retina and the primary optic cortex (also called the V1 field) are connected by a six layer subcortical structure, the corpus geniculatum laterale. The upper four parvocellular layers recognize colours, while the lower two magnocellular layers collect light. This indicates that the vision of form and colour is both functionally and anatomically decomposed to separate parts.

Recently, four perceptional paths of the visual cortex were identified. It is also known that these nervous paths exchange information among themselves. This forms a topological structure of nervous connections, applied by the colour vision mechanism. We emphasize that the visual input is *not* projected to the optic cortex, but instead, a network of active states of frequency and light intensity selective cells is generated, what we call in our model, a lattice of object-attribute relations.

References

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