TREE CONTRACTIONS AND EVOLUTIONARY TREES

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Abstract. An evolutionary tree is a rooted tree where each internal vertex has at least two children and where the leaves are labeled with distinct symbols representing species. Evolutionary trees are useful for modeling the evolutionary history of species. An agreement subtree of two evolutionary trees is an evolutionary tree which is also a topological subtree of the two given trees. We give an algorithm to determine the largest possible number of leaves in any agreement subtree of two trees T_1 and T_2 with n leaves each. If the maximum degree d of these trees is bounded by a constant, the time complexity is $O(n\log^2 n)$ and is within a $\log n$ factor of optimal. For general d, this algorithm runs in $O(nd^2\log d\log^2 n)$ time or alternatively in $O(nd\sqrt{d}\log^3 n)$ time.

Key words. minimal condensed forms, tree contractions, evolutionary trees, computational biology

AMS(MOS) subject classifications. 05C05, 05C85, 05C90, 68Q25, 92B05

1. Introduction. An evolutionary tree is a rooted tree where each internal vertex has at least two children and where the leaves are labeled with distinct symbols representing species. Evolutionary trees are useful for modeling the evolutionary history of species. Many mathematical biologists and computer scientists have been investigating how to construct and compare evolutionary trees [2, 5, 7, 10, 11, 12, 16, 17, 18, 20, 24, 26, 27, 28, 33, 34, 35, 36, 37, 43, 44, 46, 48, 49]. An agreement subtree of two evolutionary trees is an evolutionary tree which is also a topological subtree of the two given trees. A maximum agreement subtree is one with the largest possible number of leaves. Different theories about the evolutionary history of the same species often result in different evolutionary trees. A fundamental problem in computational biology is to determine how much two theories have in common. To a certain extent, this problem can be answered by computing a maximum agreement subtree of two given evolutionary trees [19].

Let T_1 and T_2 be two evolutionary trees with n leaves each. Let d be the maximum degree of these trees. Previously, Kubicka, Kubicki and McMorris [39] gave an algorithm that can compute the number of leaves in a maximum agreement subtree of T_1 and T_2 in $O(n^{(\frac{1}{2}+\epsilon)\log n})$ time for d=2. Steel and Warnow [47] gave the first polynomial-time algorithm. Their algorithm runs in $O(\min\{d!n^2, d^{2.5}n^2\log n\})$ time if d is bounded by a constant and in $O(n^{4.5}\log n)$ time for general trees. Farach and Thorup [14] later reduced the time complexity of this algorithm to $O(n^2)$ for general trees. More recently, they gave an algorithm [15] that runs in $O(n^{1.5}\log n)$ time for general trees. If d is bounded by a constant, this algorithm runs in $O(nc^{\sqrt{\log n}} + n\sqrt{d}\log n)$ time for some constant c > 1.

This paper presents an algorithm for computing a maximum agreement subtree in $O(n\log^2 n)$ time for d bounded by a constant. Since there is a lower bound of $\Omega(n\log n)$, our algorithm is within a $\log n$ factor of optimal. For general d, this algorithm runs in $O(nd^2\log d\log^2 n)$ time or alternatively in $O(nd\sqrt{d}\log^3 n)$ time. This algorithm employs new tree contraction techniques [1, 22, 38, 40, 41]. With tree contraction, we can immediately obtain an $O(n\log^5 n)$ -time algorithm for d bounded by a constant. Reducing the time bound to $O(n\log^2 n)$ requires additional techniques. We develop new results that are useful for bounding the time complexity of tree

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contraction algorithms. As in [14, 15, 47], we also explore the dynamic programming structure of the problem. We obtain some highly regular structural properties and combine these properties with the tree contraction techniques to reduce the time bound by a factor of $\log^2 n$. To remove the last $\log n$ factor, we incorporate some techniques that can compute maxima of multiple sets of sequences at multiple points, where the input sequences are in a compressed format.

We present tree contraction techniques in $\S 2$ and outline our algorithms in $\S 3$. The maximum agreement subtree problem is solved in $\S 4$ and $\S 5$ with a discussion of condensed sequence techniques in $\S 5.1$. Section $\S 6$ concludes this paper with an open problem.

2. New tree contraction techniques. Throughout this paper, all trees are rooted ones, and every nonempty tree path is a vertex-simple one from a vertex to a descendant. For a tree T and a vertex u, let T^u denote the subtree of T formed by u and all its descendants in T.

A key idea of our dynamic programming approach is to partition T_1 and T_2 into well-structured tree paths. We recursively solve our problem for T_1^x and T_2^y for all heads x and y of the tree paths in the partitions of T_1 and T_2 , respectively. The partitioning is based on new tree contraction techniques developed in this section.

A tree is homeomorphic if every internal vertex of that tree has at least two children. Note that the size of a homeomorphic tree is less than twice its number of leaves. Let S be a tree that may or may not be homeomorphic. A chain of S is a tree path in S such that every vertex of the given path has at most one child in S. A tube of S is a maximal chain of S. A root path of a tree is a tree path whose head is the root of that tree; similarly, a leaf path is one ending at a leaf. A leaf tube of S is a tube that is also a leaf path. Let $\mathcal{L}(S)$ denote the set of leaf tubes in S. Let $\mathcal{R}(S) = S - \mathcal{L}(S)$, i.e., the subtree of S obtained by deleting from S all its leaf tubes. The operation S is called the rake operation. See Figures 1 and 2 for examples of rakes and leaf tubes.

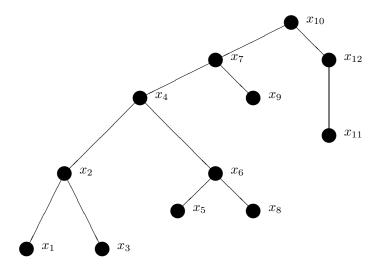
Our dynamic programming approach iteratively rakes T_1 and T_2 until they become empty. The tubes obtained in the process form the desired partitions of T_1 and T_2 . Our rake-based algorithms focus on certain sets of tubes described here. A tube system of a tree T is a set of nonempty tree paths P_1, \dots, P_m in T such that (1) the paths P_i contain no leaves of T and (2) T^{h_1}, \dots, T^{h_m} are pairwise disjoint, where h_i is the head of P_i . Condition (1) is required here because our rake-based algorithms process leaves and non-leaf vertices differently. Condition (2) holds if and only if for all i and j, h_i is not an ancestor or descendant of h_j . We can iteratively rake T to obtain tube systems. The set of tubes obtained by the first rake, i.e., $\mathcal{L}(T)$, is not a tube system of T because $\mathcal{L}(T)$ simply consists of the leaves of T and thus violates Condition (1). Every further rake produces a tube system of T until T is raked to emtpy. Our rake-based algorithms only use these systems although there may be others.

We next develop a theorem to bound the time complexities of rake-based algorithms in this paper. For a tree path P in a tree T,

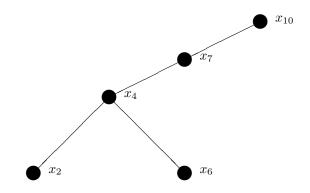
- K(P,T) denotes the set of children of P's vertices in T, excluding P's vertices;
- t(P) denotes the number of vertices in P;
- b(P,T) denotes the number of leaves in T^h where h is the head of P.

(The symbol K stands for the word kids, t for top, and b for bottom.)

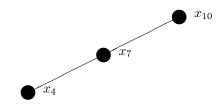
Given T, we recursively define a mapping Φ_T from the subtrees S of T to reals.



After the first rake, the above tree becomes the following tree.

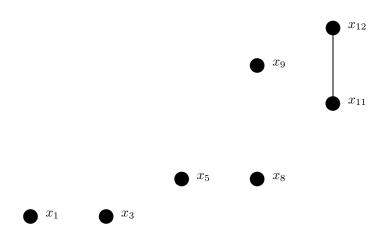


After the second rake, the above tree becomes the following tree.



After the third rake, the above tree becomes empty.

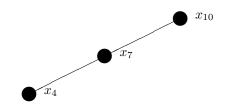
Fig. 1. An example of iterative applications of rakes.



The first rake deletes the above leaf tubes.



The second rake deletes the above leaf tubes.



The third rake deletes the above leaf tube.

Fig. 2. The leaf tubes deleted by the rakes in Figure 1.

If S is an empty tree, then $\Phi_T(S) = 0$. Otherwise,

$$\Phi_T(S) = \Phi_T(\mathcal{R}(S)) + \sum_{P \in \mathcal{L}(S)} b(P, T) \cdot \log(1 + t(P)).$$

(Note. All logarithmic functions log in this paper are in base 2.)

THEOREM 2.1. For all positive integers n and all n-leaf homeomorphic trees T, $\Phi_T(T) \leq n(1 + \log n)$.

Proof. For any given n, $\Phi_T(T)$ is maximized when T is a binary tree formed by attaching n leaves to a path of n-1 vertices. The proof is by induction.

Base Case. For n = 1, the theorem trivially holds.

Now assume n > 2.

Induction Hypothesis. For every positive integer n' < n, the theorem holds.

Induction Step. Let r be the smallest integer such that T is empty after r rakes. Then, at the end of the (r-1)-th rake, T is a path $P=x_1,\dots,x_p$. Let T_1,\dots,T_s be the subtrees of T rooted at vertices in K(P,T). Let n_i be the number of leaves in T_i . Note that

$$\Phi_T(T) = n \log(p+1) + \sum_{i=1}^{s} \Phi_{T_i}(T_i).$$

Since $1 \le n_i < n$ and T_i is homeomorphic, by the induction hypothesis,

$$\Phi_T(T) \le n \log(p+1) + \sum_{i=1}^s n_i (1 + \log n_i).$$

Since $\sum_{i=1}^{s} n_i = n$,

(1)
$$\Phi_T(T) \le n + n \log(p+1) + \sum_{i=1}^s n_i \log n_i.$$

Because T is homeomorphic, each x_i has at least one child in K(P,T). Since $n \geq 2$, $r \geq 2$. Then, x_p cannot be a leaf in T and thus has at least two children in K(P,T). Consequently, $s \geq p+1$. Next, note that for all $m_1, m_2 > 0$,

$$m_1 \log m_1 + m_2 \log m_2 \le (m_1 + m_2) \log(m_1 + m_2).$$

With this inequality and the fact that $s \ge p + 1$, we can combine the terms in the right-hand side summation of Inequality 1 to obtain the following inequality.

