Optimal Parameter Estimation of Ellipses

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Abstract. In this paper, we propose an unbiased minimum variance estimator to estimate the parameters of an ellipse. The objective of the optimization is to compute a minimum variance estimator. The experimental results show the dramatic improvement over existed weighted least sum of squares approach especially when the ellipse is occluded.

1 Introduction

The methods of estimation of the parameters of quadratic curve can be classified into two categories, the least squares curve fitting [1, 5, 7, 8], and the Kalman filtering techniques [3, 6]. The general quadratic curve can be written as follows:

$$Q(X,Y) = aX^2 + bXY + cY^2 + dX + eY + f = 0, \tag{1}$$

with $b^2 < 4ac$ corresponding to the ellipses. Suppose that points $(x_i, y_i), i = 1, 2, ..., n$ are the detected elliptical points, then the least sum of squares fitting method finds the ellipse parameters (a, b, c, d, e, f) by minimizing following objective function:

$$\sum_{i=1}^{n} \varepsilon_i = \sum_{i=1}^{n} (ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f). \tag{2}$$

The data points have a non-uniform contribution to the above objective function [1]. In order to achieve better performance, the weighted least squares approach has been used [7]. However, the optimal weights would be highly involved, complex and computationally expensive.

Kalman filtering is a sequential technique in the sense that the observation data are sequentially fed into the algorithm, and new estimates are recursively computed from previous estimates and the current new observation. The performance of the sequential techniques is relatively poor for nonlinear problems. Typically, the Kalman filter requires many data points to converge to an acceptable solution.

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In this paper, we propose an optimal unbiased minimum variance estimator using the objective function based on the normal distance of a data point to the ellipse to estimate parameters of the ellipse. The error function is non-linear. We use a parameter space decomposition technique to reduce the computation costs.

2 Estimation Criteria

Suppose that an observation vector y is related to a parameter vector m by an equation $y = Am + \delta_y$, where δ_y is a random vector with zero mean, $E\delta_y = 0$, and covariance matrix $\Gamma_y = E\delta_y\delta_y^t$. The unbiased, minimum variance estimator of m (i.e., that minimizes $E\|\hat{m} - m\|$) is also the one that minimizes

$$(y - Am)^t \Gamma_y^{-1}(y - Am). \tag{3}$$

(see, *e.g.*,[4]). The resulting estimator is

$$\hat{m} = (A^t \Gamma_y^{-1} A)^{-1} A^t \Gamma_y^{-1} y \tag{4}$$

with an error covariance matrix

$$\Gamma_{\hat{m}} = E(\hat{m} - m)(\hat{m} - m)^t = (A^t \Gamma_y^{-1} A)^{-1}. \tag{5}$$

With a nonlinear problem, the observation equation becomes

$$y = f(m) + \delta_y \tag{6}$$

where $f(m)$ is a nonlinear function. As an extension from the linear model, we minimize

$$(y - f(m))^t \Gamma_y^{-1}(y - f(m)). \tag{7}$$

In other words, the optimal parameter vector m is the one that minimizes the matrix-weighted discrepancy between the computed observation $f(m)$ and the actual observation y. At the solution that minimize (3), the estimated $\hat{m}$ has a covariance matrix

$$\begin{aligned}\Gamma_{\hat{m}} &= E(\hat{m} - m)(\hat{m} - m)^t \\ &\simeq \{\frac{\partial f(\hat{m})^t}{\partial m} \Gamma_y^{-1} \frac{\partial f(\hat{m})}{\partial m}\}^{-1}.\end{aligned} \tag{8}$$

One of the advantages of this minimum variance criterion is that we do not need to know the exact noise distribution, which is very difficult to obtain in most applications. The above discussion does not require knowledge of more than second-order statistics of the noise distribution, which often, in practice, can be estimated.

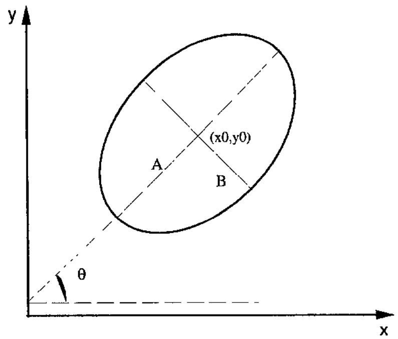

Fig. 1. Illustraion of five ellipse parameters (x_0, y_0, A, B, θ).

