# TREE CONTRACTIONS AND EVOLUTIONARY TREES 

MING-YANG KAO*


#### Abstract

An evolutionary tree is a rooted tree where each internal vertex has at least two children and where the leaves are labeled with distinct symbols representing species. Evolutionary trees are useful for modeling the evolutionary history of species. An agreement subtree of two evolutionary trees is an evolutionary tree which is also a topological subtree of the two given trees. We give an algorithm to determine the largest possible number of leaves in any agreement subtree of two trees $T_{1}$ and $T_{2}$ with $n$ leaves each. If the maximum degree $d$ of these trees is bounded by a constant, the time complexity is $O\left(n \log ^{2} n\right)$ and is within a $\log n$ factor of optimal. For general $d$, this algorithm runs in $O\left(n d^{2} \log d \log ^{2} n\right)$ time or alternatively in $O\left(n d \sqrt{d} \log ^{3} n\right)$ time.


Key words. minimal condensed forms, tree contractions, evolutionary trees, computational biology

AMS(MOS) subject classifications. 05C05, 05C85, 05C90, 68Q25, 92B05

1. Introduction. An evolutionary tree is a rooted tree where each internal vertex has at least two children and where the leaves are labeled with distinct symbols representing species. Evolutionary trees are useful for modeling the evolutionary history of species. Many mathematical biologists and computer scientists have been investigating how to construct and compare evolutionary trees [2, 5, 7, 10, 11, 12, 16, 17, 18, 20, 24, 26, 27, 28, 33, 34, 35, 36, 37, 43, 44, 46, 48, 49. An agreement subtree of two evolutionary trees is an evolutionary tree which is also a topological subtree of the two given trees. A maximum agreement subtree is one with the largest possible number of leaves. Different theories about the evolutionary history of the same species often result in different evolutionary trees. A fundamental problem in computational biology is to determine how much two theories have in common. To a certain extent, this problem can be answered by computing a maximum agreement subtree of two given evolutionary trees 19.

Let $T_{1}$ and $T_{2}$ be two evolutionary trees with $n$ leaves each. Let $d$ be the maximum degree of these trees. Previously, Kubicka, Kubicki and McMorris 39 gave an algorithm that can compute the number of leaves in a maximum agreement subtree of $T_{1}$ and $T_{2}$ in $O\left(n^{\left(\frac{1}{2}+\epsilon\right) \log n}\right)$ time for $d=2$. Steel and Warnow 47 gave the first polynomial-time algorithm. Their algorithm runs in $O\left(\min \left\{d!n^{2}, d^{2.5} n^{2} \log n\right\}\right)$ time if $d$ is bounded by a constant and in $O\left(n^{4.5} \log n\right)$ time for general trees. Farach and Thorup 14] later reduced the time complexity of this algorithm to $O\left(n^{2}\right)$ for general trees. More recently, they gave an algorithm 15 that runs in $O\left(n^{1.5} \log n\right)$ time for general trees. If $d$ is bounded by a constant, this algorithm runs in $O\left(n c^{\sqrt{\log n}}+n \sqrt{d} \log n\right)$ time for some constant $c>1$.

This paper presents an algorithm for computing a maximum agreement subtree in $O\left(n \log ^{2} n\right)$ time for $d$ bounded by a constant. Since there is a lower bound of $\Omega(n \log n)$, our algorithm is within a $\log n$ factor of optimal. For general $d$, this algorithm runs in $O\left(n d^{2} \log d \log ^{2} n\right)$ time or alternatively in $O\left(n d \sqrt{d} \log ^{3} n\right)$ time. This algorithm employs new tree contraction techniques [1, 22, 38, 40, 41]. With tree contraction, we can immediately obtain an $O\left(n \log ^{5} n\right)$-time algorithm for $d$ bounded by a constant. Reducing the time bound to $O\left(n \log ^{2} n\right)$ requires additional techniques. We develop new results that are useful for bounding the time complexity of tree

[^0]contraction algorithms. As in $14,15,47$, we also explore the dynamic programming structure of the problem. We obtain some highly regular structural properties and combine these properties with the tree contraction techniques to reduce the time bound by a factor of $\log ^{2} n$. To remove the last $\log n$ factor, we incorporate some techniques that can compute maxima of multiple sets of sequences at multiple points, where the input sequences are in a compressed format.

We present tree contraction techniques in $\S 2$ and outline our algorithms in $\oint 3$. The maximum agreement subtree problem is solved in $\S(1$ and $\S 5$ with a discussion of condensed sequence techniques in $\oint 5.1$. Section $\S 6$ concludes this paper with an open problem.
2. New tree contraction techniques. Throughout this paper, all trees are rooted ones, and every nonempty tree path is a vertex-simple one from a vertex to a descendant. For a tree $T$ and a vertex $u$, let $T^{u}$ denote the subtree of $T$ formed by $u$ and all its descendants in $T$.

A key idea of our dynamic programming approach is to partition $T_{1}$ and $T_{2}$ into well-structured tree paths. We recursively solve our problem for $T_{1}^{x}$ and $T_{2}^{y}$ for all heads $x$ and $y$ of the tree paths in the partitions of $T_{1}$ and $T_{2}$, respectively. The partitioning is based on new tree contraction techniques developed in this section.

A tree is homeomorphic if every internal vertex of that tree has at least two children. Note that the size of a homeomorphic tree is less than twice its number of leaves. Let $S$ be a tree that may or may not be homeomorphic. A chain of $S$ is a tree path in $S$ such that every vertex of the given path has at most one child in $S$. A tube of $S$ is a maximal chain of $S$. A root path of a tree is a tree path whose head is the root of that tree; similarly, a leaf path is one ending at a leaf. A leaf tube of $S$ is a tube that is also a leaf path. Let $\mathcal{L}(S)$ denote the set of leaf tubes in $S$. Let $\mathcal{R}(S)=S-\mathcal{L}(S)$, i.e., the subtree of $S$ obtained by deleting from $S$ all its leaf tubes. The operation $\mathcal{R}$ is called the rake operation. See Figures 1 and 2 for examples of rakes and leaf tubes.

Our dynamic programming approach iteratively rakes $T_{1}$ and $T_{2}$ until they become empty. The tubes obtained in the process form the desired partitions of $T_{1}$ and $T_{2}$. Our rake-based algorithms focus on certain sets of tubes described here. A tube system of a tree $T$ is a set of nonempty tree paths $P_{1}, \cdots, P_{m}$ in $T$ such that (1) the paths $P_{i}$ contain no leaves of $T$ and (2) $T^{h_{1}}, \cdots, T^{h_{m}}$ are pairwise disjoint, where $h_{i}$ is the head of $P_{i}$. Condition (1) is required here because our rake-based algorithms process leaves and non-leaf vertices differently. Condition (2) holds if and only if for all $i$ and $j, h_{i}$ is not an ancestor or descendant of $h_{j}$. We can iteratively rake $T$ to obtain tube systems. The set of tubes obtained by the first rake, i.e., $\mathcal{L}(T)$, is not a tube system of $T$ because $\mathcal{L}(T)$ simply consists of the leaves of $T$ and thus violates Condition (1). Every further rake produces a tube system of $T$ until $T$ is raked to emtpy. Our rake-based algorithms only use these systems although there may be others.

We next develop a theorem to bound the time complexities of rake-based algorithms in this paper. For a tree path $P$ in a tree $T$,

- $K(P, T)$ denotes the set of children of $P$ 's vertices in $T$, excluding $P$ 's vertices;
- $t(P)$ denotes the number of vertices in $P$;
- $b(P, T)$ denotes the number of leaves in $T^{h}$ where $h$ is the head of $P$. (The symbol $K$ stands for the word kids, $t$ for top, and $b$ for bottom.)

Given $T$, we recursively define a mapping $\Phi_{T}$ from the subtrees $S$ of $T$ to reals.


After the first rake, the above tree becomes the following tree.


After the second rake, the above tree becomes the following tree.


After the third rake, the above tree becomes empty.
FIG. 1. An example of iterative applications of rakes.


The first rake deletes the above leaf tubes.


The second rake deletes the above leaf tubes.


The third rake deletes the above leaf tube.
Fig. 2. The leaf tubes deleted by the rakes in Figure 1 .

If $S$ is an empty tree, then $\Phi_{T}(S)=0$. Otherwise,

$$
\Phi_{T}(S)=\Phi_{T}(\mathcal{R}(S))+\sum_{P \in \mathcal{L}(S)} b(P, T) \cdot \log (1+t(P))
$$

(Note. All logarithmic functions log in this paper are in base 2.)
Theorem 2.1. For all positive integers $n$ and all $n$-leaf homeomorphic trees $T$, $\Phi_{T}(T) \leq n(1+\log n)$.

Proof. For any given $n, \Phi_{T}(T)$ is maximized when $T$ is a binary tree formed by attaching $n$ leaves to a path of $n-1$ vertices. The proof is by induction.

Base Case. For $n=1$, the theorem trivially holds.
Now assume $n \geq 2$.
Induction Hypothesis. For every positive integer $n^{\prime}<n$, the theorem holds.
Induction Step. Let $r$ be the smallest integer such that $T$ is empty after $r$ rakes. Then, at the end of the $(r-1)$-th rake, $T$ is a path $P=x_{1}, \cdots, x_{p}$. Let $T_{1}, \cdots, T_{s}$ be the subtrees of $T$ rooted at vertices in $K(P, T)$. Let $n_{i}$ be the number of leaves in $T_{i}$. Note that

$$
\Phi_{T}(T)=n \log (p+1)+\sum_{i=1}^{s} \Phi_{T_{i}}\left(T_{i}\right)
$$

Since $1 \leq n_{i}<n$ and $T_{i}$ is homeomorphic, by the induction hypothesis,

$$
\Phi_{T}(T) \leq n \log (p+1)+\sum_{i=1}^{s} n_{i}\left(1+\log n_{i}\right)
$$

Since $\sum_{i=1}^{s} n_{i}=n$,

$$
\begin{equation*}
\Phi_{T}(T) \leq n+n \log (p+1)+\sum_{i=1}^{s} n_{i} \log n_{i} \tag{1}
\end{equation*}
$$

Because $T$ is homeomorphic, each $x_{i}$ has at least one child in $K(P, T)$. Since $n \geq 2$, $r \geq 2$. Then, $x_{p}$ cannot be a leaf in $T$ and thus has at least two children in $K(P, T)$. Consequently, $s \geq p+1$. Next, note that for all $m_{1}, m_{2}>0$,

$$
m_{1} \log m_{1}+m_{2} \log m_{2} \leq\left(m_{1}+m_{2}\right) \log \left(m_{1}+m_{2}\right)
$$

With this inequality and the fact that $s \geq p+1$, we can combine the terms in the right-hand side summation of Inequality 1 to obtain the following inequality.

$$
\begin{equation*}
\Phi_{T}(T) \leq n+n \log (p+1)+\sum_{i=1}^{p+1} n_{i}^{\prime} \log n_{i}^{\prime} \tag{2}
\end{equation*}
$$

where $\sum_{i=1}^{p+1} n_{i}^{\prime}=n$ and $n_{i}^{\prime} \geq 1$. For any given $p$, the summation in Inequality 2 is maximized when $n_{1}^{\prime}=n-p$ and $n_{2}^{\prime}=\cdots=n_{p+1}^{\prime}=1$. Therefore,

$$
\begin{equation*}
\Phi_{T}(T) \leq n+n \log (p+1)+(n-p) \log (n-p) \tag{3}
\end{equation*}
$$

The right-hand side of Inequality 3 is maximized when $p=n-1$. This gives the desired bound and finishes the induction proof. $\quad \square$
3. Comparing evolutionary trees. Formally, an evolutionary tree is a homeomorphic tree whose leaves are labeled by distinct labels. The label set of an evolutionary tree is the set of all the leaf labels of that tree.