(2)
$$\Phi_T(T) \le n + n \log(p+1) + \sum_{i=1}^{p+1} n_i' \log n_i',$$

where $\sum_{i=1}^{p+1} n_i' = n$ and $n_i' \ge 1$. For any given p, the summation in Inequality 2 is maximized when $n_1' = n - p$ and $n_2' = \cdots = n_{p+1}' = 1$. Therefore,

(3)
$$\Phi_T(T) < n + n \log(p+1) + (n-p) \log(n-p).$$

The right-hand side of Inequality 3 is maximized when p = n - 1. This gives the desired bound and finishes the induction proof. \square

3. Comparing evolutionary trees. Formally, an evolutionary tree is a homeomorphic tree whose leaves are labeled by distinct labels. The label set of an evolutionary tree is the set of all the leaf labels of that tree.

The homeomorphic version T' of a tree T is the homeomorphic tree constructed from T as follows. Let $W = \{w \mid w \text{ is a leaf of } T \text{ or is the lowest common ancestor of two leaves}\}$. T' is the tree over W that preserves the ancestor-descendant relationship of T. Let T_1 and T_2 be two evolutionary trees with label sets L_1 and L_2 , respectively.

- For a subset L'_1 of L_1 , $T_1||L'_1$ denotes the homeomorphic version of the tree constructed by deleting from T_1 all the leaves with labels outside L'_1 .
- Let $T_1||T_2 = T_1||(L_1 \cap L_2)$.
- For a tree path P of T_1 , $P||T_2$ denotes the tree path in $T_1||T_2$ formed by the vertices of P that remain in $T_1||T_2$.
- For a set \mathcal{P} of tree paths P_1, \dots, P_m of $T_1, \mathcal{P}||T_2$ denotes the set of all $P_i||T_2$. Formally, if L' is a maximum cardinality subset of $L_1 \cap L_2$ such that there exists a label-preserving tree isomorphism between $T_1||L'|$ and $T_2||L'|$, then $T_1||L'|$ and $T_2||L'|$ are called maximum agreement subtrees of T_1 and T_2 .
 - RR (T_1, T_2) denotes the number of leaves in a maximum agreement subtree of T_1 and T_2 .
 - RA (T_1, T_2) is the mapping from each vertex $v \in T_2 || T_1$ to RR $(T_1, (T_2 || T_1)^v)$, i.e., RA $(T_1, T_2)(v) = RR(T_1, (T_2 || T_1)^v)$.

For a tree path Q of T_2 , if Q is nonempty, let $H(Q, T_2)$ be the set of all vertices in Q and those in $K(Q, T_2)$. If Q is empty, let $H(Q, T_2)$ consist of the root of T_2 , and thus, if both T_2 and Q are empty, $H(Q, T_2) = \emptyset$.

• For a set \mathcal{Q} of tree paths Q_1, \dots, Q_m of T_2 , let $RP(T_1, T_2, \mathcal{Q})$ be the mapping from $v \in \bigcup_{i=1}^m H(Q_i||T_1, T_2||T_1)$ to $RR(T_1, (T_2||T_1)^v)$, i.e., $RP(T_1, T_2, \mathcal{Q})(v) = RR(T_1, (T_2||T_1)^v)$. For simplicity, when \mathcal{Q} consists of only one path Q, let $RP(T_1, T_2, Q)$ denote $RP(T_1, T_2, \mathcal{Q})$.

(The notations RR, RA and RP abbreviate the phrases root to root, root to all and root to path. We use RR to replace the notation MAST of previous work [14, 15, 47] for the sake of notational uniformity.)

Lemma 3.1. Let T_1, T_2, T_3 be evolutionary trees.

- $(T_1||T_2)||T_3 = T_1||(T_2||T_3).$
- If T_3 is a subtree of T_1 , then $T_3||T_1 = T_1||T_3 = T_3$.
- $RR(T_1, T_2) = RR(T_1||T_2, T_2) = RR(T_1, T_2||T_1) = RR(T_1||T_2, T_2||T_1).$

Proof. Straightforward. \square

FACT 1 ([14]). Given an n-leaf evolutionary tree T and k disjoint sets L_1, \dots, L_k of leaf labels of T, the subtrees $T||L_1, \dots, T||L_k$ can be computed in O(n) time.

Proof. The ideas are to preprocess T for answering queries of lowest common ancestors [25, 45] and to reconstruct subtrees from appropriate tree traversal numberings [4, 9]. \square

Given T_1 and T_2 , our main goal is to evaluate $RR(T_1,T_2)$ efficiently. Note that $RR(T_1,T_2) = RR(T_1||T_2,T_2||T_1)$ and that $T_1||T_2$ and $T_2||T_1$ can be computed in linear time. Thus, the remaining discussion assumes that T_1 and T_2 have the same label set. To evaluate $RR(T_1,T_2)$, we actually compute $RA(T_2,T_1)$ and divide the discussion among the five problems defined below. Each problem is named as a p-q case, where p and q are the numbers of tree paths in T_1 and T_2 contained in the input. The inputs of these problems are illustrated in Figure 3.

Problem 1 (one-one case).

Input:

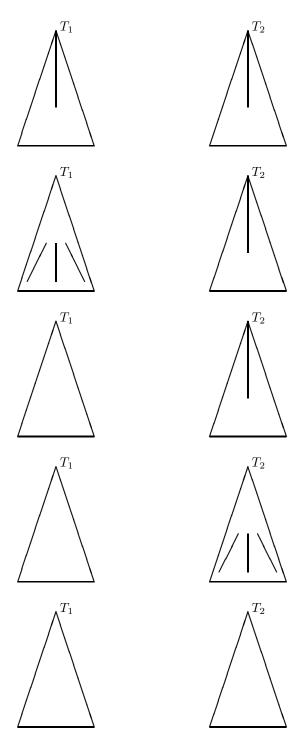


Fig. 3. Inputs of Problems 1-5.

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1. T_1 and T_2;
       2. root paths P of T_1 and Q of T_2 with no leaves from their respective trees;
      3. RP(T_1^u, T_2, Q) for all u \in K(P, T_1);
       4. RP(T_2^v, T_1, P) for all v \in K(Q, T_2).
    Output: RP(T_1, T_2, Q) and RP(T_2, T_1, P).
The next problem generalizes Problem 1.
Problem 2 (many-one case).
    Input:
       1. T_1 and T_2;
       2. a tube system \mathcal{P} = \{P_1, \dots, P_m\} of T_1 and a root path Q of T_2 with no
          leaf from T_2;
       3. RP(T_1^u, T_2, Q) for all P_i and u \in K(P_i, T_1);
       4. RP(T_2^v, T_1, P) for all v \in K(Q, T_2).
    Output:
       1. RP(T_1^{h_i}, T_2, Q) for the head h_i of each P_i;
       2. RP(T_2, T_1, \mathcal{P}).
Problem 3 (zero-one case).
    Input:
       1. T_1 and T_2;
       2. a root path Q of T_2 with no leaf from T_2;
       3. RA(T_2^v, T_1) for all v \in K(Q, T_2).
    Output: RA(T_2, T_1).
The next problem generalizes Problem 3.
PROBLEM 4 (ZERO-MANY CASE).
    Input:
       1. T_1 and T_2;
       2. a tube system Q = \{Q_1, \dots, Q_m\} of T_2;
       3. RA(T_2^v, T_1) for all Q_i and v \in K(Q_i, T_2).
    Output: RA(T_2^{h_i}, T_1) for the head h_i of each Q_i.
Our main goal is to evaluate RR(T_1, T_2). It suffices to solve the next problem.
Problem 5 (zero-zero case).
    Input: T_1 and T_2.
    Output: RA(T_2, T_1).
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Our algorithms for these problems are called *One-One, Many-One, Zero-One, Zero-Many* and *Zero-Zero*, respectively. Each algorithm except One-One uses the preceding one in this list as a subroutine. These reductions are based on the rake operation defined in §2. We give One-One in §5 and the other four in §4.1-4.4.

These five algorithms assume that the input trees T_1 and T_2 have n leaves each and d is the maximum degree. We use integer sort and radix sort [4, 9] extensively to help achieve the desired time complexity. (For brevity, from here onwards, radix sort refers to both integer and radix sorts.) For this reason, we make the following *integer indexing assumptions*:

- ullet An integer array of size O(n) is allocated to each algorithm.
- The vertices of T_1 and T_2 are indexed by integers from [1, O(n)].
- The leaf labels are indexed by integers from [1, O(n)].

We call Zero-Zero only once to compare two given trees. Consequently, we may reasonably assume that the tree vertices are indexed with integers from [1, O(n)]. When we call Zero-Zero, we simply allocate an array of size O(n). As for indexing the leaf labels, this paper considers only evolutionary trees whose leaf labels are drawn

from a total order. Before we call Zero-Zero, we can sort the leaf labels and index them with integers from [1, O(n)]. This preprocessing takes $O(n \log n)$ time, which is well within our desired time complexity for Zero-Zero.

The other four algorithms are called more than once, and their integer indexing assumptions are maintained in slightly different situations from that for Zero-Zero. When an algorithm issues subroutine calls, it is responsible for maintaining the indexing assumptions for the callees. In certain cases, the caller uses radix sort to reindex the labels and the vertices of each callee's input trees. The caller also partitions its array into segments and allocates to each callee a segment in proportion to that callee's input size. The new indices and the array segments for subroutine calls can be computed in obvious manners within the desired time complexity of each caller. For brevity of presentation, such preprocessing steps are omitted in the descriptions of the five algorithms.

Some inputs to the algorithms are mappings. We represent a mapping f by the set of all pairs (x, f(x)). With this representation, the total size of the input mappings in an algorithm is O(n). Since the input mappings have integer values at most n, this representation and the integer indexing assumptions together enable us to evaluate the input mappings at many points in a batch by means of radix sort. Other mappings that are produced within the algorithms are similarly evaluated. When these algorithms are detailed, it becomes evident that such evaluations can computed in straightforward manners in time linear in n and the number of points evaluated. The descriptions of these algorithms assume that the values of mappings are accessed by radix sort.

4. The rake-based reductions. For ease of understanding, our solutions to Problems 1–5 are presented in a different order from their logical one. This section assumes the following theorem for Problem 1 and uses it to solve Problems 2–5. In §5.6, we prove this theorem by giving an algorithm, called *One-One*, that solves Problem 1 within the theorem's stated time bounds.

THEOREM 4.1. Problem 1 can be solved in $O(nd^2 \log d + n \log(p+1) \log(q+1))$ time or alternatively in $O(nd\sqrt{d} \log n + n \log(p+1) \log(q+1))$ time.

Proof. Follows from Theorem 5.14 at the end of $\S 5.6$. \square

4.1. The many-one case. The following algorithm is for Problem 2 and uses One-One as a subroutine. Note that Problem 2 is merely a multi-path version of Problem 1.