3 The Objective Function

The ellipse can be represented by the following equation:

$$\frac{(xcos\theta + y\sin\theta - x_0cos\theta - y_0sin\theta)^2}{A^2} + \frac{(-xsin\theta + ycos\theta + x_0sin\theta - y_0cos\theta)^2}{B^2} = 1, \tag{9}$$

with $m = (A, B, \theta, x_0, y_0)$ as five free parameters. Fig. 1 illustrates the meaning of the parameters. Let $u_{i,j}$ denote the jth component of 2D vectors of the ith point ($j = 1$ for X coordinate and $j = 2$ for Y coordinate). Let the true two-dimensional position of the ith point P_i be defined by parameter α_i, where $P_i(\alpha_i) = (Acos\alpha_i + x_0, Bsin\alpha_i + y_0)$, and the collection of all such parameters be $\alpha_\bullet$. The set of direct observation pairs consists of all noise corrupted versions $\hat{u}_{i,\bullet} = (x_i, y_i)$ of u_i with i from 1 to n. Suppose that the noises between different observation points are uncorrelated and that the correlation between the errors in different components of the image coordinates is negligible. Without loss of generality, we also assume the same error variance in the different components of image points, which has the same variance σ^2. According to the criteria discussed in the above subsection, the objective function to minimize is

$$f(m, \alpha_\bullet) = \sum_{i=1}^{n}\sum_{j=1}^{2} \sigma^{-2}||u_{i,j}(m, \alpha_i) - \hat{u}_{i,j}||^2 \tag{10}$$

where $u_{i,j}(m, \alpha_\bullet)$ is the noise-free projection computed from m and $\alpha_\bullet$.

4 Minimizing the Objective Function

The objective function in equation (10) is neither linear nor quadratic in m and $\alpha_\bullet$ and an iterative algorithm is required to get a solution: m and $\alpha_\bullet$. Instead of performing a computationally expensive direct optimization, we reduce the dimension of the parameter space first. Since the objective functions are continuous, we have

$$\min_{m,\alpha_\bullet} f(m,\alpha_\bullet) = \min_m\{\min_{\alpha_\bullet} f(m,\alpha_\bullet)\} = \min_m g(m) \tag{11}$$

where

$$g(m) = \min_{\alpha_\bullet} f(m,\alpha_\bullet) \tag{12}$$

is the smallest "cost" computed by choosing the "best" points $\alpha_\bullet$, with a given ellipse parameter vector m. As illustrated in Fig. 2, this means that the space $(m, \alpha_\bullet)$ is decomposed into two subspaces corresponding to m and $\alpha_\bullet$, respectively. In the subspace of m, an iterative algorithm (*e.g.*,the Levenberg-Marguardt method or conjugate gradient method) is used. The subspace $\alpha_\bullet$ can be further decomposed. Each individual α_i can be computed noniteratively for any given m. According to the decomposition shown in equation (12), the search space in $\min_m g(m)$ is just five-dimensional.

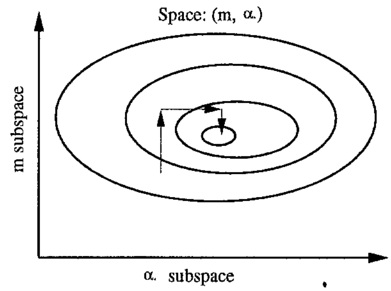

Fig. 2. Decomposition of parameter space. Iterative algorithm is used only in a small subspace corresponding to m. Given each m, the best α_i is computed indiviually. Since the dimension of $\alpha_\bullet$ is very large (n-dimensional with n points), this decomposition significantly reduces computational cost.

Now we consider how to compute the best α_i given m. The problem can be defined as follows: Given a point $P = (x_i, y_i)$ and an ellipse with parameters $m = (A, B, \theta, x_0, y_0)$, find the point P' on the ellipse which minimizes the distance $||PP'||$. For the details of computing P', the reader is referred to [2].