The homeomorphic version $T^{\prime}$ of a tree $T$ is the homeomorphic tree constructed from $T$ as follows. Let $W=\{w \mid w$ is a leaf of $T$ or is the lowest common ancestor of two leaves $\}. T^{\prime}$ is the tree over $W$ that preserves the ancestor-descendant relationship of $T$. Let $T_{1}$ and $T_{2}$ be two evolutionary trees with label sets $L_{1}$ and $L_{2}$, respectively.

- For a subset $L_{1}^{\prime}$ of $L_{1}, T_{1} \| L_{1}^{\prime}$ denotes the homeomorphic version of the tree constructed by deleting from $T_{1}$ all the leaves with labels outside $L_{1}^{\prime}$.
- Let $T_{1}\left\|T_{2}=T_{1}\right\|\left(L_{1} \cap L_{2}\right)$.
- For a tree path $P$ of $T_{1}, P \| T_{2}$ denotes the tree path in $T_{1} \| T_{2}$ formed by the vertices of $P$ that remain in $T_{1} \| T_{2}$.
- For a set $\mathcal{P}$ of tree paths $P_{1}, \cdots, P_{m}$ of $T_{1}, \mathcal{P} \| T_{2}$ denotes the set of all $P_{i} \| T_{2}$.

Formally, if $L^{\prime}$ is a maximum cardinality subset of $L_{1} \cap L_{2}$ such that there exists a label-preserving tree isomorphism between $T_{1} \| L^{\prime}$ and $T_{2} \| L^{\prime}$, then $T_{1} \| L^{\prime}$ and $T_{2} \| L^{\prime}$ are called maximum agreement subtrees of $T_{1}$ and $T_{2}$.

- $\operatorname{RR}\left(T_{1}, T_{2}\right)$ denotes the number of leaves in a maximum agreement subtree of $T_{1}$ and $T_{2}$.
- $\operatorname{RA}\left(T_{1}, T_{2}\right)$ is the mapping from each vertex $v \in T_{2} \| T_{1}$ to $\operatorname{RR}\left(T_{1},\left(T_{2} \| T_{1}\right)^{v}\right)$, i.e., $\operatorname{RA}\left(T_{1}, T_{2}\right)(v)=\operatorname{RR}\left(T_{1},\left(T_{2} \| T_{1}\right)^{v}\right)$.

For a tree path $Q$ of $T_{2}$, if $Q$ is nonempty, let $H\left(Q, T_{2}\right)$ be the set of all vertices in $Q$ and those in $K\left(Q, T_{2}\right)$. If $Q$ is empty, let $H\left(Q, T_{2}\right)$ consist of the root of $T_{2}$, and thus, if both $T_{2}$ and $Q$ are empty, $H\left(Q, T_{2}\right)=\emptyset$.

- For a set $\mathcal{Q}$ of tree paths $Q_{1}, \cdots, Q_{m}$ of $T_{2}$, let $\operatorname{RP}\left(T_{1}, T_{2}, \mathcal{Q}\right)$ be the mapping from $v \in \cup_{i=1}^{m} H\left(Q_{i}\left\|T_{1}, T_{2}\right\| T_{1}\right)$ to $\operatorname{RR}\left(T_{1},\left(T_{2} \| T_{1}\right)^{v}\right)$, i.e., $\operatorname{RP}\left(T_{1}, T_{2}, \mathcal{Q}\right)(v)=$ $\operatorname{RR}\left(T_{1},\left(T_{2} \| T_{1}\right)^{v}\right)$. For simplicity, when $\mathcal{Q}$ consists of only one path $Q$, let $\operatorname{RP}\left(T_{1}, T_{2}, Q\right)$ denote $\operatorname{RP}\left(T_{1}, T_{2}, \mathcal{Q}\right)$.
(The notations RR, RA and RP abbreviate the phrases root to root, root to all and root to path. We use RR to replace the notation MAST of previous work [14, 15, 47] for the sake of notational uniformity.)

Lemma 3.1. Let $T_{1}, T_{2}, T_{3}$ be evolutionary trees.

- $\left(T_{1} \| T_{2}\right)\left\|T_{3}=T_{1}\right\|\left(T_{2} \| T_{3}\right)$.
- If $T_{3}$ is a subtree of $T_{1}$, then $T_{3}\left\|T_{1}=T_{1}\right\| T_{3}=T_{3}$.
$\bullet \operatorname{RR}\left(T_{1}, T_{2}\right)=\operatorname{RR}\left(T_{1} \| T_{2}, T_{2}\right)=\operatorname{RR}\left(T_{1}, T_{2} \| T_{1}\right)=\operatorname{RR}\left(T_{1}\left\|T_{2}, T_{2}\right\| T_{1}\right)$.
Proof. Straightforward.
FACT 1 (14]). Given an n-leaf evolutionary tree $T$ and $k$ disjoint sets $L_{1}, \cdots, L_{k}$ of leaf labels of $T$, the subtrees $T\left\|L_{1}, \cdots, T\right\| L_{k}$ can be computed in $O(n)$ time.

Proof. The ideas are to preprocess $T$ for answering queries of lowest common ancestors 25, 45 and to reconstruct subtrees from appropriate tree traversal numberings 44, 9|. $\square$

Given $T_{1}$ and $T_{2}$, our main goal is to evaluate $\operatorname{RR}\left(T_{1}, T_{2}\right)$ efficiently. Note that $\operatorname{RR}\left(T_{1}, T_{2}\right)=\operatorname{RR}\left(T_{1}\left\|T_{2}, T_{2}\right\| T_{1}\right)$ and that $T_{1} \| T_{2}$ and $T_{2} \| T_{1}$ can be computed in linear time. Thus, the remaining discussion assumes that $T_{1}$ and $T_{2}$ have the same label set. To evaluate $\operatorname{Rr}\left(T_{1}, T_{2}\right)$, we actually compute $\operatorname{RA}\left(T_{2}, T_{1}\right)$ and divide the discussion among the five problems defined below. Each problem is named as a $p-q$ case, where $p$ and $q$ are the numbers of tree paths in $T_{1}$ and $T_{2}$ contained in the input. The inputs of these problems are illustrated in Figure 3.

Problem 1 (one-one case).

## Input:



Fig. 3. Inputs of Problems $1-5$

1. $T_{1}$ and $T_{2}$;
2. root paths $P$ of $T_{1}$ and $Q$ of $T_{2}$ with no leaves from their respective trees;
3. $\operatorname{RP}\left(T_{1}^{u}, T_{2}, Q\right)$ for all $u \in K\left(P, T_{1}\right)$;
4. $\operatorname{RP}\left(T_{2}^{v}, T_{1}, P\right)$ for all $v \in K\left(Q, T_{2}\right)$.

Output: $\operatorname{RP}\left(T_{1}, T_{2}, Q\right)$ and $\operatorname{RP}\left(T_{2}, T_{1}, P\right)$. The next problem generalizes Problem 11 .
Problem 2 (many-one case).

## Input:

1. $T_{1}$ and $T_{2}$;
2. a tube system $\mathcal{P}=\left\{P_{1}, \cdots, P_{m}\right\}$ of $T_{1}$ and a root path $Q$ of $T_{2}$ with no leaf from $T_{2}$;
3. $\operatorname{RP}\left(T_{1}^{u}, T_{2}, Q\right)$ for all $P_{i}$ and $u \in K\left(P_{i}, T_{1}\right)$;
4. $\operatorname{RP}\left(T_{2}^{v}, T_{1}, \mathcal{P}\right)$ for all $v \in K\left(Q, T_{2}\right)$.

## Output:

1. $\operatorname{RP}\left(T_{1}^{h_{i}}, T_{2}, Q\right)$ for the head $h_{i}$ of each $P_{i}$;
2. $\operatorname{RP}\left(T_{2}, T_{1}, \mathcal{P}\right)$.

Problem 3 (ZERO-ONE CASE).

## Input:

1. $T_{1}$ and $T_{2}$;
2. a root path $Q$ of $T_{2}$ with no leaf from $T_{2}$;
3. $\operatorname{RA}\left(T_{2}^{v}, T_{1}\right)$ for all $v \in K\left(Q, T_{2}\right)$.

Output: $\operatorname{RA}\left(T_{2}, T_{1}\right)$.
The next problem generalizes Problem 3 .
Problem 4 (ZERO-MANY CASE).

## Input:

1. $T_{1}$ and $T_{2}$;
2. a tube system $\mathcal{Q}=\left\{Q_{1}, \cdots, Q_{m}\right\}$ of $T_{2}$;
3. $\operatorname{RA}\left(T_{2}^{v}, T_{1}\right)$ for all $Q_{i}$ and $v \in K\left(Q_{i}, T_{2}\right)$.

Output: $\operatorname{RA}\left(T_{2}^{h_{i}}, T_{1}\right)$ for the head $h_{i}$ of each $Q_{i}$.
Our main goal is to evaluate $\operatorname{RR}\left(T_{1}, T_{2}\right)$. It suffices to solve the next problem.
Problem 5 (zero-zero case).
Input: $T_{1}$ and $T_{2}$.
Output: $\operatorname{RA}\left(T_{2}, T_{1}\right)$.
Our algorithms for these problems are called One-One, Many-One, Zero-One, Zero-Many and Zero-Zero, respectively. Each algorithm except One-One uses the preceding one in this list as a subroutine. These reductions are based on the rake operation defined in $\S \approx$. We give One-One in $\S()^{2}$ and the other four in $\S 4.1-4.4$.

These five algorithms assume that the input trees $T_{1}$ and $T_{2}$ have $n$ leaves each and $d$ is the maximum degree. We use integer sort and radix sort [4, 9] extensively to help achieve the desired time complexity. (For brevity, from here onwards, radix sort refers to both integer and radix sorts.) For this reason, we make the following integer indexing assumptions:

- An integer array of size $O(n)$ is allocated to each algorithm.
- The vertices of $T_{1}$ and $T_{2}$ are indexed by integers from $[1, O(n)]$.
- The leaf labels are indexed by integers from $[1, O(n)]$.

We call Zero-Zero only once to compare two given trees. Consequently, we may reasonably assume that the tree vertices are indexed with integers from $[1, O(n)]$. When we call Zero-Zero, we simply allocate an array of size $O(n)$. As for indexing the leaf labels, this paper considers only evolutionary trees whose leaf labels are drawn
from a total order. Before we call Zero-Zero, we can sort the leaf labels and index them with integers from $[1, O(n)]$. This preprocessing takes $O(n \log n)$ time, which is well within our desired time complexity for Zero-Zero.

The other four algorithms are called more than once, and their integer indexing assumptions are maintained in slightly different situations from that for Zero-Zero. When an algorithm issues subroutine calls, it is responsible for maintaining the indexing assumptions for the callees. In certain cases, the caller uses radix sort to reindex the labels and the vertices of each callee's input trees. The caller also partitions its array into segments and allocates to each callee a segment in proportion to that callee's input size. The new indices and the array segments for subroutine calls can be computed in obvious manners within the desired time complexity of each caller. For brevity of presentation, such preprocessing steps are omitted in the descriptions of the five algorithms.