Algorithm Many-One; begin

- 1. For all P_i , compute $T_{1,i} = T_1^{h_i}$, $T_{2,i} = T_2 || T_{1,i}$, and $Q_i = Q || T_{1,i}$;
- 2. For all empty Q_i , compute part of the output as follows:
 - (a) Compute the root \hat{v} of $T_{2,i}$ and $v \in K(Q, T_2)$ such that $\hat{v} \in T_2^v$;
 - (b) $RP(T_1^{h_i}, T_2, Q)(\hat{v}) \leftarrow RP(T_2^{v}, T_1, \mathcal{P})(h_i); (Note. \ H(Q_i, T_{2,i}) = \{\hat{v}\}.$ This is part of the output.)
 - (c) For all $x \in H(P_i, T_1)$, $RP(T_2, T_1, \mathcal{P})(x) \leftarrow RP(T_2^v, T_1, \mathcal{P})(x)$; (Note. This is part of the output.)
- 3. For all nonempty Q_i , compute the remaining output as follows: (*Note*. The many-one case is reduced to the one-one case with input $T_{1,i}$, $T_{2,i}$, P_i and Q_i .)
 - (a) For all $u \in K(P_i, T_{1,i})$, $\text{RP}(T_{1,i}^u, T_{2,i}, Q_i) \leftarrow \text{RP}(T_1^u, T_2, Q)$;
 - (b) For all $\hat{v} \in K(Q_i, T_{2,i})$, compute $RP(T_{2,i}^{\hat{v}}, T_{1,i}, P_i)$ as follows:
 - i. Compute the vertex $v \in K(Q, T_2)$ such that $\hat{v} \in T_2^v$;

- ii. $RP(T_{2,i}^{\hat{v}}, T_{1,i}, P_i)(x) \leftarrow RP(T_2^{\hat{v}}, T_1, \mathcal{P})(x)$ for all $x \in H(P_i, T_{1,i})$;
- (c) Compute $RP(T_{1,i}, T_{2,i}, Q_i)$ and $RP(T_{2,i}, T_{1,i}, P_i)$ by applying One-One to $T_{1,i}, T_{2,i}, P_i, Q_i$ and the mappings computed at Steps 3a and 3b;
- (d) $RP(T_1^{h_i}, T_2, Q) \leftarrow RP(T_{1,i}, T_{2,i}, Q_i)$; (Note. This is part of the output.)
- (e) For all $x \in H(P_i, T_{1,i})$, $RP(T_2, T_1, \mathcal{P})(x) \leftarrow RP(T_{2,i}, T_{1,i}, P_i)(x)$; (Note. This is part of the output.)

end.

THEOREM 4.2. Many-One solves Problem 2 with the following time complexities:

$$O(nd^2 \log d + \log(1 + t(Q)) \cdot \sum_{i=1}^{m} b(P_i, T_1) \log(1 + t(P_i))),$$

or alternatively

$$O(nd\sqrt{d}\log n + \log(1 + t(Q)) \cdot \sum_{i=1}^{m} b(P_i, T_1) \log(1 + t(P_i))).$$

Proof. Since T_1 and T_2 have the same label set, all $T_{2,i}$ are nonempty. To compute the output RP, there are two cases depending on whether Q_i is empty or nonempty. These cases are computed by Steps 2 and 3. The correctness of Many-One is then determined by that of Steps 2b, 2c, 3a, 3b, 3(b)ii, 3d and 3e. These steps can be verified using Lemma 3.1. As for the time complexity, these steps take O(n) time using radix sort to evaluate RP. Step 1 uses Fact 1 and takes O(n) time. Steps 2a and 3(b)i take O(n) time using tree traversal and radix sort. As discussed in §3, Step 3c preprocesses the input of its One-One calls to maintain their integer indexing assumptions. We reindex the labels and vertices of $T_{1,i}$ and $T_{2,i}$ and pass the new indices to the calls. We also partition Many-One's O(n)-size array to allocate a segment of size $|T_{1,i}|$ to the call with input $T_{1,i}$. Since the total input size of the calls is O(n), this preprocessing takes O(n) time in an obvious manner. After this preprocessing, the running time of Step 3c dominates that of Many-One. The stated time bounds follow from Theorem 4.1 and the fact that Q_i is not longer than Q and the degrees of $T_{2,i}$ are at most d. \square

4.2. The zero-one case. The following algorithm is for Problem 3. It uses Many-One as a subroutine to recursively compare T_2 with the subtrees of T_1 rooted at the heads of the tubes obtained by iteratively raking T_1 . The tubes obtained by the first rake are compared with T_2 first, and the tube obtained by the last rake is compared last.

Algorithm Zero-One;

- begin
 - 1. $S \leftarrow T_1$;
 - 2. $LF \leftarrow \mathcal{L}(S)$; (Note. LF consists of the leaves of T_1 .)
 - 3. For all $x \in LF$, $RA(T_2, T_1)(x) \leftarrow 1$; (Note. This is part of the output.)
 - 4. For all $u \in LF$, $RP(T_1^u, T_2, Q)(y) \leftarrow 1$, where y is the unique vertex of $T_2||T_1^u$; (Note. This is the base case of rake-based recursion.)
 - 5. $S \leftarrow S \mathcal{L}(S)$;
 - 6. while S is not empty do the following steps:
 - (a) Compute $\mathcal{L}(S) = \{P_1, \dots, P_m\};$
 - (b) Gather the mappings $RP(T_1^u, T_2, Q)$ for all P_i and $u \in K(P_i, T_1)$; (Note. These mappings are either initialized at Step 4 or computed at previous iterations of Step 6d.)

- (c) $RP(T_2^v, T_1, \mathcal{L}(S))(x) \leftarrow RA(T_2^v, T_1)(x)$ for all $v \in K(Q, T_2)$ and $x \in$ $\bigcup_{i=1}^m H(P_i, T_1);$
- (d) Compute $RP(T_1^{h_i}, T_2, Q)$ for the head h_i of each P_i and $RP(T_2, T_1, \mathcal{L}(S))$ by applying Many-One to $T_1, T_2, \mathcal{L}(S), Q$ and the mappings obtained at Steps 6b and 6c; (Note. This is the recursion step of rake-based
- (e) For all $x \in \bigcup_{i=1}^m K(P_i, T_1)$, RA $(T_2, T_1)(x) \leftarrow \text{RP}(T_2, T_1, \mathcal{L}(S))(x)$; (Note. This is part of the output.)
- (f) $S \leftarrow S \mathcal{L}(S)$;

end.

Theorem 4.3. Zero-One solves Problem 3 with the following time complexities:

$$O(nd^2 \log d \log n + n \log n \log(1 + t(Q))),$$

or alternatively

$$O(nd\sqrt{d}\log^2 n + n\log n\log(1+t(Q))).$$

Proof. The $\mathcal{L}(S)$ at Step 6a is a tube system. The heads of the tubes in $\mathcal{L}(S)$ become children of the tubes in future $\mathcal{L}(S)$. The vertices $u \in K(P_i, T_1)$ at Step 6b are either leaves of T_1 or heads of the tubes in previous $\mathcal{L}(S)$. These properties ensure the correctness of the rake-based recursion. The remaining correctness proof uses Lemma 3.1 to verify the correctness of Steps 3, 4, 6c and 6e. Steps 1-5, 6a, 6b and 6f are straightforward and take O(n) time. Step 6c and 6e take O(n) time using radix sort to access RP and RA. At Step 6d, to maintain the integer indexing assumptions for the call to Many-One, we simply pass to Many-One the indices of T_1 and T_2 and the whole array of Zero-One. Step 6d has the same time complexity as Zero-One. The desired time bounds follow from Theorems 2.1 and Theorem 4.2. \square

4.3. The zero-many case. The following algorithm is for Problem 4 and uses Zero-One as a subroutine. Note that Problem 4 is merely a multi-path version of Problem 3.

Algorithm Zero-Many;

begin

- 1. For all Q_i , compute $T_{2,i} = T_2^{h_i}$ and $T_{1,i} = T_1 || T_{2,i}$; 2. For all Q_i and $v \in K(Q_i, T_{2,i})$, $\text{RA}(T_{2,i}^v, T_{1,i}) \leftarrow \text{RA}(T_2^v, T_1)$; 3. For all Q_i , compute $\text{RA}(T_{2,i}, T_{1,i})$ by applying Zero-One to $T_{1,i}, T_{2,i}, Q_i$ and the mapping computed at Step 2;
- 4. For all Q_i , $RA(T_2^{h_i}, T_1) \leftarrow RA(T_{2,i}, T_{1,i})$; (Note. This is the output.)

end.

THEOREM 4.4. Zero-Many solves Problem 4 with the following time complexities:

$$O(nd^2 \log d \log n + \log n \cdot \sum_{i=1}^{m} b(Q_i, T_2) \log(1 + t(Q_i))),$$

or alternatively

$$O(nd\sqrt{d}\log^2 n + \log n \cdot \sum_{i=1}^m b(Q_i, T_2) \log(1 + t(Q_i))).$$

Proof. The proof is similar to that of Theorem 4.2. The time bounds follow from Theorem 4.3. \square

4.4. The zero-zero case. The following algorithm is for Problem 5. It uses Zero-Many as a subroutine to recursively compare T_1 with the subtrees of T_2 rooted at the heads of the tubes obtained by iteratively raking T_2 . The tubes obtained by the first rake are compared with T_1 first, and the tube obtained by the last rake is compared last.

Algorithm Zero-Zero;

begin

- 1. $S \leftarrow T_2$;
- 2. $LF \leftarrow \mathcal{L}(S)$; (Note. LF consists of the leaves of T_2 .)
- 3. For all $v \in LF$, $RA(T_2^v, T_1)(x) \leftarrow 1$, where x is the only vertex in $T_1||T_2^v$; (Note. This is the base case of rake-based recursion.)
- 4. $S \leftarrow S \mathcal{L}(S)$;
- 5. while S is not empty do
 - (a) Compute $\mathcal{L}(S) = \{Q_1, \dots, Q_m\};$
 - (b) Gather the mappings $RA(T_2^v, T_1)$ for all Q_i and $v \in K(Q_i, T_2)$; (Note. These mappings are either initialized at Step 3 or computed at previous iterations of Step 5c.)
 - (c) Compute $RA(T_2^{h_i}, T_1)$ for the head h_i of each Q_i by applying Zero-Many to $T_1, T_2, \mathcal{L}(S)$ and the mappings obtained at Step 5b. (*Note.* This is the recursion step of rake-based recursion.)
 - (d) $S \leftarrow S \mathcal{L}(S)$;
- 6. $RA(T_2, T_1) \leftarrow RA(T_2^h, T_1)$, where h is the root of T_2 ; (Note. This is the output. If T_2 has only one vertex, $RA(T_2^h, T_1)$ is computed at Step 3; otherwise it is computed at the last iteration of Step 5c.)

end.