5 Simulation Results

The simulation experiments carry out a comparative study of three different estimation approaches, which are the least sum of squares fitting approach, Safaee-Rad etc.'s weighted least sum of squares fitting approach [7] and our unbiased minimum variance estimator. The measure for "Goodness" of fit is defined as "the sum of normal distances of all the data points to the optimal ellipse", which is considered as an objective and independent measurement [7].

A perfect ellipse is defined by given the ellipse parameters, (A, B, θ, x_0, y_0). The two-dimensional edge points were generated randomly for each trial. The noise was simulated as the digitization error. Three kinds of experiments were conducted during the simulation. The first set of experimental data was based on the edge points which are uniformly distributed on the entire curve of the ellipse. In the second and third set of experiments, the given edge points only covered one-half or one-quarter of the ellipse respectively. During each set of experiments, different numbers of data points were used.

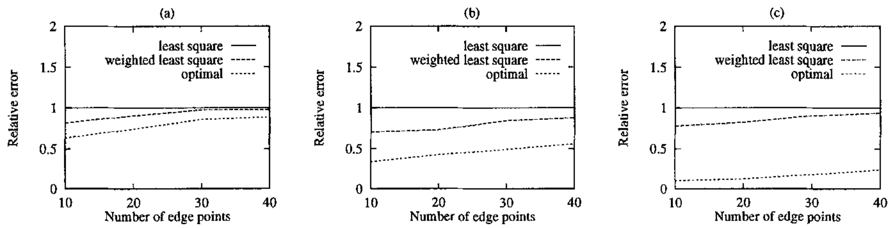

Fig. 3. Relative error of the sum of normal distances of all the data points to the optimal ellipse. (a) Edge points are uniformly distributed on the entire ellipse, (b) Edge points are uniformly distributed on the half of the ellipse, (c) Edge points are uniformly distributed on the quarter of the ellipse.

Fig. 3 shows the average error of the sum of normal distances of all the data points to the optimal ellipse based on 100 trials. The error is plotted as a relative ratio, where the error of the least sum of squares fitting is treated as the base. Fig. 3(c) presents the simulation results when the edge points are coming from one quarter of the ellipse. The improvement of the optimal approach is dramatic. This is because that least squares fitting is statistically biased, the estimation results of these approaches based on the edge points from partial ellipse are far from accurate.

6 Experiments with Real Images

The optimal unbiased minimum variance estimation approach has been applied to a project investigating posture in automobile seats. The body posture of the human driver is estimated by recovering the three dimensional structure of natural body feature points using multiple calibrated cameras. The body feature points are marked with round targets, which appear to be ellipses due to perspective projection. The approach proposed by Wu and Wang [8] was used to detect the boundary points.

After the boundary detection, the optimal estimation approach was used to obtain (x_0, y_0, θ, A, B). Fig. 4 shows the results of two input images. The results are quite good despite the facts: 1) very noisy boundary since the nonplanar feature of the surface of the human body makes the surface of the tag no longer planar; 2) preprocessing errors, such as quantization errors, the edge detection errors and boundary detection errors.

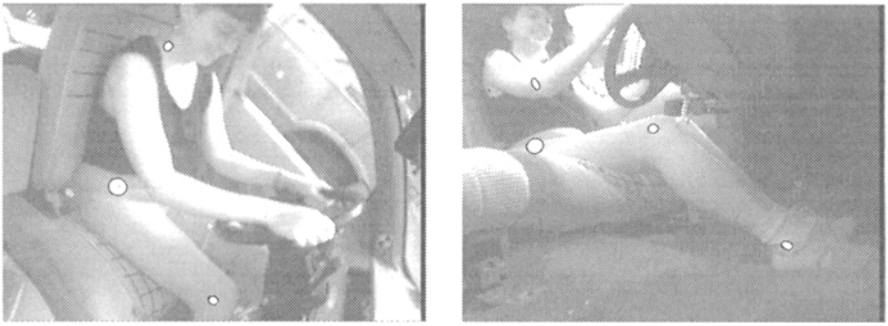

Fig. 4. Detected ellipses are highlighted with dark curves.

7 Conclusions

In this paper, we propose a new unbiased minimum variance estimator for the ellipse parameters (x_0, y_0, θ, A, B). The comparative study shows that this approach can achieve much better performance than least sum of squares fitting and weighted least sum of squares fitting especially when the edge points are coming from the partial boundary of the ellipse and the number of the edge points is relatively small.

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