Some inputs to the algorithms are mappings. We represent a mapping $f$ by the set of all pairs $(x, f(x))$. With this representation, the total size of the input mappings in an algorithm is $O(n)$. Since the input mappings have integer values at most $n$, this representation and the integer indexing assumptions together enable us to evaluate the input mappings at many points in a batch by means of radix sort. Other mappings that are produced within the algorithms are similarly evaluated. When these algorithms are detailed, it becomes evident that such evaluations can computed in straightforward manners in time linear in $n$ and the number of points evaluated. The descriptions of these algorithms assume that the values of mappings are accessed by radix sort.
4. The rake-based reductions. For ease of understanding, our solutions to Problems 115 are presented in a different order from their logical one. This section assumes the following theorem for Problem 11 and uses it to solve Problems 2.5. In $\oint$.6, we prove this theorem by giving an algorithm, called One-One, that solves Problem 11 within the theorem's stated time bounds.

Theorem 4.1. Problem time or alternatively in $O(n d \sqrt{d} \log n+n \log (p+1) \log (q+1))$ time.

Proof. Follows from Theorem 5.14 at the end of $\S 5.6$.
4.1. The many-one case. The following algorithm is for Problem 2 and uses One-One as a subroutine. Note that Problem 2 is merely a multi-path version of Problem 1.
Algorithm Many-One; begin

1. For all $P_{i}$, compute $T_{1, i}=T_{1}^{h_{i}}, T_{2, i}=T_{2} \| T_{1, i}$, and $Q_{i}=Q \| T_{1, i}$;
2. For all empty $Q_{i}$, compute part of the output as follows:
(a) Compute the root $\hat{v}$ of $T_{2, i}$ and $v \in K\left(Q, T_{2}\right)$ such that $\hat{v} \in T_{2}^{v}$;
(b) $\operatorname{RP}\left(T_{1}^{h_{i}}, T_{2}, Q\right)(\hat{v}) \leftarrow \operatorname{RP}\left(T_{2}^{v}, T_{1}, \mathcal{P}\right)\left(h_{i}\right)$; (Note. $H\left(Q_{i}, T_{2, i}\right)=\{\hat{v}\}$. This is part of the output.)
(c) For all $x \in H\left(P_{i}, T_{1}\right), \operatorname{RP}\left(T_{2}, T_{1}, \mathcal{P}\right)(x) \leftarrow \operatorname{RP}\left(T_{2}^{v}, T_{1}, \mathcal{P}\right)(x)$; (Note. This is part of the output.)
3. For all nonempty $Q_{i}$, compute the remaining output as follows: (Note. The many-one case is reduced to the one-one case with input $T_{1, i}, T_{2, i}, P_{i}$ and $Q_{i}$.)
(a) For all $u \in K\left(P_{i}, T_{1, i}\right), \operatorname{RP}\left(T_{1, i}^{u}, T_{2, i}, Q_{i}\right) \leftarrow \operatorname{RP}\left(T_{1}^{u}, T_{2}, Q\right)$;
(b) For all $\hat{v} \in K\left(Q_{i}, T_{2, i}\right)$, compute $\operatorname{RP}\left(T_{2, i}^{\hat{v}}, T_{1, i}, P_{i}\right)$ as follows:
i. Compute the vertex $v \in K\left(Q, T_{2}\right)$ such that $\hat{v} \in T_{2}^{v}$;
ii. $\operatorname{RP}\left(T_{2, i}^{\hat{v}}, T_{1, i}, P_{i}\right)(x) \leftarrow \operatorname{RP}\left(T_{2}^{v}, T_{1}, \mathcal{P}\right)(x)$ for all $x \in H\left(P_{i}, T_{1, i}\right)$;
(c) Compute $\operatorname{RP}\left(T_{1, i}, T_{2, i}, Q_{i}\right)$ and $\operatorname{RP}\left(T_{2, i}, T_{1, i}, P_{i}\right)$ by applying One-One to $T_{1, i}, T_{2, i}, P_{i}, Q_{i}$ and the mappings computed at Steps 3 and 3b;
(d) $\operatorname{RP}\left(T_{1}^{h_{i}}, T_{2}, Q\right) \leftarrow \operatorname{RP}\left(T_{1, i}, T_{2, i}, Q_{i}\right) ;$ (Note. This is part of the output.)
(e) For all $x \in H\left(P_{i}, T_{1, i}\right), \operatorname{RP}\left(T_{2}, T_{1}, \mathcal{P}\right)(x) \leftarrow \operatorname{RP}\left(T_{2, i}, T_{1, i}, P_{i}\right)(x)$; (Note. This is part of the output.)
end.
Theorem 4.2. Many-One solves Problem 圆 with the following time complexities:

$$
O\left(n d^{2} \log d+\log (1+t(Q)) \cdot \sum_{i=1}^{m} b\left(P_{i}, T_{1}\right) \log \left(1+t\left(P_{i}\right)\right)\right)
$$

or alternatively

$$
O\left(n d \sqrt{d} \log n+\log (1+t(Q)) \cdot \sum_{i=1}^{m} b\left(P_{i}, T_{1}\right) \log \left(1+t\left(P_{i}\right)\right)\right)
$$

Proof. Since $T_{1}$ and $T_{2}$ have the same label set, all $T_{2, i}$ are nonempty. To compute the output RP, there are two cases depending on whether $Q_{i}$ is empty or nonempty. These cases are computed by Steps 2 and 3. The correctness of Many-One is then determined by that of Steps 2b, 2d, 3a, 3b, 3(b)ii, 3d and 3d. These steps can be verified using Lemma 3.1. As for the time complexity, these steps take $O(n)$ time using radix sort to evaluate RP. Step 1 uses Fact 1 and takes $O(n)$ time. Steps 2a and $3(\mathrm{~b}) \mathrm{i}$ take $O(n)$ time using tree traversal and radix sort. As discussed in $\$ 3$, Step 30 preprocesses the input of its One-One calls to maintain their integer indexing assumptions. We reindex the labels and vertices of $T_{1, i}$ and $T_{2, i}$ and pass the new indices to the calls. We also partition Many-One's $O(n)$-size array to allocate a segment of size $\left|T_{1, i}\right|$ to the call with input $T_{1, i}$. Since the total input size of the calls is $O(n)$, this preprocessing takes $O(n)$ time in an obvious manner. After this preprocessing, the running time of Step 30 dominates that of Many-One. The stated time bounds follow from Theorem 4.1 and the fact that $Q_{i}$ is not longer than $Q$ and the degrees of $T_{2, i}$ are at most $d$. $\square$
4.2. The zero-one case. The following algorithm is for Problem 3. It uses Many-One as a subroutine to recursively compare $T_{2}$ with the subtrees of $T_{1}$ rooted at the heads of the tubes obtained by iteratively raking $T_{1}$. The tubes obtained by the first rake are compared with $T_{2}$ first, and the tube obtained by the last rake is compared last.
Algorithm Zero-One;
begin

1. $S \leftarrow T_{1}$;
2. $L F \leftarrow \mathcal{L}(S) ;\left(\right.$ Note. $L F$ consists of the leaves of $T_{1}$.)
3. For all $x \in L F, \operatorname{RA}\left(T_{2}, T_{1}\right)(x) \leftarrow 1$; (Note. This is part of the output.)
4. For all $u \in L F, \operatorname{RP}\left(T_{1}^{u}, T_{2}, Q\right)(y) \leftarrow 1$, where $y$ is the unique vertex of $T_{2} \| T_{1}^{u}$; (Note. This is the base case of rake-based recursion.)
5. $S \leftarrow S-\mathcal{L}(S)$;
6. while $S$ is not empty do the following steps:
(a) Compute $\mathcal{L}(S)=\left\{P_{1}, \cdots, P_{m}\right\}$;
(b) Gather the mappings $\operatorname{RP}\left(T_{1}^{u}, T_{2}, Q\right)$ for all $P_{i}$ and $u \in K\left(P_{i}, T_{1}\right)$; (Note. These mappings are either initialized at Step 4 or computed at previous iterations of Step 6d.)
(c) $\operatorname{RP}\left(T_{2}^{v}, T_{1}, \mathcal{L}(S)\right)(x) \leftarrow \operatorname{RA}\left(T_{2}^{v}, T_{1}\right)(x)$ for all $v \in K\left(Q, T_{2}\right)$ and $x \in$ $\cup_{i=1}^{m} H\left(P_{i}, T_{1}\right)$;
(d) Compute $\operatorname{RP}\left(T_{1}^{h_{i}}, T_{2}, Q\right)$ for the head $h_{i}$ of each $P_{i}$ and $\operatorname{RP}\left(T_{2}, T_{1}, \mathcal{L}(S)\right)$ by applying Many-One to $T_{1}, T_{2}, \mathcal{L}(S), Q$ and the mappings obtained at Steps 6 and 6 ; (Note. This is the recursion step of rake-based recursion.)
(e) For all $x \in \cup_{i=1}^{m} K\left(P_{i}, T_{1}\right), \operatorname{RA}\left(T_{2}, T_{1}\right)(x) \leftarrow \operatorname{RP}\left(T_{2}, T_{1}, \mathcal{L}(S)\right)(x)$; (Note. This is part of the output.)
(f) $S \leftarrow S-\mathcal{L}(S)$;
end.
Theorem 4.3. Zero-One solves Problem 3 with the following time complexities:

$$
O\left(n d^{2} \log d \log n+n \log n \log (1+t(Q))\right)
$$

or alternatively

$$
O\left(n d \sqrt{d} \log ^{2} n+n \log n \log (1+t(Q))\right)
$$

Proof. The $\mathcal{L}(S)$ at Step 6a is a tube system. The heads of the tubes in $\mathcal{L}(S)$ become children of the tubes in future $\mathcal{L}(S)$. The vertices $u \in K\left(P_{i}, T_{1}\right)$ at Step 6 b are either leaves of $T_{1}$ or heads of the tubes in previous $\mathcal{L}(S)$. These properties ensure the correctness of the rake-based recursion. The remaining correctness proof uses Lemma 3.1 to verify the correctness of Steps 3, 4, 6c and 6c. Steps 115, 6a, 6b and 6 are straightforward and take $O(n)$ time. Step 60 and 6 e take $O(n)$ time using radix sort to access RP and RA. At Step 6d, to maintain the integer indexing assumptions for the call to Many-One, we simply pass to Many-One the indices of $T_{1}$ and $T_{2}$ and the whole array of Zero-One. Step 6d has the same time complexity as Zero-One. The desired time bounds follow from Theorems 2.1 and Theorem 4.2. $\quad$.
4.3. The zero-many case. The following algorithm is for Problem 4 and uses Zero-One as a subroutine. Note that Problem is merely a multi-path version of Problem 3.
Algorithm Zero-Many; begin

1. For all $Q_{i}$, compute $T_{2, i}=T_{2}^{h_{i}}$ and $T_{1, i}=T_{1} \| T_{2, i}$;
2. For all $Q_{i}$ and $v \in K\left(Q_{i}, T_{2, i}\right), \operatorname{RA}\left(T_{2, i}^{v}, T_{1, i}\right) \leftarrow \operatorname{RA}\left(T_{2}^{v}, T_{1}\right)$;
3. For all $Q_{i}$, compute $\operatorname{RA}\left(T_{2, i}, T_{1, i}\right)$ by applying Zero-One to $T_{1, i}, T_{2, i}, Q_{i}$ and the mapping computed at Step 2;
4. For all $Q_{i}, \operatorname{RA}\left(T_{2}^{h_{i}}, T_{1}\right) \leftarrow \operatorname{RA}\left(T_{2, i}, T_{1, i}\right) ;$ (Note. This is the output.)
end.
Theorem 4.4. Zero-Many solves Problem with the following time complexities:

$$
O\left(n d^{2} \log d \log n+\log n \cdot \sum_{i=1}^{m} b\left(Q_{i}, T_{2}\right) \log \left(1+t\left(Q_{i}\right)\right)\right)
$$

or alternatively

$$
O\left(n d \sqrt{d} \log ^{2} n+\log n \cdot \sum_{i=1}^{m} b\left(Q_{i}, T_{2}\right) \log \left(1+t\left(Q_{i}\right)\right)\right)
$$