THEOREM 4.5. Zero-Zero solves Problem 5 within $O(nd^2 \log d \log^2 n)$ time or alternatively within $O(nd\sqrt{d} \log^3 n)$ time.

Proof. The proof is similar to that of Theorem 4.3. The time bounds follow from Theorems 2.1 and 4.4. \square

- 5. The one-one case. Our algorithm for Problem 1 makes extensive use of bisection-based dynamic programming and implicit computation in compressed formats. This problem generalizes the longest common subsequence problem [6, 23, 29, 30, 32, which has efficient dynamic programming solutions. A direct dynamic programming approach to our problem would recursively solve the problem with T_1^x and T_2^y in place of T_1 and T_2 for all vertices $x \in P$ and $y \in Q$. This approach may require solving $\Omega(n^2)$ subproblems. To improve the time complexity, observe that the number of leaves in a maximum agreement subtree of T_1^x and T_2^y can range only from 0 to n. Moreover, this number never increases when x moves from the root of T_1 along P to P's endpoint, and y remains fixed, or vice versa. Compared to the length of P, $RR(T_1^x, T_2^y)$ often assumes relatively few different values. Thus, to compute this number along P, it is useful to compute the locations at P where the number decreases. We can find those locations with a bisection scheme and use them to implicitly solve the $O(n^2)$ subproblems in certain compressed formats. We first describe basic techniques used in such implicit computation in §5.1 and then proceed to discuss bisection-based dynamic programming techniques in §5.2–§5.5. We combine all these techniques to give an algorithm to solve Problem 1 in §5.6.
- **5.1. Condensed sequences.** For integers k_1 and k_2 with $k_1 \leq k_2$, let $[k_1, k_2] = \{k_1, \dots, k_2\}$, i.e., the integer interval between k_1 and k_2 . The *length* of an integer interval is the number of its integers. The *upper* and *lower halves* of an even length

 $[k_1, k_2]$ are $[k_1, \frac{k_1+k_2-1}{2}]$ and $[\frac{k_1+k_2+1}{2}, k_2]$, respectively. The regular integer intervals are defined recursively. For all integers $\alpha \geq 0$, $[1, 2^{\alpha}]$ is regular. The upper and lower halves of an even length regular interval are also regular.

For example, [1, 8] is regular. Its regular subintervals are [1, 4], [5, 8], [1, 2], [3, 4], [5, 6], [7, 8], and the singletons [1, 1], [2, 2], ..., [8, 8].

A normal sequence is a nonincreasing sequence $\{f(j)\}_{j=1}^l$ of nonnegative numbers. A normal sequence is nontrivial if it has at least one nonzero term.

For example, 5, 4, 4, 0 is a nontrivial normal sequence, whereas 0, 0, 0 is a trivial one.

Let f_1, \dots, f_k be k normal sequences of length l. An interval query for f_1, \dots, f_k is a pair $([k_1, k_2], j)$ where $[k_1, k_2] \subseteq [1, k]$ and $j \in [1, l]$. If $k_1 = k_2$, $([k_1, k_2], j)$ is also called a point query. The value of a query $([k_1, k_2], j)$ is $\max_{k_1 \le i \le k_2} f_i(j)$. A query $([k_1, k_2], j)$ is regular if $[k_1, k_2]$ is a regular integer interval.

For example, let

$$f_1 = 5, 4, 4, 3, 2;$$

 $f_2 = 8, 7, 4, 2, 0;$
 $f_3 = 9, 9, 5, 0, 0.$

Then, f_1 , f_2 and f_3 are normal sequences of length 5. Here, k = 3 and l = 5. Thus, ([1,3],2) is an interval query; its value is $\max\{f_1(2), f_2(2), f_3(2)\} = 9$. The pair ([1,1],3) is a point query; its value is $f_1(3) = 4$. The pair ([1,2],2) is a regular query; its values is $\max\{f_1(2), f_2(2)\} = 7$.

The *joint* of f_1, \dots, f_k is the normal sequence \hat{f} also of length l such that $\hat{f}(j) = \max\{f_1(j), \dots, f_k(j)\}.$

Continuing the above example, the joint of f_1, f_2, f_3 is

$$\hat{f} = 9, 9, 5, 3, 2.$$

The minimal condensed form of a normal sequence $\{f(j)\}_{j=1}^l$ is the set of all pairs (j,f(j)) where $f(j)\neq 0$ and j is the largest index of any f(j') with f(j')=f(j). A condensed form is a set of pairs (j,f(j)) that includes the minimal condensed form. The size of a condensed form is the number of pairs in it. The total size of a collection of condensed forms is the sum of the sizes of those forms.

Continuing the above example, the minimal condensed form of f_3 is $\{(2,9), (3,5)\}$; its size is 2. The set $\{(1,9), (2,9), (3,5), (5,0)\}$ is a condensed form of f_3 ; its size is 4. The total size of these two forms is 6.

LEMMA 5.1. Let F_1, \dots, F_k be sets of nontrivial normal sequences of length l. Let \hat{f}_i be the joint of the sequences in F_i . Given a condensed form of each sequence in each F_i , we can compute the minimal condensed forms of all \hat{f}_i in O(l+s) time where s is the total size of the input forms.

Proof. The desired minimal forms can be computed by the two steps below:

- 1. Sort the pairs in the given condensed forms for F_i into a sequence in the increasing order of the first components of these pairs.
- 2. Go through this sequence to delete all unnecessary pairs to obtain the minimal condensed form of \hat{f}_i .

We can use radix sort to implement Step 1 in O(l+s) time for all F_i . Step 2 can be easily implemented in O(s) time for all F_i . \square

LEMMA 5.2. Let f_1, \dots, f_k be nontrivial normal sequences of length l. Assume that the input consists of a condensed form of each f_i with a total size of s.

- 1. We can evaluate m point queries in O(m+l+s) time.
- 2. We can evaluate m_1 regular queries and m_2 irregular queries in a total of $O(m_1 + (m_2 + l + s) \log(k + 1))$ time.

Proof. The proof of Statement 1 uses radix sort in an obvious manner. To prove Statement 2, we assume without loss of generality that k is a power of two. The input queries can be evaluated by the following three stages within the desired time bound.

- Stage 1. For each regular interval $[k_1, k_2] \subseteq [1, k]$, let $f[k_1, k_2]$ be the joint of f_{k_1}, \dots, f_{k_2} . We use Lemma 5.1 $O(\log(k+1))$ times to compute the minimal condensed forms of all $f[k_1, k_2]$. The total size of these forms is $O(s \log(k+1))$. This stage takes $O((l+s)\log(k+1))$ time.
- Stage 2. For each irregular input query $([i_1, i_2], j)$, we partition $[i_1, i_2]$ into $O(\log(k+1))$ regular subintervals $[h_1, h_2], [h_2+1, h_3], \dots, [h_{r-1}+1, h_r]$. Then, the value of $([i_1, i_2], j)$ is the maximum of those of $([h_1, h_2], j), \dots, ([h_{r-1}+1, h_r], j)$. These regular queries are point queries for $f[h_1, h_2], \dots, f[h_{r-1}+1, h_r]$. Together with the given m_1 regular queries, we have now generated $O(m_1 + m_2 \log(k+1))$ point queries for all $f[k_1, k_2]$. This stage takes $O(m_1 + m_2 \log(k+1))$ time.
- Stage 3. We use Statement 1 and the minimal condensed forms of $f[k_1.k_2]$ to evaluate the points queries generated at Stage 2. Once the values of these point queries are obtained, we can easily compute the values of the input queries. This stage takes $O(m_1 + m_2 \log(k+1) + l + s \log(k+1))$ time. \square
- **5.2.** Normalizing the input. To solve Problem 1, we first augment its input T_1, T_2, P and Q in order to simplify our discussion. Let $P = x_1, \dots, x_p$ and $Q = y_1, \dots, y_q$. Without loss of generality, we assume that $p \ge q$.
 - 1. Let α and β be the smallest positive integers such that $p'=2^{\alpha}+1, \ q'=2^{\beta}+1, \ p'\geq q', \ p'>p \ \text{and} \ q'>q.$ (Note. The conditions $p'>p \ \text{and} \ q'>q \ \text{are}$ employed for technical simplicity. They can be changed to $p'\geq p \ \text{and} \ q'\geq q$ with some modification on Algorithm One-One.)
 - 2. Attach to x_p the path $x_{p+1}, \dots, x_{p'}$ and to y_q the path $y_{q+1}, \dots, y_{q'}$.
 - 3. Let $P' = x_1, \dots, x_{p'}$ and $Q' = y_1, \dots, y_{q'}$.
 - 4. Attach a leaf to each of $x_{p+1}, \dots, x_{p'-1}$ and $y_{q+1}, \dots, y_{q'-1}$, two leaves to $x_{p'}$, and two leaves to $y_{q'}$.
 - 5. Assign distinct labels to the new leaves which also differ from the existing labels of T_1 and T_2 .
 - 6. Let S_1 be T_1 together with P' and the new leaves of P'. Let S_2 be T_2 together with Q' and the new leaves of Q'.

 S_1 and S_2 are evolutionary trees. P' and Q' contain no leaves from S_1 and S_2 , and are root paths of these trees. Let $n' = \max\{n_1, n_2\}$ where n_i is the number of leaves in S_i . Let d' be the maximum degree in S_1 and S_2 .

Lemma 5.3

- n' = O(n), p' = O(p), q' = O(q), and $d' \le d + 1$.
- $RP(T_1, T_2, Q) = RP(S_1, S_2, Q')$ and $RP(T_2, T_1, P) = RP(S_2, S_1, P')$.

Proof. Straightforward.

In light of Lemma 5.3, our discussion below mainly works with S_1, S_2, P' and Q'. Let $G = G_P \cup G_Q$ where G_P is the set of all pairs (x_i, y_1) and G_Q is the set of all (x_1, y_j) . To solve Problem 1, a main task is to evaluate $RR(S_1^x, S_2^y)$ for $(x, y) \in G$. The output RP values that are excluded here can be retrieved directly from the input RP mappings.

5.3. Predecessors. A pair $(x_{i'}, y_{j'})$ is a *predecessor* of a distinct (x_i, y_j) if $i \leq i'$ and $j \leq j'$. One-One proceeds by recursively reducing the problem of computing

 $RR(S_1^x, S_2^y)$ to that of computing the RR values of the *P-predecessor*, *Q-predecessor* and PQ-predecessor defined below.