Proof. The proof is similar to that of Theorem 4.2. The time bounds follow from Theorem 4.3. $\quad$ I
4.4. The zero-zero case. The following algorithm is for Problem 5. It uses Zero-Many as a subroutine to recursively compare $T_{1}$ with the subtrees of $T_{2}$ rooted at the heads of the tubes obtained by iteratively raking $T_{2}$. The tubes obtained by the first rake are compared with $T_{1}$ first, and the tube obtained by the last rake is compared last.
Algorithm Zero-Zero; begin

1. $S \leftarrow T_{2}$;
2. $L F \leftarrow \mathcal{L}(S) ;\left(\right.$ Note. $L F$ consists of the leaves of $T_{2}$.)
3. For all $v \in L F, \operatorname{RA}\left(T_{2}^{v}, T_{1}\right)(x) \leftarrow 1$, where $x$ is the only vertex in $T_{1} \| T_{2}^{v}$; (Note. This is the base case of rake-based recursion.)
4. $S \leftarrow S-\mathcal{L}(S)$;
5. while $S$ is not empty do
(a) Compute $\mathcal{L}(S)=\left\{Q_{1}, \cdots, Q_{m}\right\}$;
(b) Gather the mappings $\operatorname{Ra}\left(T_{2}^{v}, T_{1}\right)$ for all $Q_{i}$ and $v \in K\left(Q_{i}, T_{2}\right)$; (Note. These mappings are either initialized at Step 3 or computed at previous iterations of Step 5c.)
(c) Compute $\operatorname{RA}\left(T_{2}^{h_{i}}, T_{1}\right)$ for the head $h_{i}$ of each $Q_{i}$ by applying Zero-Many to $T_{1}, T_{2}, \mathcal{L}(S)$ and the mappings obtained at Step 5b. (Note. This is the recursion step of rake-based recursion.)
(d) $S \leftarrow S-\mathcal{L}(S)$;
6. $\operatorname{RA}\left(T_{2}, T_{1}\right) \leftarrow \operatorname{RA}\left(T_{2}^{h}, T_{1}\right)$, where $h$ is the root of $T_{2} ;($ Note. This is the output. If $T_{2}$ has only one vertex, $\operatorname{RA}\left(T_{2}^{h}, T_{1}\right)$ is computed at Step 3 ; otherwise it is computed at the last iteration of Step 5 .)
end.
Theorem 4.5. Zero-Zero solves Problem 5 within $O\left(n d^{2} \log d \log ^{2} n\right)$ time or alternatively within $O\left(n d \sqrt{d} \log ^{3} n\right)$ time.

Proof. The proof is similar to that of Theorem 4.3. The time bounds follow from Theorems 2.1 and 4.4. $\square$
5. The one-one case. Our algorithm for Problem 1 makes extensive use of bisection-based dynamic programming and implicit computation in compressed formats. This problem generalizes the longest common subsequence problem [66, 23, 29 , 30, 32], which has efficient dynamic programming solutions. A direct dynamic programming approach to our problem would recursively solve the problem with $T_{1}^{x}$ and $T_{2}^{y}$ in place of $T_{1}$ and $T_{2}$ for all vertices $x \in P$ and $y \in Q$. This approach may require solving $\Omega\left(n^{2}\right)$ subproblems. To improve the time complexity, observe that the number of leaves in a maximum agreement subtree of $T_{1}^{x}$ and $T_{2}^{y}$ can range only from 0 to $n$. Moreover, this number never increases when $x$ moves from the root of $T_{1}$ along $P$ to $P$ 's endpoint, and $y$ remains fixed, or vice versa. Compared to the length of $P, \operatorname{RR}\left(T_{1}^{x}, T_{2}^{y}\right)$ often assumes relatively few different values. Thus, to compute this number along $P$, it is useful to compute the locations at $P$ where the number decreases. We can find those locations with a bisection scheme and use them to implicitly solve the $O\left(n^{2}\right)$ subproblems in certain compressed formats. We first describe basic techniques used in such implicit computation in $\$ 5.1$ and then proceed to discuss bisection-based dynamic programming techniques in $\$ 5.2 \$ 5.5$. We combine all these techniques to give an algorithm to solve Problem 11 in $\$ 5.6$.
5.1. Condensed sequences. For integers $k_{1}$ and $k_{2}$ with $k_{1} \leq k_{2}$, let $\left[k_{1}, k_{2}\right]=$ $\left\{k_{1}, \cdots, k_{2}\right\}$, i.e., the integer interval between $k_{1}$ and $k_{2}$. The length of an integer interval is the number of its integers. The upper and lower halves of an even length
[ $\left.k_{1}, k_{2}\right]$ are $\left[k_{1}, \frac{k_{1}+k_{2}-1}{2}\right]$ and $\left[\frac{k_{1}+k_{2}+1}{2}, k_{2}\right]$, respectively. The regular integer intervals are defined recursively. For all integers $\alpha \geq 0,\left[1,2^{\alpha}\right]$ is regular. The upper and lower halves of an even length regular interval are also regular.

For example, $[1,8]$ is regular. Its regular subintervals are $[1,4],[5,8],[1,2],[3,4]$, $[5,6],[7,8]$, and the singletons $[1,1],[2,2], \ldots,[8,8]$.

A normal sequence is a nonincreasing sequence $\{f(j)\}_{j=1}^{l}$ of nonnegative numbers. A normal sequence is nontrivial if it has at least one nonzero term.

For example, $5,4,4,0$ is a nontrivial normal sequence, whereas $0,0,0$ is a trivial one.

Let $f_{1}, \cdots, f_{k}$ be $k$ normal sequences of length $l$. An interval query for $f_{1}, \cdots, f_{k}$ is a pair $\left(\left[k_{1}, k_{2}\right], j\right)$ where $\left[k_{1}, k_{2}\right] \subseteq[1, k]$ and $j \in[1, l]$. If $k_{1}=k_{2},\left(\left[k_{1}, k_{2}\right], j\right)$ is also called a point query. The value of a query $\left(\left[k_{1}, k_{2}\right], j\right)$ is $\max _{k_{1} \leq i \leq k_{2}} f_{i}(j)$. A query ( $\left[k_{1}, k_{2}\right], j$ ) is regular if $\left[k_{1}, k_{2}\right]$ is a regular integer interval.

For example, let

$$
\begin{aligned}
& f_{1}=5,4,4,3,2 \\
& f_{2}=8,7,4,2,0 \\
& f_{3}=9,9,5,0,0
\end{aligned}
$$

Then, $f_{1}, f_{2}$ and $f_{3}$ are normal sequences of length 5 . Here, $k=3$ and $l=5$. Thus, $([1,3], 2)$ is an interval query; its value is $\max \left\{f_{1}(2), f_{2}(2), f_{3}(2)\right\}=9$. The pair $([1,1], 3)$ is a point query; its value is $f_{1}(3)=4$. The pair $([1,2], 2)$ is a regular query; its values is $\max \left\{f_{1}(2), f_{2}(2)\right\}=7$.

The joint of $f_{1}, \cdots, f_{k}$ is the normal sequence $\hat{f}$ also of length $l$ such that $\hat{f}(j)=$ $\max \left\{f_{1}(j), \cdots, f_{k}(j)\right\}$.

Continuing the above example, the joint of $f_{1}, f_{2}, f_{3}$ is

$$
\hat{f}=9,9,5,3,2
$$

The minimal condensed form of a normal sequence $\{f(j)\}_{j=1}^{l}$ is the set of all pairs $(j, f(j))$ where $f(j) \neq 0$ and $j$ is the largest index of any $f\left(j^{\prime}\right)$ with $f\left(j^{\prime}\right)=f(j)$. A condensed form is a set of pairs $(j, f(j))$ that includes the minimal condensed form. The size of a condensed form is the number of pairs in it. The total size of a collection of condensed forms is the sum of the sizes of those forms.

Continuing the above example, the minimal condensed form of $f_{3}$ is $\{(2,9),(3,5)\}$; its size is 2 . The set $\{(1,9),(2,9),(3,5),(5,0)\}$ is a condensed form of $f_{3}$; its size is 4 . The total size of these two forms is 6 .

LEMMA 5.1. Let $F_{1}, \cdots, F_{k}$ be sets of nontrivial normal sequences of length $l$. Let $\hat{f}_{i}$ be the joint of the sequences in $F_{i}$. Given a condensed form of each sequence in each $F_{i}$, we can compute the minimal condensed forms of all $\hat{f}_{i}$ in $O(l+s)$ time where $s$ is the total size of the input forms.

Proof. The desired minimal forms can be computed by the two steps below:

1. Sort the pairs in the given condensed forms for $F_{i}$ into a sequence in the increasing order of the first components of these pairs.
2. Go through this sequence to delete all unnecessary pairs to obtain the minimal condensed form of $\hat{f}_{i}$.
We can use radix sort to implement Step 1 in $O(l+s)$ time for all $F_{i}$. Step 2 can be easily implemented in $O(s)$ time for all $F_{i}$.

Lemma 5.2. Let $f_{1}, \cdots, f_{k}$ be nontrivial normal sequences of length $l$. Assume that the input consists of a condensed form of each $f_{i}$ with a total size of $s$.

1. We can evaluate $m$ point queries in $O(m+l+s)$ time.
2. We can evaluate $m_{1}$ regular queries and $m_{2}$ irregular queries in a total of $O\left(m_{1}+\left(m_{2}+l+s\right) \log (k+1)\right)$ time.
Proof. The proof of Statement 1 uses radix sort in an obvious manner. To prove Statement 2, we assume without loss of generality that $k$ is a power of two. The input queries can be evaluated by the following three stages within the desired time bound.

Stage 1. For each regular interval $\left[k_{1}, k_{2}\right] \subseteq[1, k]$, let $f\left[k_{1}, k_{2}\right]$ be the joint of $f_{k_{1}}, \cdots, f_{k_{2}}$. We use Lemma $5.1 O(\log (k+1))$ times to compute the minimal condensed forms of all $f\left[k_{1}, k_{2}\right]$. The total size of these forms is $O(s \log (k+1))$. This stage takes $O((l+s) \log (k+1))$ time.

Stage 2. For each irregular input query $\left(\left[i_{1}, i_{2}\right], j\right)$, we partition $\left[i_{1}, i_{2}\right]$ into $O(\log (k+1))$ regular subintervals $\left[h_{1}, h_{2}\right],\left[h_{2}+1, h_{3}\right], \cdots,\left[h_{r-1}+1, h_{r}\right]$. Then, the value of $\left(\left[i_{1}, i_{2}\right], j\right)$ is the maximum of those of $\left(\left[h_{1}, h_{2}\right], j\right), \cdots,\left(\left[h_{r-1}+1, h_{r}\right], j\right)$. These regular queries are point queries for $f\left[h_{1}, h_{2}\right], \cdots, f\left[h_{r-1}+1, h_{r}\right]$. Together with the given $m_{1}$ regular queries, we have now generated $O\left(m_{1}+m_{2} \log (k+1)\right)$ point queries for all $f\left[k_{1} \cdot k_{2}\right]$. This stage takes $O\left(m_{1}+m_{2} \log (k+1)\right)$ time.

Stage 3. We use Statement 1 and the minimal condensed forms of $f\left[k_{1} \cdot k_{2}\right]$ to evaluate the points queries generated at Stage 2. Once the values of these point queries are obtained, we can easily compute the values of the input queries. This stage takes $O\left(m_{1}+m_{2} \log (k+1)+l+s \log (k+1)\right)$ time. $\quad \square$
5.2. Normalizing the input. To solve Problem 1, we first augment its input $T_{1}, T_{2}, P$ and $Q$ in order to simplify our discussion. Let $P=x_{1}, \cdots, x_{p}$ and $Q=$ $y_{1}, \cdots, y_{q}$. Without loss of generality, we assume that $p \geq q$.

1. Let $\alpha$ and $\beta$ be the smallest positive integers such that $p^{\prime}=2^{\alpha}+1, q^{\prime}=2^{\beta}+1$, $p^{\prime} \geq q^{\prime}, p^{\prime}>p$ and $q^{\prime}>q$. (Note. The conditions $p^{\prime}>p$ and $q^{\prime}>q$ are employed for technical simplicity. They can be changed to $p^{\prime} \geq p$ and $q^{\prime} \geq q$ with some modification on Algorithm One-One.)
2. Attach to $x_{p}$ the path $x_{p+1}, \cdots, x_{p^{\prime}}$ and to $y_{q}$ the path $y_{q+1}, \cdots, y_{q^{\prime}}$.
3. Let $P^{\prime}=x_{1}, \cdots, x_{p^{\prime}}$ and $Q^{\prime}=y_{1}, \cdots, y_{q^{\prime}}$.
4. Attach a leaf to each of $x_{p+1}, \cdots, x_{p^{\prime}-1}$ and $y_{q+1}, \cdots, y_{q^{\prime}-1}$, two leaves to $x_{p^{\prime}}$, and two leaves to $y_{q^{\prime}}$.
5. Assign distinct labels to the new leaves which also differ from the existing labels of $T_{1}$ and $T_{2}$.
6. Let $S_{1}$ be $T_{1}$ together with $P^{\prime}$ and the new leaves of $P^{\prime}$. Let $S_{2}$ be $T_{2}$ together with $Q^{\prime}$ and the new leaves of $Q^{\prime}$.
$S_{1}$ and $S_{2}$ are evolutionary trees. $P^{\prime}$ and $Q^{\prime}$ contain no leaves from $S_{1}$ and $S_{2}$, and are root paths of these trees. Let $n^{\prime}=\max \left\{n_{1}, n_{2}\right\}$ where $n_{i}$ is the number of leaves in $S_{i}$. Let $d^{\prime}$ be the maximum degree in $S_{1}$ and $S_{2}$.

Lemma 5.3 .

- $n^{\prime}=O(n), p^{\prime}=O(p), q^{\prime}=O(q)$, and $d^{\prime} \leq d+1$.
- $\operatorname{RP}\left(T_{1}, T_{2}, Q\right)=\operatorname{RP}\left(S_{1}, S_{2}, Q^{\prime}\right)$ and $\operatorname{RP}\left(T_{2}, T_{1}, P\right)=\operatorname{RP}\left(S_{2}, S_{1}, P^{\prime}\right)$.

Proof. Straightforward. $\square$
In light of Lemma 5.3, our discussion below mainly works with $S_{1}, S_{2}, P^{\prime}$ and $Q^{\prime}$. Let $G=G_{P} \cup G_{Q}$ where $G_{P}$ is the set of all pairs $\left(x_{i}, y_{1}\right)$ and $G_{Q}$ is the set of all $\left(x_{1}, y_{j}\right)$. To solve Problem 1, a main task is to evaluate $\operatorname{RR}\left(S_{1}^{x}, S_{2}^{y}\right)$ for $(x, y) \in G$. The output RP values that are excluded here can be retrieved directly from the input RP mappings.
5.3. Predecessors. A pair $\left(x_{i^{\prime}}, y_{j^{\prime}}\right)$ is a predecessor of a distinct $\left(x_{i}, y_{j}\right)$ if $i \leq i^{\prime}$ and $j \leq j^{\prime}$. One-One proceeds by recursively reducing the problem of computing
$\operatorname{RR}\left(S_{1}^{x}, S_{2}^{y}\right)$ to that of computing the RR values of the $P$-predecessor, $Q$-predecessor and $P Q$-predecessor defined below.

Let $P\left[i, i^{\prime}\right]$ be the path $x_{i}, \cdots, x_{i^{\prime}}$, where $i \leq i^{\prime}$. Let $X_{i}$ be the set of the children of $x_{i}$ in $S_{1}$ that are not in $P^{\prime}$. We similarly define $Q\left[j, j^{\prime}\right]$ and $Y_{j}$. A pair $\left(x_{i}, y_{j}\right)$ is intersecting if $S_{1}^{u}$ and $S_{2}^{v}$ have at least one common leaf label for some $u \in X_{i}$ and $v \in Y_{j} .\left(P\left[i, i^{\prime}\right], Q\left[y_{j}, y_{j^{\prime}}\right]\right)$ is intersecting if some $x_{i^{\prime \prime}} \in P\left[i, i^{\prime}\right]$ and $y_{i^{\prime \prime}} \in Q\left[j, j^{\prime}\right]$ form an intersecting pair.

The lengths of $P\left[i, i^{\prime}\right]$ and $Q\left[j, j^{\prime}\right]$ are those of $\left[i, i^{\prime}\right]$ and $\left[j, j^{\prime}\right]$, respectively. A path $P\left[i, i^{\prime}\right]$ is regular if $\left[i, i^{\prime}\right]$ is a regular interval. A regular $Q\left[j, j^{\prime}\right]$ is similarly defined. We now construct a tree $\Psi$ over pairs of regular paths; this tree is slightly different from that of 15]. The root of $\Psi$ is $\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)$. A pair $\left(P\left[i, i^{\prime}\right], Q\left[j, j^{\prime}\right]\right) \in \Psi$ is a leaf if and only if either (1) $i=i^{\prime}, j=j^{\prime}$ and $\left(x_{i}, y_{j}\right)$ is intersecting, or (2) this pair is nonintersecting. For a nonleaf $\left(P\left[i, i^{\prime}\right], Q\left[j, j^{\prime}\right]\right) \in$ $\Psi$, if $j=j^{\prime}$, then its children are $\left(P\left[i, \frac{i+i^{\prime}-1}{2}\right], y_{j}\right)$ and $\left(P\left[\frac{i+i^{\prime}+1}{2}, i^{\prime}\right], y_{j}\right)$. Otherwise, this pair has four children $\left(P\left[i, \frac{i+i^{\prime}-1}{2}\right], Q\left[j, \frac{j+j^{\prime}-1}{2}\right]\right),\left(P\left[i, \frac{i+i^{\prime}-1}{2}\right], Q\left[\frac{j+j^{\prime}+1}{2}, j^{\prime}\right]\right)$, $\left(P\left[\frac{i+i^{\prime}+1}{2}, i^{\prime}\right], Q\left[j, \frac{j+j^{\prime}-1}{2}\right]\right),\left(P\left[\frac{i+i^{\prime}+1}{2}, i^{\prime}\right], Q\left[\frac{j+j^{\prime}+1}{2}, j^{\prime}\right]\right)$.

The ceiling of $\left(P\left[i, i^{\prime}\right], Q\left[j, j^{\prime}\right]\right)$ is $\left(x_{i}, y_{j}\right)$; its floor is $\left(x_{i^{\prime}+1}, y_{j^{\prime}+1}\right)$ 15. Its $P_{-}$ diagonal is $\left(x_{i^{\prime}+1}, y_{j}\right)$; its $Q$-diagonal is $\left(x_{i}, y_{j^{\prime}+1}\right)$. Let $E$ be the set of all ceilings, diagonals, floors of the leaves of $\Psi$. Let $B=\left\{\left(x_{i}, y_{q^{\prime}}\right) \mid i \in\left[1, p^{\prime}\right]\right\} \cup\left\{\left(x_{p^{\prime}}, y_{j}\right) \mid j \in\right.$ $\left.\left[1, q^{\prime}\right]\right\}$. Due to its recursive nature, One-One evaluates $\operatorname{RR}\left(S_{1}^{x}, S_{2}^{y}\right)$ for all $(x, y) \in$ $G \cup E \cup B$.

Given $\left(x_{i}, y_{j}\right)$, if $\left(x_{i+1}, y_{i+1}\right) \in G \cup E \cup B$, then this pair is the $P Q$-predecessor of $\left(x_{i}, y_{j}\right)$. Let $i^{\prime}$ be the smallest index that is larger than $i$ such that $\left(x_{i^{\prime}}, y_{j}\right) \in G \cup E \cup B$. This $\left(x_{i^{\prime}}, y_{j}\right)$ is the $P$-predecessor of $\left(x_{i}, y_{j}\right)$. Let $j^{\prime}$ be the smallest index larger than $j$ such that $\left(x_{i}, y_{j^{\prime}}\right) \in G \cup E \cup B$. This $\left(x_{i}, y_{j^{\prime}}\right)$ is the $Q$-predecessor of $\left(x_{i}, y_{j}\right)$.

Lemma 5.4.

1. Each intersecting $\left(x_{i}, y_{j}\right) \in(G \cup E)-B$ has a P-predecessor $\left(x_{i+1}, y_{j}\right)$, a $Q$-predecessor $\left(x_{i}, y_{j+1}\right)$ and a $P Q$-predecessor $\left(x_{i+1}, y_{j+1}\right)$.
2. Each nonintersecting $\left(x_{i}, y_{j}\right) \in E-B$ has a $P$-predecessor $\left(x_{i^{\prime}}, y_{j}\right)$ and a $Q$-predecessor $\left(x_{i}, y_{j^{\prime}}\right)$. Also, $\left(P\left[i, i^{\prime}-1\right], Q\left[j, j^{\prime}-1\right]\right)$ is nonintersecting.
3. Each nonintersecting $\left(x_{i}, y_{1}\right) \in G_{P}-B$ has a $P$-predecessor $\left(x_{i+1}, y_{1}\right)$ and a $Q$-predecessor $\left(x_{i}, y_{j}\right)$. Moreover, $\left(x_{i}, Q[1, j-1]\right)$ is nonintersecting.
4. Each nonintersecting $\left(x_{1}, y_{j}\right) \in G_{Q}-B$ has a $P$-predecessor $\left(x_{i}, y_{j}\right)$ and a $Q$-predecessor $\left(x_{1}, y_{j+1}\right)$. Moreover, $\left(P[1, i-1], y_{j}\right)$ is nonintersecting.
Proof. Statement 1 follows from the definitions of $\Psi$ and $E$. The proofs of Statements 3 and 4 are similar to Case 3 in the proof of Statement 2 below.

As for Statement 2, by the definition of $B, x_{i^{\prime}}$ and $y_{j^{\prime}}$ exist. To show $\left(P\left[i, i^{\prime}-\right.\right.$ $\left.1], Q\left[j, j^{\prime}-1\right]\right)$ is nonintersecting, we consider the following four cases. The proofs of their symmetric cases are similar to theirs and are omitted for brevity.

Case 1: $\left(x_{i}, y_{j}\right)$ is the ceiling of a nonintersecting leaf $\left(P\left[i, i_{2}\right], Q\left[j, j_{2}\right]\right) \in \Psi$. Since $\left(x_{i}, y_{j_{2}+1}\right)$ and $\left(x_{i_{2}+1}, y_{j}\right)$ are in $E, i^{\prime} \leq i_{2}+1$ and $j^{\prime} \leq j_{2}+1$. Then because $\left(P\left[i, i_{2}\right], Q\left[j, j_{2}\right]\right)$ is nonintersecting, so is $\left(P\left[i, i^{\prime}-1\right], Q\left[j, j^{\prime}-1\right]\right)$.