Let P[i,i'] be the path $x_i, \dots, x_{i'}$, where $i \leq i'$. Let X_i be the set of the children of x_i in S_1 that are not in P'. We similarly define Q[j,j'] and Y_j . A pair (x_i,y_j) is intersecting if S_1^u and S_2^v have at least one common leaf label for some $u \in X_i$ and $v \in Y_j$. $(P[i,i'],Q[y_j,y_{j'}])$ is intersecting if some $x_{i''} \in P[i,i']$ and $y_{i''} \in Q[j,j']$ form an intersecting pair.

The lengths of P[i,i'] and Q[j,j'] are those of [i,i'] and [j,j'], respectively. A path P[i,i'] is regular if [i,i'] is a regular interval. A regular Q[j,j'] is similarly defined. We now construct a tree Ψ over pairs of regular paths; this tree is slightly different from that of [15]. The root of Ψ is (P[1,p'-1],Q[1,q'-1]). A pair $(P[i,i'],Q[j,j']) \in \Psi$ is a leaf if and only if either (1) $i=i',\ j=j'$ and (x_i,y_j) is intersecting, or (2) this pair is nonintersecting. For a nonleaf $(P[i,i'],Q[j,j']) \in \Psi$, if j=j', then its children are $(P[i,\frac{i+i'-1}{2}],y_j)$ and $(P[\frac{i+i'+1}{2},i'],y_j)$. Otherwise, this pair has four children $(P[i,\frac{i+i'-1}{2}],Q[j,\frac{j+j'-1}{2}]),(P[i,\frac{i+i'-1}{2}],Q[j,\frac{j+j'+1}{2},j'])$, $(P[\frac{i+i'+1}{2},i'],Q[j,\frac{j+j'-1}{2}]),(P[\frac{i+i'+1}{2},i'],Q[j,j'])$ is (x_i,y_j) ; its floor is $(x_{i'+1},y_{j'+1})$ [15]. Its P-

The ceiling of (P[i,i'],Q[j,j']) is (x_i,y_j) ; its floor is $(x_{i'+1},y_{j'+1})$ [15]. Its P-diagonal is $(x_{i'+1},y_j)$; its Q-diagonal is $(x_i,y_{j'+1})$. Let E be the set of all ceilings, diagonals, floors of the leaves of Ψ . Let $B = \{(x_i,y_{q'}) \mid i \in [1,p']\} \cup \{(x_{p'},y_j) \mid j \in [1,q']\}$. Due to its recursive nature, One-One evaluates $RR(S_1^x,S_2^y)$ for all $(x,y) \in G \cup E \cup B$.

Given (x_i, y_j) , if $(x_{i+1}, y_{i+1}) \in G \cup E \cup B$, then this pair is the PQ-predecessor of (x_i, y_j) . Let i' be the smallest index that is larger than i such that $(x_{i'}, y_j) \in G \cup E \cup B$. This $(x_{i'}, y_j)$ is the P-predecessor of (x_i, y_j) . Let j' be the smallest index larger than j such that $(x_i, y_{j'}) \in G \cup E \cup B$. This $(x_i, y_{j'})$ is the Q-predecessor of (x_i, y_j) .

Lemma 5.4.

- 1. Each intersecting $(x_i, y_j) \in (G \cup E) B$ has a P-predecessor (x_{i+1}, y_j) , a Q-predecessor (x_i, y_{j+1}) and a PQ-predecessor (x_{i+1}, y_{j+1}) .
- 2. Each nonintersecting $(x_i, y_j) \in E B$ has a P-predecessor $(x_{i'}, y_j)$ and a Q-predecessor $(x_i, y_{j'})$. Also, (P[i, i'-1], Q[j, j'-1]) is nonintersecting.
- 3. Each nonintersecting $(x_i, y_1) \in G_P B$ has a P-predecessor (x_{i+1}, y_1) and a Q-predecessor (x_i, y_j) . Moreover, $(x_i, Q[1, j-1])$ is nonintersecting.
- 4. Each nonintersecting $(x_1, y_j) \in G_Q B$ has a P-predecessor (x_i, y_j) and a Q-predecessor (x_1, y_{j+1}) . Moreover, $(P[1, i-1], y_j)$ is nonintersecting.

Proof. Statement 1 follows from the definitions of Ψ and E. The proofs of Statements 3 and 4 are similar to Case 3 in the proof of Statement 2 below.

As for Statement 2, by the definition of B, $x_{i'}$ and $y_{j'}$ exist. To show (P[i, i'-1], Q[j, j'-1]) is nonintersecting, we consider the following four cases. The proofs of their symmetric cases are similar to theirs and are omitted for brevity.

Case 1: (x_i, y_j) is the ceiling of a nonintersecting leaf $(P[i, i_2], Q[j, j_2]) \in \Psi$. Since (x_i, y_{j_2+1}) and (x_{i_2+1}, y_j) are in $E, i' \leq i_2+1$ and $j' \leq j_2+1$. Then because $(P[i, i_2], Q[j, j_2])$ is nonintersecting, so is (P[i, i'-1], Q[j, j'-1]).

Case 2: (x_i, y_j) is the Q-diagonal of a nonintersecting leaf $(P[i, i_2], Q[j_1, j-1])$ (or symmetrically, (x_i, y_j) is the P-diagonal of a nonintersecting leaf $(P[i_1, i-1], Q[j, j_2])$). Since (x_{i_2+1}, y_j) is the floor of $(P[i, i_2], Q[j_1, j-1]), (x_{i_2+1}, y_j) \in E$ and thus $i' \leq i_2+1$. Let j'' be the smallest index such that $j \leq j''$ and $(P[i, i_2], y_{j''})$ is intersecting. There are two subcases.

Case 2a: j'' does not exist. Then, $(P[i, i_2], Q[j, q'])$ is nonintersecting and therefore (P[i, i'-1], Q[j, j'-1]) is nonintersecting.

Case 2b. j'' exists. Let $Q[j_3, j_4]$ be a regular path that contains $y_{j''}$ and is of the same length as $Q[j_1, j-1]$. Note that $j \leq j_3$ and $(P[i, i_2], Q[j_3, j_4]) \in \Psi$. There are two subcases.

Case 2b(1): $j_3 = j$. Then (x_i, y_j) is the ceiling of $(P[i, i_2], Q[j_3, j_4])$. Since (x_i, y_j) is nonintersecting, it is the ceiling of a nonintersecting leaf in Ψ which is a descendant of $(P[i, i_2], Q[j_3, j_4])$. Therefore, Case 2b(1) is reduced to Case 1.

Case 2b(2): $j_3 > j$. By the construction of Ψ , $(x_i, y_{j_3}) \in E$ and thus $j' \leq j_3$. By the choice of $Q[j_3, j_4]$, $(P[i, i_2], Q[j, j_3 - 1])$ is nonintersecting and so is (P[i, i' - 1], Q[j, j' - 1]).

Case 3: (x_i, y_j) is the Q-diagonal of an intersecting leaf (x_i, y_{j-1}) (or symmetrically, (x_i, y_j) is the P-diagonal of an intersecting leaf (x_{i-1}, y_j)). Since $(x_{i+1}, y_j) \in E$, i' = i+1 and $P[i, i'-1] = x_i$. Let j'' be the smallest index such that j < j'' and $(x_i, y_{j''})$ is intersecting. There are two subcases.

Case 3a: j'' does not exist. Then, $(x_i, Q[j, q'])$ is nonintersecting and therefore (P[i, i'-1], Q[j, j'-1]) is nonintersecting.

Case 3b: j'' exists. Then, $(x_i, y_{j''}) \in E$ and $j' \leq j''$. By the choice of j'', $(x_i, Q[j, j''-1])$ is nonintersecting. Thus, (P[i, i'-1], Q[j, j'-1]) is nonintersecting. Case 4: (x_i, y_j) is the floor of a leaf $(P[i_1, i-1], Q[j_1, j-1])$, which may or may not be intersecting. Let $(P[i_3, i_4], Q[j_3, j_4])$ be the lowest ancestor of $(P[i_1, i-1], Q[j_1, j-1])$ in Ψ such that (x_i, y_j) is not the floor of $(P[i_3, i_4], Q[j_3, j_4])$. This ancestor exists because $(x_i, y_j) \notin B$. There are two subcases.

Case 4a: $j_3 = j_4$ and $i_3 < i_4$. Then, $P[i_1, i-1]$ is a subpath of $P[i_3, \frac{i_3+i_4-1}{2}]$ and $i = \frac{i_3+i_4+1}{2}$. Also, $j_3 = j_1 = j-1$. Thus, (x_i, y_j) is the Q-diagonal of $(P[i, i_4], y_{j-1}) \in \Psi$. By the construction of Ψ , (x_i, y_j) is the Q-diagonal of a leaf which is either $(P[i, i_4], y_{j-1})$ itself or its descendant. Depending on whether this leaf is nonintersecting or intersecting, Case 4a is reduced to Case 2 or 3.

Case 4b: $j_3 < j_4$ and $i_3 < i_4$. There are two subcases.

Case 4b(1): $P[i_1, i-1] \subset P[i_3, \frac{i_3+i_4-1}{2}]$ and $Q[j_1, j-1] \subset Q[j_3, \frac{j_3+j_4-1}{2}]$. Note that $i = \frac{i_3+i_4+1}{2}, j = \frac{j_3+j_4+1}{2}$, and (x_i, y_j) is the ceiling of $(P[\frac{i_3+i_4+1}{2}, i_4], Q[\frac{j_3+j_4+1}{2}, j_4]) \in \Psi$. Since (x_i, y_j) is nonintersecting, (x_i, y_j) is the ceiling of a nonintersecting leaf in Ψ which is $(P[\frac{i_3+i_4+1}{2}, i_4], Q[\frac{j_3+j_4+1}{2}, j_4])$ itself or a descendant. This reduces Case 4b(1) to Case 1.