Case 2: $\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal of a nonintersecting leaf $\left(P\left[i, i_{2}\right], Q\left[j_{1}, j-1\right]\right)$ (or symmetrically, $\left(x_{i}, y_{j}\right)$ is the $P$-diagonal of a nonintersecting leaf $\left.\left(P\left[i_{1}, i-1\right], Q\left[j, j_{2}\right]\right)\right)$. Since $\left(x_{i_{2}+1}, y_{j}\right)$ is the floor of $\left(P\left[i, i_{2}\right], Q\left[j_{1}, j-1\right]\right),\left(x_{i_{2}+1}, y_{j}\right) \in E$ and thus $i^{\prime} \leq i_{2}+1$. Let $j^{\prime \prime}$ be the smallest index such that $j \leq j^{\prime \prime}$ and $\left(P\left[i, i_{2}\right], y_{j^{\prime \prime}}\right)$ is intersecting. There are two subcases.

Case 2a: $j^{\prime \prime}$ does not exist. Then, $\left(P\left[i, i_{2}\right], Q\left[j, q^{\prime}\right]\right)$ is nonintersecting and therefore $\left(P\left[i, i^{\prime}-1\right], Q\left[j, j^{\prime}-1\right]\right)$ is nonintersecting.

Case 2b. $j^{\prime \prime}$ exists. Let $Q\left[j_{3}, j_{4}\right]$ be a regular path that contains $y_{j^{\prime \prime}}$ and is of the same length as $Q\left[j_{1}, j-1\right]$. Note that $j \leq j_{3}$ and $\left(P\left[i, i_{2}\right], Q\left[j_{3}, j_{4}\right]\right) \in \Psi$. There are two subcases.

Case $2 \mathrm{~b}(1): j_{3}=j$. Then $\left(x_{i}, y_{j}\right)$ is the ceiling of $\left(P\left[i, i_{2}\right], Q\left[j_{3}, j_{4}\right]\right)$. Since $\left(x_{i}, y_{j}\right)$ is nonintersecting, it is the ceiling of a nonintersecting leaf in $\Psi$ which is a descendant of $\left(P\left[i, i_{2}\right], Q\left[j_{3}, j_{4}\right]\right)$. Therefore, Case $2 \mathrm{~b}(1)$ is reduced to Case 1 .

Case $2 \mathrm{~b}(2): j_{3}>j$. By the construction of $\Psi,\left(x_{i}, y_{j_{3}}\right) \in E$ and thus $j^{\prime} \leq j_{3}$. By the choice of $Q\left[j_{3}, j_{4}\right],\left(P\left[i, i_{2}\right], Q\left[j, j_{3}-1\right]\right)$ is nonintersecting and so is ( $P\left[i, i^{\prime}-\right.$ 1], $Q\left[j, j^{\prime}-1\right]$ ).

Case 3: $\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal of an intersecting leaf $\left(x_{i}, y_{j-1}\right)$ (or symmetrically, $\left(x_{i}, y_{j}\right)$ is the $P$-diagonal of an intersecting leaf $\left.\left(x_{i-1}, y_{j}\right)\right)$. Since $\left(x_{i+1}, y_{j}\right) \in E$, $i^{\prime}=i+1$ and $P\left[i, i^{\prime}-1\right]=x_{i}$. Let $j^{\prime \prime}$ be the smallest index such that $j<j^{\prime \prime}$ and $\left(x_{i}, y_{j^{\prime \prime}}\right)$ is intersecting. There are two subcases.

Case 3a: $j^{\prime \prime}$ does not exist. Then, $\left(x_{i}, Q\left[j, q^{\prime}\right]\right)$ is nonintersecting and therefore $\left(P\left[i, i^{\prime}-1\right], Q\left[j, j^{\prime}-1\right]\right)$ is nonintersecting.

Case 3b: $j^{\prime \prime}$ exists. Then, $\left(x_{i}, y_{j^{\prime \prime}}\right) \in E$ and $j^{\prime} \leq j^{\prime \prime}$. By the choice of $j^{\prime \prime}$, $\left(x_{i}, Q\left[j, j^{\prime \prime}-1\right]\right)$ is nonintersecting. Thus, $\left(P\left[i, i^{\prime}-1\right], Q\left[j, j^{\prime}-1\right]\right)$ is nonintersecting.

Case 4: $\left(x_{i}, y_{j}\right)$ is the floor of a leaf $\left(P\left[i_{1}, i-1\right], Q\left[j_{1}, j-1\right]\right)$, which may or may not be intersecting. Let $\left(P\left[i_{3}, i_{4}\right], Q\left[j_{3}, j_{4}\right]\right)$ be the lowest ancestor of $\left(P\left[i_{1}, i-1\right], Q\left[j_{1}, j-\right.\right.$ $1])$ in $\Psi$ such that $\left(x_{i}, y_{j}\right)$ is not the floor of $\left(P\left[i_{3}, i_{4}\right], Q\left[j_{3}, j_{4}\right]\right)$. This ancestor exists because $\left(x_{i}, y_{j}\right) \notin B$. There are two subcases.

Case 4a: $j_{3}=j_{4}$ and $i_{3}<i_{4}$. Then, $P\left[i_{1}, i-1\right]$ is a subpath of $P\left[i_{3}, \frac{i_{3}+i_{4}-1}{2}\right]$ and $i=\frac{i_{3}+i_{4}+1}{2}$. Also, $j_{3}=j_{1}=j-1$. Thus, $\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal of $\left(P\left[i, i_{4}\right], y_{j-1}\right) \in \Psi$. By the construction of $\Psi,\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal of a leaf which is either $\left(P\left[i, i_{4}\right], y_{j-1}\right)$ itself or its descendant. Depending on whether this leaf is nonintersecting or intersecting, Case 4 a is reduced to Case 2 or 3.

Case 4b: $j_{3}<j_{4}$ and $i_{3}<i_{4}$. There are two subcases.
Case $4 \mathrm{~b}(1): P\left[i_{1}, i-1\right] \subset P\left[i_{3}, \frac{i_{3}+i_{4}-1}{2}\right]$ and $Q\left[j_{1}, j-1\right] \subset Q\left[j_{3}, \frac{j_{3}+j_{4}-1}{2}\right]$. Note that $i=\frac{i_{3}+i_{4}+1}{2}, j=\frac{j_{3}+j_{4}+1}{2}$, and $\left(x_{i}, y_{j}\right)$ is the ceiling of $\left(P\left[\frac{i_{3}+i_{4}+1}{2}, i_{4}\right], Q\left[\frac{j_{3}+j_{4}+1}{2}, j_{4}\right]\right) \in$ $\Psi$. Since $\left(x_{i}, y_{j}\right)$ is nonintersecting, $\left(x_{i}, y_{j}\right)$ is the ceiling of a nonintersecting leaf in $\Psi$ which is $\left(P\left[\frac{i_{3}+i_{4}+1}{2}, i_{4}\right], Q\left[\frac{j_{3}+j_{4}+1}{2}, j_{4}\right]\right)$ itself or a descendant. This reduces Case $4 \mathrm{~b}(1)$ to Case 1.

Case $4 \mathrm{~b}(2): P\left[i_{1}, i-1\right] \subset P\left[i_{3}, \frac{i_{3}+i_{4}-1}{2}\right]$ and $Q\left[j_{1}, j-1\right] \subset Q\left[\frac{j_{3}+j_{4}+1}{2}, j_{4}\right]$ (or symmetrically, $P\left[i_{1}, i-1\right] \subset P\left[\frac{i_{3}+i_{4}+1}{2}, i_{4}\right]$ and $\left.Q\left[j_{1}, j-1\right] \subset Q\left[j_{3}, \frac{j_{3}+j_{4}-1}{2}\right]\right)$. Note that $i=$ $\frac{i_{3}+i_{4}+1}{2}, j=j_{4}+1$, and $\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal of $\left(P\left[\frac{i_{3}+i_{4}+1}{2}, i_{4}\right], Q\left[\frac{j_{3}+j_{4}+1}{2}, j_{4}\right]\right) \in$ $\Psi$. Then, $\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal of a leaf which is $\left(P\left[\frac{i_{3}+i_{4}+1}{2}, i_{4}\right], Q\left[\frac{j_{3}+j_{4}+1}{2}, j_{4}\right]\right)$ itself or a descendant. Depending on whether this leaf is nonintersecting or intersecting, Case $4 b(2)$ is reduced to Case 2 or 3 .
5.4. Counting lemmas. We now give some counting lemmas that are used in $\oint 5.6$ to bound One-One's time complexity.

For all $\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right) \in \Psi$,

- $C\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)$ denotes the set of all ceilings of the leaves in $\Psi$ which are either $\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)$ itself or its descendants;
- $D\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)$ denotes the set of all $Q$-diagonals of the leaves in $\Psi$ which are either $\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)$ itself or its descendants;
- $I\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)=\left\{\left(x_{i}, y_{j}\right) \mid x_{i} \in P\left[i_{1}, i_{2}\right], y_{j} \in Q\left[j_{1}, j_{2}\right]\right.$ and $\left(x_{i}, y_{j}\right)$ is intersecting\}.
Lemma 5.5.

1. $\left|I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)\right| \leq n$.
2. $\Psi$ has $O(n \log (q+1))$ leaves of the form $\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)$ where $j_{1}<j_{2}$.
3. $\Psi$ has $O(n \log (q+1))$ pairs of the form $\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$ where $P\left[i_{1}, i_{2}\right]$ is of length $\frac{p^{\prime}-1}{q^{\prime}-1}$.
4. $|E|=O(n \log (p+1))$.

Proof. Statements 1-3 are proved below. The proof of Statment 4 is similar to those of Statements 2 and 3.

Statement 1. For all distinct intersecting pairs $\left(x_{i}, y_{j}\right)$ and $\left(x_{i^{\prime}}, y_{j^{\prime}}\right)$, the leaf labels shared by the subtrees $T_{1}^{u}$ where $u \in X_{i}$ and the subtrees $T_{2}^{v}$ where $v \in Y_{i}$ are different from the shared labels for $X_{i^{\prime}}$ and $Y_{j^{\prime}}$. Statement 1 then follows from the fact that $S_{1}$ and $S_{2}$ share $n$ leaf labels.

Statements 2 and 3. On each level of $\Psi$, for all distinct pairs ( $\left.P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right)$ and $\left(P\left[i_{1}^{\prime}, i_{2}^{\prime}\right], Q\left[j_{1}^{\prime}, j_{2}^{\prime}\right]\right), I\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right) \cap I\left(P\left[i_{1}^{\prime}, i_{2}^{\prime}\right], Q\left[j_{1}^{\prime}, j_{2}^{\prime}\right]\right)=\emptyset$. Thus, each level has at most $\left|I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)\right|$ nonleaf pairs. Consequently, from the second level downwards, each level has at most $4 \cdot\left|I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)\right|$ pairs. These two statements then follows from Statement 1 and the fact that the pairs specified in these two statements are within the top $1+\log \left(q^{\prime}-1\right)$ levels of $\Psi$.

A pair $\left(x_{i}, y_{j}\right)$ is $P$-regular if $\left[i, i^{\prime}-1\right]$ is a regular interval where $\left(x_{i^{\prime}}, y_{j}\right)$ is the $P$-predecessor of $\left(x_{i}, y_{j}\right)$. (We do not need the notion of $Q$-regular because $p^{\prime} \geq q^{\prime}$.)

Given a regular $\left[i_{1}, i_{2}\right]$, a set $\left\{h_{1}, \cdots, h_{k}\right\}$ regularly partitions $\left[i_{1}, i_{2}\right]$ if $h_{1}=\overline{i_{1}}$ and the intervals $\left[h_{1}, h_{2}-1\right],\left[h_{2}, h_{3}-1\right], \cdots,\left[h_{k-1}, h_{k}-1\right],\left[h_{k}, i_{2}\right]$ are all regular.

Lemma 5.6.