Case 4b(2): $P[i_1,i-1] \subset P[i_3,\frac{i_3+i_4-1}{2}]$ and $Q[j_1,j-1] \subset Q[\frac{j_3+j_4+1}{2},j_4]$ (or symmetrically, $P[i_1,i-1] \subset P[\frac{i_3+i_4+1}{2},i_4]$ and $Q[j_1,j-1] \subset Q[j_3,\frac{j_3+j_4-1}{2}]$). Note that $i=\frac{i_3+i_4+1}{2},\ j=j_4+1,$ and (x_i,y_j) is the Q-diagonal of $(P[\frac{i_3+i_4+1}{2},i_4],Q[\frac{j_3+j_4+1}{2},j_4]) \in \Psi$. Then, (x_i,y_j) is the Q-diagonal of a leaf which is $(P[\frac{i_3+i_4+1}{2},i_4],Q[\frac{j_3+j_4+1}{2},j_4])$ itself or a descendant. Depending on whether this leaf is nonintersecting or intersecting, Case 4b(2) is reduced to Case 2 or 3. \square

5.4. Counting lemmas. We now give some counting lemmas that are used in §5.6 to bound One-One's time complexity.

For all $(P[i_1, i_2], Q[j_1, j_2]) \in \Psi$,

- $C(P[i_1,i_2],Q[j_1,j_2])$ denotes the set of all ceilings of the leaves in Ψ which are either $(P[i_1,i_2],Q[j_1,j_2])$ itself or its descendants;
- $D(P[i_1, i_2], Q[j_1, j_2])$ denotes the set of all Q-diagonals of the leaves in Ψ which are either $(P[i_1, i_2], Q[j_1, j_2])$ itself or its descendants;
- $I(P[i_1, i_2], Q[j_1, j_2]) = \{(x_i, y_j) \mid x_i \in P[i_1, i_2], y_j \in Q[j_1, j_2] \text{ and } (x_i, y_j) \text{ is intersecting} \}.$

Lemma 5.5.

- 1. $|I(P[1, p'-1], Q[1, q'-1])| \le n$.
- 2. Ψ has $O(n \log(q+1))$ leaves of the form $(P[i_1, i_2], Q[j_1, j_2])$ where $j_1 < j_2$.
- 3. Ψ has $O(n \log(q+1))$ pairs of the form $(P[i_1, i_2], y_j)$ where $P[i_1, i_2]$ is of length $\frac{p'-1}{q'-1}$.
- 4. $|E| = \hat{O}(n \log(p+1))$.

Proof. Statements 1-3 are proved below. The proof of Statement 4 is similar to those of Statements 2 and 3.

Statement 1. For all distinct intersecting pairs (x_i, y_j) and $(x_{i'}, y_{j'})$, the leaf labels shared by the subtrees T_1^u where $u \in X_i$ and the subtrees T_2^v where $v \in Y_i$ are different from the shared labels for $X_{i'}$ and $Y_{j'}$. Statement 1 then follows from the fact that S_1 and S_2 share n leaf labels.

Statements 2 and 3. On each level of Ψ , for all distinct pairs $(P[i_1,i_2],Q[j_1,j_2])$ and $(P[i'_1,i'_2],Q[j'_1,j'_2])$, $I(P[i_1,i_2],Q[j_1,j_2]) \cap I(P[i'_1,i'_2],Q[j'_1,j'_2]) = \emptyset$. Thus, each level has at most |I(P[1,p'-1],Q[1,q'-1])| nonleaf pairs. Consequently, from the second level downwards, each level has at most $4 \cdot |I(P[1,p'-1],Q[1,q'-1])|$ pairs. These two statements then follows from Statement 1 and the fact that the pairs specified in these two statements are within the top $1 + \log(q'-1)$ levels of Ψ . \square

A pair (x_i, y_j) is P-regular if [i, i'-1] is a regular interval where $(x_{i'}, y_j)$ is the P-predecessor of (x_i, y_j) . (We do not need the notion of Q-regular because $p' \geq q'$.)

Given a regular $[i_1, i_2]$, a set $\{h_1, \dots, h_k\}$ regularly partitions $[i_1, i_2]$ if $h_1 = i_1$ and the intervals $[h_1, h_2 - 1], [h_2, h_3 - 1], \dots, [h_{k-1}, h_k - 1], [h_k, i_2]$ are all regular.

Lemma 5.6.

- 1. Assume that j > 1 and $P([i_1, i_2], y_j) \in \Psi$. If the P-predecessor (x_i, y_j) of some $(x_{i'}, y_j) \in C(P[i_1, i_2], y_j)$ is not in $\{(x_{i_2+1}, y_j)\} \cup C(P[i_1, i_2], y_j)$, then $P([i_1, i_2], y_{j-1}) \in \Psi$ and $(x_i, y_j) \in D(P[i_1, i_2], y_{j-1})$.
- 2. Assume that j < q' and $P([i_1, i_2], y_{j-1}) \in \Psi$. If the P-predecessor (x_i, y_j) of some $(x_{i'}, y_j) \in D(P[i_1, i_2], y_{j-1})$ is not in $\{(x_{i_2+1}, y_j) \cup D(P[i_1, i_2], y_{j-1}), then <math>P([i_1, i_2], y_j) \in \Psi$ and $(x_i, y_j) \in C(P[i_1, i_2], y_j)$.
- 3. For every $(P[i_1, i_2], y_j) \in \Psi$, the set $\{i \mid (x_i, y_j) \in C(P[i_1, i_2], y_j)\}$ regularly partitions $[i_1, i_2]$ and so does the set $\{i \mid (x_i, y_j) \in D(P[i_1, i_2], y_j)\}$.
- 4. For all $(P[i_1, i_2], y_j) \in \Psi$, every pair in $C(P[i_1, i_2], y_j) \cup D(P[i_1, i_2], y_j)$ is P-regular.
- 5. At most $O(n \log(q+1))$ of the nonintersecting pairs of E are P-irregular.

Proof. The proofs of Statements 1 and 5 are detailed below. The proof of Statement 2 is similar to that of Statement 1 and is omitted. Statement 3 is obvious. Statement 4 follows from the first three statements and the fact that if two sets regularly partition $[i_1, i_2]$, then so does their union.

Statement 1. Note that $i_1 < i \le i_2$ and q' > j > 1. The pair (x_i, y_j) can be the ceiling, the P-diagonal, the Q-diagonal, or the floor of some leaf $(P[i_3, i_4], Q[j_3, j_4]) \in \Psi$. These four cases are discussed below.

Case 1: (x_i, y_j) is the ceiling. Then $i = i_3$ and $j = j_3$. Since $i_1 < i \le i_2$ and both $[i, i_4]$ and $[i_1, i_2]$ are regular, $[i, i_4] \subset [i_1, i_2]$. Since the length of $P[i_1, i_2]$ is at most $\frac{p'-1}{q'-1}$, so is the length of $P[i, i_4]$. Thus $Q[j_3, j_4] = y_j$ and $(P[i, i_4], y_j)$ is a descendant of $(P[i_1, i_2], y_j)$. This contradicts the assumption that $(x_i, y_j) \notin C(P[i_1, i_2], y_j)$ and this case cannot exist.

Case 2: (x_i, y_j) is the P-diagonal. Then $i = i_4 + 1$ and $j = j_3$. As in Case 1, $Q[j_3, j_4] = y_j$ and $(P[i_3, i-1], y_j)$ is a descendant of $(P[i_1, i_2], y_j)$. Thus, there exists a leaf $(P[i, i_6], y_j)$ that is a descendant of $(P[i_1, i_2], y_j)$. Because (x_i, y_j) is the ceiling of this leaf, the existence of this leaf contradicts the assumption that

 $(x_i, y_j) \notin C(P[i_1, i_2], y_j)$ and this case cannot exist.

Case 3: (x_i, y_j) is the Q-diagonal. Then, $i = i_3$ and $j = j_4 + 1$. As in Case 1, $[i, i_4] \subset [i_1, i_2]$ and $Q[j_3, j_4] = y_{j-1}$. Since $(P[i, i_4], y_{j-1}) \in \Psi$, $(P[i_1, i_2], y_{j-1}) \in \Psi$. Then $(P[i, i_4], y_{j-1})$ is a descendant of $(P[i_1, i_2], y_{j-1})$ and $(x_i, y_j) \in D(P[i_1, i_2], y_{j-1})$.

Case 4: (x_i, y_j) is the floor. Then, $i = i_4 + 1$ and $j = j_4 + 1$. As in Case 3, $(P[i_1, i_2], y_{j-1}) \in \Psi, \ Q[j_3, j-1] = y_{j-1} \text{ and } (P[i_3, i-1], y_{j-1}) \text{ is a descendant}$ of $(P[i_1, i_2], y_{j-1})$. Thus, there is a leaf $(P[i, i_6], y_{j-1})$ which is a descendant of $(P[i_1,i_2],y_{j-1})$. Since (x_i,y_j) is this leaf's Q-diagonal, it is in $D(P[i_1,i_2],y_{j-1})$.

Statement 5. Note that E consists of the following three types of pairs:

- 1. the ceiling, diagonals and floor of a leaf $(P[i_1, i_2], Q[j_1, j_2]) \in \Psi$ where $j_1 < j_2$.
- 2. the P-diagonal and floor of $(P[i_1, i_2], y_i]) \in \Psi$ where $P[i_1, i_2]$ is of length
- $\frac{p'-1}{q'-1}$.

 3. the pairs in $C(P[i_1,i_2],j]) \cup D(P[i_1,i_2],y_j])$ where $(P[i_1,i_2],j]) \in \Psi$ and $P[i_1,i_2]$ is of length $\frac{p'-1}{q'-1}$.

By Statement 4, only the pairs of the first two types may be P-irregular. This statement then follows from Lemmas 5.5(2) and 5.5(3). \square

5.5. Recurrences. One-One uses the following formulas to recursively compute $RR(S_1^{x_i}, S_2^{y_j})$ for $(x_i, y_j) \in G \cup E \cup B$ in terms of the RR values of the appropriate P-predecessor, Q-predecessor and PQ-predecessor of (x_i, y_j) .

For vertex subsets U of S_1 and V of S_2 , M(U, V) denotes the maximum weight of any matching of the bipartite graph $(U, V, U \times V)$ where the weight of an edge (u, v)is $RR(S_1^u, S_2^v)$. Let $M(U, v) = M(U, \{v\})$ and $M(u, V) = M(\{u\}, V)$. Given two vertices $x \in S_1$ and $y \in S_2$, let $\overline{\mathrm{M}}(U,V,x,y)$ be the maximum weight of any matching of the same graph without the edge (x, y).

LEMMA 5.7. For each $(x_i, y_j) \in B$, $RR(S_1^x, S_2^y) = 0$.