1. Assume that $j>1$ and $P\left(\left[i_{1}, i_{2}\right], y_{j}\right) \in \Psi$. If the $P$-predecessor $\left(x_{i}, y_{j}\right)$ of some $\left(x_{i^{\prime}}, y_{j}\right) \in C\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$ is not in $\left\{\left(x_{i_{2}+1}, y_{j}\right)\right\} \cup C\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$, then $P\left(\left[i_{1}, i_{2}\right], y_{j-1}\right) \in \Psi$ and $\left(x_{i}, y_{j}\right) \in D\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$.
2. Assume that $j<q^{\prime}$ and $P\left(\left[i_{1}, i_{2}\right], y_{j-1}\right) \in \Psi$. If the $P$-predecessor $\left(x_{i}, y_{j}\right)$ of some $\left(x_{i^{\prime}}, y_{j}\right) \in D\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$ is not in $\left\{\left(x_{i_{2}+1}, y_{j}\right) \cup D\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)\right.$, then $P\left(\left[i_{1}, i_{2}\right], y_{j}\right) \in \Psi$ and $\left(x_{i}, y_{j}\right) \in C\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$.
3. For every $\left(P\left[i_{1}, i_{2}\right], y_{j}\right) \in \Psi$, the set $\left\{i \mid\left(x_{i}, y_{j}\right) \in C\left(P\left[i_{1}, i_{2}\right], y_{j}\right)\right\}$ regularly partitions $\left[i_{1}, i_{2}\right]$ and so does the set $\left\{i \mid\left(x_{i}, y_{j}\right) \in D\left(P\left[i_{1}, i_{2}\right], y_{j}\right)\right\}$.
4. For all $\left(P\left[i_{1}, i_{2}\right], y_{j}\right) \in \Psi$, every pair in $C\left(P\left[i_{1}, i_{2}\right], y_{j}\right) \cup D\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$ is $P$-regular.
5. At most $O(n \log (q+1))$ of the nonintersecting pairs of $E$ are $P$-irregular.

Proof. The proofs of Statements 1 and 5 are detailed below. The proof of Statement 2 is similar to that of Statement 1 and is omitted. Statement 3 is obvious. Statement 4 follows from the first three statements and the fact that if two sets regularly partition $\left[i_{1}, i_{2}\right]$, then so does their union.

Statement 1. Note that $i_{1}<i \leq i_{2}$ and $q^{\prime}>j>1$. The pair $\left(x_{i}, y_{j}\right)$ can be the ceiling, the $P$-diagonal, the $Q$-diagonal, or the floor of some leaf $\left(P\left[i_{3}, i_{4}\right], Q\left[j_{3}, j_{4}\right]\right) \in$ $\Psi$. These four cases are discussed below.

Case 1: $\left(x_{i}, y_{j}\right)$ is the ceiling. Then $i=i_{3}$ and $j=j_{3}$. Since $i_{1}<i \leq i_{2}$ and both $\left[i, i_{4}\right]$ and $\left[i_{1}, i_{2}\right]$ are regular, $\left[i, i_{4}\right] \subset\left[i_{1}, i_{2}\right]$. Since the length of $P\left[i_{1}, i_{2}\right]$ is at most $\frac{p^{\prime}-1}{q^{\prime}-1}$, so is the length of $P\left[i, i_{4}\right]$. Thus $Q\left[j_{3}, j_{4}\right]=y_{j}$ and $\left(P\left[i, i_{4}\right], y_{j}\right)$ is a descendant of $\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$. This contradicts the assumption that $\left(x_{i}, y_{j}\right) \notin C\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$ and this case cannot exist.

Case 2: $\left(x_{i}, y_{j}\right)$ is the $P$-diagonal. Then $i=i_{4}+1$ and $j=j_{3}$. As in Case 1, $Q\left[j_{3}, j_{4}\right]=y_{j}$ and $\left(P\left[i_{3}, i-1\right], y_{j}\right)$ is a descendant of $\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$. Thus, there exists a leaf $\left(P\left[i, i_{6}\right], y_{j}\right)$ that is a descendant of $\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$. Because $\left(x_{i}, y_{j}\right)$ is the ceiling of this leaf, the existence of this leaf contradicts the assumption that
$\left(x_{i}, y_{j}\right) \notin C\left(P\left[i_{1}, i_{2}\right], y_{j}\right)$ and this case cannot exist.
Case 3: $\left(x_{i}, y_{j}\right)$ is the $Q$-diagonal. Then, $i=i_{3}$ and $j=j_{4}+1$. As in Case 1, $\left[i, i_{4}\right] \subset\left[i_{1}, i_{2}\right]$ and $Q\left[j_{3}, j_{4}\right]=y_{j-1}$. Since $\left(P\left[i, i_{4}\right], y_{j-1}\right) \in \Psi,\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right) \in \Psi$. Then $\left(P\left[i, i_{4}\right], y_{j-1}\right)$ is a descendant of $\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$ and $\left(x_{i}, y_{j}\right) \in D\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$.

Case 4: $\left(x_{i}, y_{j}\right)$ is the floor. Then, $i=i_{4}+1$ and $j=j_{4}+1$. As in Case 3, $\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right) \in \Psi, Q\left[j_{3}, j-1\right]=y_{j-1}$ and $\left(P\left[i_{3}, i-1\right], y_{j-1}\right)$ is a descendant of $\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$. Thus, there is a leaf $\left(P\left[i, i_{6}\right], y_{j-1}\right)$ which is a descendant of $\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$. Since $\left(x_{i}, y_{j}\right)$ is this leaf's $Q$-diagonal, it is in $D\left(P\left[i_{1}, i_{2}\right], y_{j-1}\right)$.

Statement 5 . Note that $E$ consists of the following three types of pairs:

1. the ceiling, diagonals and floor of a leaf $\left(P\left[i_{1}, i_{2}\right], Q\left[j_{1}, j_{2}\right]\right) \in \Psi$ where $j_{1}<j_{2}$.
2. the $P$-diagonal and floor of $\left.\left(P\left[i_{1}, i_{2}\right], y_{j}\right]\right) \in \Psi$ where $P\left[i_{1}, i_{2}\right]$ is of length $\frac{p^{\prime}-1}{q^{\prime}-1}$.
3. the pairs in $\left.\left.C\left(P\left[i_{1}, i_{2}\right], j\right]\right) \cup D\left(P\left[i_{1}, i_{2}\right], y_{j}\right]\right)$ where $\left.\left(P\left[i_{1}, i_{2}\right], j\right]\right) \in \Psi$ and $P\left[i_{1}, i_{2}\right]$ is of length $\frac{p^{\prime}-1}{q^{\prime}-1}$.
By Statement 4, only the pairs of the first two types may be $P$-irregular. This statement then follows from Lemmas 5.5(2) and 5.5(3).
5.5. Recurrences. One-One uses the following formulas to recursively compute $\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j}}\right)$ for $\left(x_{i}, y_{j}\right) \in G \cup E \cup B$ in terms of the RR values of the appropriate $P$-predecessor, $Q$-predecessor and $P Q$-predecessor of $\left(x_{i}, y_{j}\right)$.

For vertex subsets $U$ of $S_{1}$ and $V$ of $S_{2}, \mathrm{~m}(U, V)$ denotes the maximum weight of any matching of the bipartite graph $(U, V, U \times V)$ where the weight of an edge $(u, v)$ is $\operatorname{Rr}\left(S_{1}^{u}, S_{2}^{v}\right)$. Let $\mathrm{m}(U, v)=\mathrm{m}(U,\{v\})$ and $\mathrm{m}(u, V)=\mathrm{m}(\{u\}, V)$. Given two vertices $x \in S_{1}$ and $y \in S_{2}$, let $\overline{\mathrm{M}}(U, V, x, y)$ be the maximum weight of any matching of the same graph without the edge $(x, y)$.

Lemma 5.7. For each $\left(x_{i}, y_{j}\right) \in B, \operatorname{RR}\left(S_{1}^{x}, S_{2}^{y}\right)=0$.
Proof. This lemma follows from the fact that $p^{\prime}>p, q>q$ and the new labels of $S_{1}$ and $S_{2}$ are different from one another and the labels of $T_{1}$ and $T_{2}$.

FACt 2 (47]). For all vertices $u \in S_{1}$ and $v \in S_{2}$,

$$
\operatorname{RR}\left(S_{1}^{u}, S_{2}^{v}\right)=\max \left\{\begin{array}{l}
\mathrm{m}\left(K\left(u, S_{1}\right), K\left(v, S_{2}\right)\right), \\
\mathrm{m}\left(u, K\left(v, S_{2}\right)\right), \\
\mathrm{m}\left(K\left(u, S_{1}\right), v\right)
\end{array}\right\} .
$$

Proof. To form maximum agreement subtrees of $S_{1}^{u}$ and $S_{2}^{v}$, there are three cases. (1) $\mathrm{m}\left(K\left(u, S_{1}\right), K\left(v, S_{2}\right)\right)$ accounts for matching $u$ to $v$. (2) $\mathrm{M}\left(u, K\left(v, S_{2}\right)\right)$ accounts for matching $u$ to a proper descendant of $v$. (3) $\mathrm{m}\left(K\left(u, S_{1}\right), v\right)$ accounts for matching $v$ to a proper descendant of $u$.

Lemma 5.8. For all $\left(x_{i}, y_{j}\right)$ where $i<p^{\prime}$ and $j<q^{\prime}$, regardless of whether $\left(x_{i}, y_{j}\right)$ is intersecting or nonintersecting,

$$
\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j}}\right)=\max \left\{\begin{array}{l}
\mathrm{M}\left(X_{i}, Y_{j}\right)+\operatorname{RR}\left(S_{1}^{x_{i+1}}, S_{2}^{y_{j+1}}\right), \\
\overline{\mathrm{M}}\left(X_{i} \cup\left\{x_{i+1}\right\}, Y_{j} \cup\left\{y_{j+1}\right\}, x_{i+1}, y_{j+1}\right), \\
\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j+1}}\right), \\
\mathrm{M}\left(x_{i}, Y_{j}\right), \\
\operatorname{RR}\left(S_{1}^{S_{i+1}}, S_{2}^{y_{j}}\right), \\
\mathrm{M}\left(X_{i}, y_{j}\right)
\end{array}\right\} .
$$

Proof. This lemma follows from Fact 2 with a finer case analysis for the cases in the proof of Fact 2 .

LEMMA 5.9. For each nonintersecting $\left(x_{i}, y_{j}\right) \in E-B$ with $P$-predecessor $\left(x_{i^{\prime}}, y_{j}\right)$ and $Q$-predecessor $\left(x_{i}, y_{j^{\prime}}\right)$,

$$
\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j}}\right)=\max \left\{\begin{array}{l}
\max _{j^{\prime \prime} \in\left[j, j^{\prime}-1\right]} \mathrm{M}\left(x_{i^{\prime}}, Y_{j^{\prime \prime}}\right)+\max _{i^{\prime \prime} \in\left[i, i^{\prime}-1\right]} \mathrm{M}\left(X_{i^{\prime \prime}}, y_{j^{\prime}}\right), \\
\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j^{\prime}}}\right), \\
\operatorname{RR}\left(S_{1}^{x_{i^{\prime}}}, S_{2}^{y_{j}}\right)
\end{array}\right\}
$$

Proof. This lemma follows from Lemma $5.4(2)$ and is obtained by iterative applications of Lemma 5.8. The following properties are used. Since $\left(P\left[i, i^{\prime}-1\right], Q\left[j, j^{\prime}-1\right]\right)$ is nonintersecting, for $i^{\prime \prime} \in\left[i, i^{\prime}-1\right]$ and $j^{\prime \prime} \in\left[j, j^{\prime}-1\right]$,

- $\mathrm{m}\left(X_{i^{\prime \prime}}, Y_{j^{\prime \prime}}\right)=0$;
- $\overline{\mathrm{m}}\left(X_{i^{\prime \prime}} \cup\left\{x_{i^{\prime \prime}+1}\right\}, Y_{j^{\prime \prime}} \cup\left\{y_{j^{\prime \prime}+1}\right\}, x_{i^{\prime \prime}+1}, y_{j^{\prime \prime}+1}\right)=\mathrm{m}\left(x_{i^{\prime \prime}}, Y_{j^{\prime \prime}}\right)+\mathrm{m}\left(X_{i^{\prime \prime}}, y_{j^{\prime \prime}}\right)$;
- $\mathrm{M}\left(x_{i^{\prime \prime}}, Y_{j^{\prime \prime}}\right)=\mathrm{m}\left(x_{i^{\prime}}, Y_{j^{\prime \prime}}\right)$;
- $\mathrm{M}\left(X_{i^{\prime \prime}}, y_{j^{\prime \prime}}\right)=\mathrm{m}\left(X_{i^{\prime \prime}}, y_{j^{\prime}}\right)$.