Proof. This lemma follows from the fact that p' > p, q > q and the new labels of S_1 and S_2 are different from one another and the labels of T_1 and T_2 . \square

FACT 2 ([47]). For all vertices $u \in S_1$ and $v \in S_2$,

$$RR(S_1^u, S_2^v) = \max \left\{ \begin{array}{l} M(K(u, S_1), K(v, S_2)), \\ M(u, K(v, S_2)), \\ M(K(u, S_1), v) \end{array} \right\}.$$

Proof. To form maximum agreement subtrees of S_1^u and S_2^v , there are three cases. (1) $M(K(u, S_1), K(v, S_2))$ accounts for matching u to v. (2) $M(u, K(v, S_2))$ accounts for matching u to a proper descendant of v. (3) $M(K(u, S_1), v)$ accounts for matching v to a proper descendant of u. \square

LEMMA 5.8. For all (x_i, y_j) where i < p' and j < q', regardless of whether (x_i, y_j) is intersecting or nonintersecting,

$$\operatorname{RR}(S_1^{x_i}, S_2^{y_j}) = \max \left\{ \begin{array}{l} \underbrace{\mathbf{M}(X_i, Y_j) + \operatorname{RR}(S_1^{x_{i+1}}, S_2^{y_{j+1}}),} \\ \overline{\mathbf{M}}(X_i \cup \{x_{i+1}\}, Y_j \cup \{y_{j+1}\}, x_{i+1}, y_{j+1}), \\ \operatorname{RR}(S_1^{x_i}, S_2^{y_{j+1}}), \\ \mathbf{M}(x_i, Y_j), \\ \operatorname{RR}(S_1^{x_{i+1}}, S_2^{y_j}), \\ \mathbf{M}(X_i, y_j) \end{array} \right\}.$$

Proof. This lemma follows from Fact 2 with a finer case analysis for the cases in the proof of Fact 2. \square

LEMMA 5.9. For each nonintersecting $(x_i, y_j) \in E-B$ with P-predecessor $(x_{i'}, y_j)$ and Q-predecessor $(x_i, y_{j'})$,

$$\operatorname{RR}(S_1^{x_i}, S_2^{y_j}) = \max \left\{ \begin{array}{l} \max_{j'' \in [j, j'-1]} \operatorname{M}(x_{i'}, Y_{j''}) + \max_{i'' \in [i, i'-1]} \operatorname{M}(X_{i''}, y_{j'}), \\ \operatorname{RR}(S_1^{x_i}, S_2^{y_j}), \\ \operatorname{RR}(S_1^{x_i'}, S_2^{y_j}) \end{array} \right\}.$$

Proof. This lemma follows from Lemma 5.4(2) and is obtained by iterative applications of Lemma 5.8. The following properties are used. Since (P[i, i'-1], Q[j, j'-1]) is nonintersecting, for $i'' \in [i, i'-1]$ and $j'' \in [j, j'-1]$,

- $M(X_{i''}, Y_{j''}) = 0;$
- $\overline{\mathrm{M}}(X_{i''} \cup \{x_{i''+1}\}, Y_{j''} \cup \{y_{j''+1}\}, x_{i''+1}, y_{j''+1}) = \mathrm{M}(x_{i''}, Y_{j''}) + \mathrm{M}(X_{i''}, y_{j''});$
- $M(x_{i''}, Y_{j''}) = M(x_{i'}, Y_{j''});$
- $M(X_{i''}, y_{j''}) = M(X_{i''}, y_{j'}).$

For brevity, the symmetric statement of the next lemma for G_Q is omitted.

LEMMA 5.10. For all nonintersecting pairs $(x_i, y_1) \in G_P - B$ with Q-predecessor (x_i, y_j) ,

$$\operatorname{RR}(S_1^{x_i}, S_2^{y_1}) = \max \left\{ \begin{array}{l} \operatorname{RR}(S_1^{x_i}, S_2^{y_j}), \\ \operatorname{RR}(S_1^{x_{i+1}}, S_2^{y_1}), \\ \operatorname{M}(X_i, y_j) + \max_{j' \in [1, j-1]} \operatorname{M}(x_{i+1}, Y_{j'}) \end{array} \right\}.$$

Proof. The proof is similar to that of Lemma 5.9 and follows from Lemma 5.4(3).

5.6. The algorithm for Problem 1. We combine the discussion in $\S5.3-\S5.5$ to give the following algorithm to solve Problem 1.

Algorithm One-One;

begin

- 1. Compute $S_1, S_2, P', Q', RP(S_1^u, S_2, Q')$ for $u \in K(P', S_1)$, and $RP(S_2^v, S_1, P')$ $v \in K(Q', S_2)$;
- 2. Compute $G \cup E \cup B$, B, I(P[1,p'-1],Q[1,q'-1]) B, the set of all non-intersecting pairs in E B, and the sets of nonintersecting pairs in $G_P B$ and $G_Q B$, respectively;
- 3. Compute the following predecessors:
 - the *P*-predecessor, *Q*-predecessor and *PQ*-predecessor of each pair in I(P[1, p'-1], Q[1, q'-1]) B;
 - the P-predecessor and Q-predecessor of each nonintersecting pair in E-B;
 - the Q-predecessor of each nonintersecting pair in $G_P B$ and the P-predecessor of each nonintersecting pair in $G_Q B$;
- 4. For all pairs in $G \cup E \cup B$, compute the non-RR terms in the appropriate recurrence formulas of §5.5:
 - Lemma 5.7 for B;
 - Lemma 5.8 for (I(P[1, p'-1], Q[1, q'-1]) B;
 - Lemma 5.9 for the nonintersecting pairs in E-B;
 - Lemma 5.10 for the nonintersecting pairs in $G_P B$ and its symmetric statement for the nonintersecting pairs in $G_Q B$;
- 5. Compute the $RR(S_1^{x_i}, S_2^{y_j})$ for all $(x_i, y_j) \in G \cup E \cup B$ using the appropriate recurrence formulas given in §5.5 and the non-RR terms computed at Step 4;

- 6. Compute the output as follows:
 - For all $y_j \in Q$, $\operatorname{RP}(T_1, T_2, Q)(y_j) \leftarrow \operatorname{RR}(S_1^{x_1}, S_2^{y_j});$ For all $x_i \in P$, $\operatorname{RP}(T_2, T_1, P)(x_i) \leftarrow \operatorname{RR}(S_1^{x_i}, S_2^{y_1});$

 - For every $v \in K(Q, T_2)$, $RP(T_1, T_2, Q)(v) \leftarrow RP(T_2^v, T_1, P)(h)$ where h is the root of $T_1||T_2^v|$;
 - For every $u \in K(P, T_1)$, $RP(T_2, T_1, P)(u) \leftarrow RP(T_1^u, T_2, Q)(h)$ where h is the root of $T_2||T_1^u$;

end.

To analyze One-One, we first focus on Step 4. The recurrences of §5.5 contain only four types of non-RR terms other than the constant 0 in Lemma 5.7:

- 1. $M(X_i, y_j)$ and $M(x_i, Y_j)$;
- 2. $\max_{i \in [i_1, i_2]} M(X_i, y_j)$ and $\max_{j \in [j_1, j_2]} M(x_i, Y_j)$;
- 3. $M(X_i, Y_j);$
- 4. $\overline{M}(X_i \cup \{x_{i+1}\}, Y_j \cup \{y_{j+1}\}, x_{i+1}, y_{j+1}).$

It is important to notice that these non-RR terms can be simultaneously evaluated. In light of this observation, we compute these terms by using the techniques of §5.1 to process the normal sequences A_i, A_u, B_i, B_v defined below:

- $A_i(j) = M(X_i, y_j)$ for all x_i and y_j .
- $B_j(i) = M(x_i, Y_j)$ for all y_j and x_i .
- $A_u(j) = RR(S_1^u, S_2^{y_j})$ for all $u \in K(P', S_1)$ and y_j .
- $B_v(i) = \operatorname{RR}(S_2^v, S_1^{\tilde{x}_i})$ for all $v \in K(Q', S_2)$ and x_i .

Note that A_i and A_u have length q', and A_i is the joint of all A_u where $u \in X_i$. Similarly, B_j and B_v have length p', and B_j is the joint of all B_v where $v \in Y_j$.

Lemma 5.11.

- 1. The minimal condensed forms of the sequences A_u and B_v have a total size of O(n) and can be computed in O(n) time.
- 2. The minimal condensed forms of the sequences A_i and B_j have a total size of O(n) and can be computed in O(n) time.

Proof. Statement 2 follows from Statement 1 and Lemma 5.1. Below we only prove Statement 1 for A_u ; Statement 1 for B_v is similarly proved. We first compute a condensed form \overline{A}_u for each A_u as follows:

- 1. For all $u \in K(P', S_1)$, compute $S_{2,u} = S_2 || S_1^u$ and $Q_u = Q' || S_1^u$.
- 2. For all u where Q_u is nonempty, do the following steps:

 - (a) $\overline{A}_u \leftarrow \{(j, w) \mid y_j \in Q_u, w = \text{RP}(S_1^u, S_2, Q')(y_j)\}.$ (b) Compute all tuples (\hat{v}, v, y_j) where $\hat{v} \in K(Q_u, S_{2,u}), v \in K(Q', S_2),$ $\hat{v} \in S_2^v$, and $v \in Y_i$.
 - (c) Find the smallest s such that some (\hat{v}, v, y_s) is obtained at Step 2b.
 - (d) If there is only one (\hat{v}, v, y_s) , then add to A_u the pair (s, w) where w = $RP(S_1^u, S_2, Q')(\hat{v}).$
- 3. For all u where $S_{2,u}$ is nonempty and Q_u is empty, do the following steps:
 - (a) Compute \hat{v} , v and y_s where \hat{v} is the root of $S_{2,u}$, $v \in K(Q', S_2)$, $\hat{v} \in S_2^v$ and $v \in Y_s$.
 - (b) $\overline{A}_u \leftarrow \{(s, w)\}$, where $w = \text{RP}(S_1^u, S_2, Q')(\hat{v})$.
- 4. For all u where $S_{2,u}$ is empty, $\overline{A}_u \leftarrow \emptyset$.

The correctness proof of this algorithm has three cases.

Case 1: Q_u is nonempty. Let $y_{j_1}, y_{j_2}, \cdots, y_{j_k} = Q_u$. Let $j_0 = 0$. Then, for all $k' \in [1, k]$ and all $j \in [j_{k'-1} + 1, j_{k'}], S_2^{y_j} || S_1^u = S_{2,u}^{y_{k'}}$ and by Lemma 3.1, $A_u(j) = S_{2,u}^{y_{k'}}$ $RP(S_1^u, S_2, Q')(y_{k'})$. There are two subcases for $j > j_k$.

Case 1a: Step 2b finds two or more (\hat{v}, v, y_s) . Then $y_s \in Q_u$, $s = j_k$, and for all

 $j \in [j_k + 1, q'], S_2^{y_j} || S_2^u \text{ is empty and } A_u(j) = 0.$

Case 1b: Step 2b finds only one (\hat{v}, v, y_s) . Then $y_s \notin Q_u$ and $s > j_k$. For all $j \in [j_k + 1, s], S_2^{y_j} || S_1^u = S_{2,u}^{\hat{v}}$ and $A_u(j) = \text{RP}(S_1^u, S_2, Q')(\hat{v})$. For all $j \in [s + 1, q'], S_2^{y_j} || S_1^u$ is empty and $A_u(j) = 0$.

Thus, the \overline{A}_u of Step 2 is a condensed form of A_u for Case 1.

Case 2: $S_{2,u}$ is nonempty and Q_u is empty. This case is similar to Case 1b, and Step 3 computes a correct condensed form \overline{A}_u for this case.

Case 3: $S_{2,u}$ is empty. This case is obvious, and Step 4 correctly computes a condensed form \overline{A}_u of A_u for this case.

The total size of all \overline{A}_u is at most that of the RP mappings of S_1, S_2, P' and Q', which is the desired O(n). Step 1 takes O(n) time using Fact 1. The other steps can be implemented in O(n) time in straightforward manners using radix sort and tree traversal. As discussed in §3, the RP mappings are evaluated by radix sort. Once the forms \overline{A}_u are obtained, we can in O(n) time radix sort the pairs in all \overline{A}_u and then delete all unnecessary pairs to obtain the desired minimal condensed forms. \square

LEMMA 5.12. All the non-RR terms of the first two types for the pairs in $G \cup E \cup B$ can be evaluated in $O(n \log(p+1) \log(q+1))$ time.

Proof. The value of $M(X_i, y_j)$ is that of the point query ([i, i], j) for $A_1, \dots, A_{q'}$, and the value of $\max_{i \in [i_1, i_2]} M(X_i, y_j)$ is that of the interval query $([i_1, i_2], j)$. By Lemma 5.5(4), there are $O(n \log(p+1))$ such terms required for the pairs in $G \cup E \cup B$. Given the results of Steps 2 and 3 of One-One, we can determine all such terms and the corresponding queries in $O(n \log(p+1))$ time. By Lemma 5.6(5), only $O(n \log(q+1))$ of these queries are not P-regular. By Lemmas 5.11(2) and 5.2(2), we can evaluate these queries in $O(n \log(p+1) \log(q+1))$ time. The terms $M(x_i, Y_j)$ and $\max_{j \in [j_1, j_2]} M(x_i, Y_j)$ are similarly evaluated is $O(n \log(p+1) \log(q+1))$ time. The analysis for these terms is easier because p' > q' and it does not involve the notion of Q-regularity. \square

LEMMA 5.13. The non-RR terms of the third and the fourth type for the pairs in $G \cup E \cup B$ can be evaluated within the following time complexity:

- 1. $O(nd \log d)$ or alternatively $O(n\sqrt{d} \log n)$ for the third type;
- 2. $O(nd^2 \log d)$ or alternatively $O(nd\sqrt{d} \log n)$ for the fourth type.

Proof. To prove Statement 1, we consider the graphs $(X_i, Y_j, X_i \times Y_j)$ on which the desired terms $M(X_i, Y_j)$ are defined. Let $Z_{i,j}$ be the subgraph of $(X_i, Y_j, X_i \times Y_j)$ constructed by removing all zero-weight edges and all resulting isolated vertices. The edges of $Z_{i,j}$ are computed as follows:

- 1. For all $u \in K(P', S_1)$, compute $S_{2,u} = S_2 || S_1^u$ and $Q_u = Q' || S_1^u$.
- 2. For all $S_{2,u}$ is nonempty, do the following steps:
 - (a) If Q_u is nonempty, compute all tuples (u, v, w) where $\hat{v} \in K(Q_u, S_{2,u})$, $v \in K(Q', S_2)$, $\hat{v} \in S_2^v$ and $w = \text{RP}(S_1^u, S_2, Q')(\hat{v})$.
 - (b) If Q_u is empty, compute the tuple (u, v, w) where \hat{v} is the root of $S_{2,u}$, $v \in K(Q', S_2)$, $\hat{v} \in S_2^v$ and $w = \text{RP}(S_1^u, S_2, Q')(\hat{v})$.

This algorithm captures all the nonzero-weight (u,v). At Step 2, $S_{2,u}^{\hat{v}} = S_2^v||S_1^u$ and by Lemma 3.1 $\text{RR}(S_1^u, S_2^v) = \text{RP}(S_1^u, S_2, Q')(\hat{v})$. Thus, the first two components of the obtained tuples form the edges of all desired $Z_{i,j}$ and the third components are the weights of these edges. We use Fact 1 to implement Step 1 in O(n) time. We can implement Step 2 in O(n) time using radix sort and tree traversal. Note that Step 2 uses radix sort to evaluate RP mappings. With the tuples (u, v, w) obtained, we use radix sort to construct all desired $Z_{i,j}$ in O(n) time. Let $m_{i,j}$ and $n_{i,j}$ be the numbers of edges and vertices in $Z_{i,j}$, respectively. Since an edge weighs at most n, we can compute $M(X_i, Y_j)$ in $O(n_{i,j} \cdot m_{i,j} + n_{i,j}^2 \cdot \log n_{i,j})$ and alternatively in

 $O(m_{i,j}\cdot\sqrt{n_{i,j}}\cdot\log(n\cdot n_{i,j}))$ time [21, 42]. Statement 1 then follows from the fact that $n_{i,j} \leq 2d', n_{i,j} \leq 2m_{i,j}$, and by Lemma 5.5(1) the sum of all $m_{i,j}$ is at most n.

To prove Statement 2, we similarly process the bipartite graphs on which the desired terms $\overline{\mathbf{M}}(X_i \cup \{x_{i+1}\}, Y_j \cup \{y_{j+1}\}, x_{i+1}, y_{j+1})$ are defined. The key difference from the third type is that in addition to some of the edges in $Z_{i,j}$, we need certain nonzero-weight (u, y_{j+1}) for $u \in X_i$ and (x_{i+1}, v) for $v \in Y_j$. Since these edges are required only for intersecting (x_i, y_j) , by Lemma 5.5(1), O(dn) such edges are needed. We use Lemma 5.11(1) to compute the weights of these edges in O(dn) time. Due to these edges, the total time complexity for the fourth type is O(d) times that for the third type. \square

The next theorem serves to prove Theorem 4.1 given at the start of §4.

THEOREM 5.14. One-One solves Problem 1 with the following time complexities:

$$O(nd^2 \log d + n \log(p+1) \log(q+1)),$$

or alternatively

$$O(nd\sqrt{d}\log n + n\log(p+1)\log(q+1)).$$

Proof. The correctness of One-One follows from Lemma 5.3 and §5.3–§5.5. As for the time complexity, Step 1 is obvious and takes O(n) time. By computing Ψ , we can compute the sets E and I(P[1,p'-1],Q[1,q'-1]). Since the leaf labels of S_1 and S_2 are from [1,O(n)], each level of Ψ can be computed in O(n) time. Since Ψ has $O(\log(p+1))$ levels, E and I(P[1,p'-1],Q[1,q'-1]) can be computed in $O(n\log(p+1))$ time. With these two sets obtained, we can compute all the desired sets in $O(n\log(p+1))$ time. Thus, Step 2 takes $O(n\log(p+1))$ time. Step 3 takes $O(n\log(p+1))$ time using radix sort. The time complexity of Step 4 dominates that of One-One. This step uses Lemmas 5.12 and 5.13 and takes $O(n\log(p+1)\log(q+1)+nd^2\log d)$ time or alternatively $O(n\log(p+1)\log(q+1)+nd\sqrt{d}\log n)$ time. Step 5 spends $O(n\log(p+1))$ time using radix sort to create pointers from the pairs in $G \cup E \cup B$ to appropriate predecessors. Step 5 then takes O(1) time per pair in $G \cup E \cup B$ and $O(n\log(p+1))$ time in total. Step 6 takes $O(n\log(p+1))$ time. It uses radix sort to access the desired RR values and evaluate the input mappings. It also uses Fact 1 to compute all $T_1||T_2^v$ and $T_2||T_1^u$. \square

6. Discussions. We answer the main problem of this paper with the following theorem and conclude with an open problem.

THEOREM 6.1. Let T_1 and T_2 be two evolutionary trees with n leaves each. Let d be their maximum degree. Given T_1 and T_2 , a maximum agreement subtree of T_1 and T_2 can be computed in $O(nd^2 \log d \log^2 n)$ time or alternatively in $O(nd\sqrt{d} \log^3 n)$ time. Thus, if d is bounded by a constant, a maximum agreement subtree can be computed in $O(n \log^2 n)$ time.

Proof. By Theorem 4.5, the algorithms in $\S4-5$ compute $RR(T_1, T_2)$ within the desired time bounds. With straightforward modifications, these algorithms can compute a maximum agreement subtree within the same time bounds. \square

The next lemma establishes a reduction from the longest common subsequence problem to that of computing a maximum agreement subtree.

LEMMA 6.2. Let $M_1 = x_1, \ldots, x_n$ and $M_2 = y_1, \ldots, y_n$ be two sequences. Assume that the symbols x_i are all distinct and so are the symbols y_j . Then, the problem of computing a longest common subsequence of M_1 and M_2 can be reduced in linear time to that of computing a maximum agreement subtree of two binary evolutionary trees.

Proof. Given M_1 and M_2 , we construct two binary evolutionary trees T_1 and T_2 as follows. Let z_1 and z_2 be two distinct symbols different from all x_i and y_i . Next, we construct two paths $P_1 = u_1, \ldots, u_{n+1}$ and $P_2 = v_1, \ldots, v_{n+1}$. T_1 is formed by making u_1 the root, attaching x_i to u_i as a leaf, and attaching z_1 and z_2 to u_{n+1} as leaves. Symmetrically, T_2 is formed by making v_1 the root, attaching y_i to v_i , and attaching z_1 and z_2 to v_{n+1} . The lemma follows from the straightforward one-to-one onto correspondence between the longest common subsequences of M_1 and M_2 and the maximum agreement subtrees of T_1 and T_2 . \square

We can use Lemma 6.2 to derive lower complexity bounds for computing a maximum agreement subtree from known bounds for the longest common subsequence problem in various models of computation [3, 6, 23, 29, 30, 32, 50]. This paper assumes a comparison model where two labels x and y can be compared to determine whether x is smaller than y or x = y or x is greater than y. Since the longest common subsequence problem in Lemma 6.2 requires $\Omega(n \log n)$ time in this model [31], the same bound holds for the problem of computing a maximum agreement subtree of two evolutionary trees where d is bounded by a constant. It would be significant to close the gap between this lower bound and the upper bound of $O(n \log^2 n)$ stated in Theorem 6.1. Recently, Farach, Przytycka and Thorup [13] independently developed an algorithm that runs in $O(n\sqrt{d}\log^3 n)$ time. For binary trees, Cole and Hariharan [8] gave an $O(n \log n)$ -time algorithm. It may be possible to close the gap by incorporating ideas used in those two results and this paper.

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