■
For brevity, the symmetric statement of the next lemma for $G_{Q}$ is omitted.
Lemma 5.10. For all nonintersecting pairs $\left(x_{i}, y_{1}\right) \in G_{P}-B$ with $Q$-predecessor $\left(x_{i}, y_{j}\right)$,

$$
\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{1}}\right)=\max \left\{\begin{array}{l}
\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j}}\right), \\
\operatorname{RR}\left(S_{1}^{x_{i}+1}, S_{2}^{y_{1}}\right), \\
\operatorname{M}\left(X_{i}, y_{j}\right)+\max _{j^{\prime} \in[1, j-1]} \mathrm{M}\left(x_{i+1}, Y_{j^{\prime}}\right)
\end{array}\right\}
$$

Proof. The proof is similar to that of Lemma 5.9 and follows from Lemma 5.4(3).
■
5.6. The algorithm for Problem 1. We combine the discussion in $5.3 \$ 5.5$ to give the following algorithm to solve Problem in.
Algorithm One-One; begin

1. Compute $S_{1}, S_{2}, P^{\prime}, Q^{\prime}, \operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)$ for $u \in K\left(P^{\prime}, S_{1}\right)$, and $\operatorname{RP}\left(S_{2}^{v}, S_{1}, P^{\prime}\right)$ $v \in K\left(Q^{\prime}, S_{2}\right) ;$
2. Compute $G \cup E \cup B, B, I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)-B$, the set of all nonintersecting pairs in $E-B$, and the sets of nonintersecting pairs in $G_{P}-B$ and $G_{Q}-B$, respectively;
3. Compute the following predecessors:

- the $P$-predecessor, $Q$-predecessor and $P Q$-predecessor of each pair in $I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)-B$;
- the $P$-predecessor and $Q$-predecessor of each nonintersecting pair in $E-$ $B$;
- the $Q$-predecessor of each nonintersecting pair in $G_{P}-B$ and the $P$ predecessor of each nonintersecting pair in $G_{Q}-B$;

4. For all pairs in $G \cup E \cup B$, compute the non-RR terms in the appropriate recurrence formulas of $\S 5.5$ :

- Lemma 5.7 for $B$;
- Lemma 5.8 for $\left(I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)-B\right.$;
- Lemma 5.9 for the nonintersecting pairs in $E-B$;
- Lemma 5.10 for the nonintersecting pairs in $G_{P}-B$ and its symmetric statement for the nonintersecting pairs in $G_{Q}-B$;

5. Compute the $\operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{j}}\right)$ for all $\left(x_{i}, y_{j}\right) \in G \cup E \cup B$ using the appropriate recurrence formulas given in $\$ 5.5$ and the non-RR terms computed at Step 4;
6. Compute the output as follows:

- For all $y_{j} \in Q, \operatorname{RP}\left(T_{1}, T_{2}, Q\right)\left(y_{j}\right) \leftarrow \operatorname{RR}\left(S_{1}^{x_{1}}, S_{2}^{y_{j}}\right)$;
- For all $x_{i} \in P, \operatorname{RP}\left(T_{2}, T_{1}, P\right)\left(x_{i}\right) \leftarrow \operatorname{RR}\left(S_{1}^{x_{i}}, S_{2}^{y_{1}}\right)$;
- For every $v \in K\left(Q, T_{2}\right), \operatorname{RP}\left(T_{1}, T_{2}, Q\right)(v) \leftarrow \operatorname{RP}\left(T_{2}^{v}, T_{1}, P\right)(h)$ where $h$ is the root of $T_{1} \| T_{2}^{v}$;
- For every $u \in K\left(P, T_{1}\right), \operatorname{RP}\left(T_{2}, T_{1}, P\right)(u) \leftarrow \operatorname{RP}\left(T_{1}^{u}, T_{2}, Q\right)(h)$ where $h$ is the root of $T_{2} \| T_{1}^{u}$;
end.
To analyze One-One, we first focus on Step 4. The recurrences of $\S 5.5$ contain only four types of non-RR terms other than the constant 0 in Lemma 5.7:

1. $\mathrm{m}\left(X_{i}, y_{j}\right)$ and $\mathrm{m}\left(x_{i}, Y_{j}\right)$;
2. $\max _{i \in\left[i_{1}, i_{2}\right]} \mathrm{M}\left(X_{i}, y_{j}\right)$ and $\max _{j \in\left[j_{1}, j_{2}\right]} \mathrm{M}\left(x_{i}, Y_{j}\right)$;
3. $\mathrm{m}\left(X_{i}, Y_{j}\right)$;
4. $\overline{\mathrm{M}}\left(X_{i} \cup\left\{x_{i+1}\right\}, Y_{j} \cup\left\{y_{j+1}\right\}, x_{i+1}, y_{j+1}\right)$.

It is important to notice that these non-RR terms can be simultaneously evaluated. In light of this observation, we compute these terms by using the techniques of $\$ 5.1$ to process the normal sequences $A_{i}, A_{u}, B_{j}, B_{v}$ defined below:

- $A_{i}(j)=\mathrm{m}\left(X_{i}, y_{j}\right)$ for all $x_{i}$ and $y_{j}$.
- $B_{j}(i)=\mathrm{M}\left(x_{i}, Y_{j}\right)$ for all $y_{j}$ and $x_{i}$.
- $A_{u}(j)=\operatorname{RR}\left(S_{1}^{u}, S_{2}^{y_{j}}\right)$ for all $u \in K\left(P^{\prime}, S_{1}\right)$ and $y_{j}$.
- $B_{v}(i)=\operatorname{RR}\left(S_{2}^{v}, S_{1}^{x_{i}}\right)$ for all $v \in K\left(Q^{\prime}, S_{2}\right)$ and $x_{i}$.

Note that $A_{i}$ and $A_{u}$ have length $q^{\prime}$, and $A_{i}$ is the joint of all $A_{u}$ where $u \in X_{i}$. Similarly, $B_{j}$ and $B_{v}$ have length $p^{\prime}$, and $B_{j}$ is the joint of all $B_{v}$ where $v \in Y_{j}$.

Lemma 5.11.

1. The minimal condensed forms of the sequences $A_{u}$ and $B_{v}$ have a total size of $O(n)$ and can be computed in $O(n)$ time.
2. The minimal condensed forms of the sequences $A_{i}$ and $B_{j}$ have a total size of $O(n)$ and can be computed in $O(n)$ time.
Proof. Statement 2 follows from Statement 1 and Lemma 5.1. Below we only prove Statement 1 for $A_{u}$; Statement 1 for $B_{v}$ is similarly proved. We first compute a condensed form $\bar{A}_{u}$ for each $A_{u}$ as follows:
3. For all $u \in K\left(P^{\prime}, S_{1}\right)$, compute $S_{2, u}=S_{2} \| S_{1}^{u}$ and $Q_{u}=Q^{\prime} \| S_{1}^{u}$.
4. For all $u$ where $Q_{u}$ is nonempty, do the following steps:
(a) $\bar{A}_{u} \leftarrow\left\{(j, w) \mid y_{j} \in Q_{u}, w=\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)\left(y_{j}\right)\right\}$.
(b) Compute all tuples $\left(\hat{v}, v, y_{j}\right)$ where $\hat{v} \in K\left(Q_{u}, S_{2, u}\right), v \in K\left(Q^{\prime}, S_{2}\right)$, $\hat{v} \in S_{2}^{v}$, and $v \in Y_{j}$.
(c) Find the smallest $s$ such that some $\left(\hat{v}, v, \underline{y_{s}}\right)$ is obtained at Step 2b.
(d) If there is only one $\left(\hat{v}, v, y_{s}\right)$, then add to $\bar{A}_{u}$ the pair $(s, w)$ where $w=$ $\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)(\hat{v})$.
5. For all $u$ where $S_{2, u}$ is nonempty and $Q_{u}$ is empty, do the following steps:
(a) Compute $\hat{v}, v$ and $y_{s}$ where $\hat{v}$ is the root of $S_{2, u}, v \in K\left(Q^{\prime}, S_{2}\right), \hat{v} \in S_{2}^{v}$ and $v \in Y_{s}$.
(b) $\bar{A}_{u} \leftarrow\{(s, w)\}$, where $w=\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)(\hat{v})$.
6. For all $u$ where $S_{2, u}$ is empty, $\bar{A}_{u} \leftarrow \emptyset$.

The correctness proof of this algorithm has three cases.
Case 1: $Q_{u}$ is nonempty. Let $y_{j_{1}}, y_{j_{2}}, \cdots, y_{j_{k}}=Q_{u}$. Let $j_{0}=0$. Then, for all $k^{\prime} \in[1, k]$ and all $j \in\left[j_{k^{\prime}-1}+1, j_{k^{\prime}}\right], S_{2}^{y_{j}} \| S_{1}^{u}=S_{2, u}^{y_{k^{\prime}}}$ and by Lemma 3.1, $A_{u}(j)=$ $\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)\left(y_{k^{\prime}}\right)$. There are two subcases for $j>j_{k}$.

Case 1a: Step 2 b finds two or more $\left(\hat{v}, v, y_{s}\right)$. Then $y_{s} \in Q_{u}, s=j_{k}$, and for all
$j \in\left[j_{k}+1, q^{\prime}\right], S_{2}^{y_{j}} \mid S_{2}^{u}$ is empty and $A_{u}(j)=0$.
Case 1b: Step 2b finds only one $\left(\hat{v}, v, y_{s}\right)$. Then $y_{s} \notin Q_{u}$ and $s>j_{k}$. For all $j \in\left[j_{k}+1, s\right], S_{2}^{y_{j}} \| S_{1}^{u}=S_{2, u}^{\hat{v}}$ and $A_{u}(j)=\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)(\hat{v})$. For all $j \in\left[s+1, q^{\prime}\right]$, $S_{2}^{y_{j}} \| S_{1}^{u}$ is empty and $A_{u}(j)=0$.

Thus, the $\bar{A}_{u}$ of Step 2 is a condensed form of $A_{u}$ for Case 1.
Case 2: $S_{2, u}$ is nonempty and $Q_{u}$ is empty. This case is similar to Case 1b, and Step 3 computes a correct condensed form $\bar{A}_{u}$ for this case.

Case 3: $S_{2, u}$ is empty. This case is obvious, and Step $\pi^{1}$ correctly computes a condensed form $\bar{A}_{u}$ of $A_{u}$ for this case.

The total size of all $\bar{A}_{u}$ is at most that of the RP mappings of $S_{1}, S_{2}, P^{\prime}$ and $Q^{\prime}$, which is the desired $O(n)$. Step 1 takes $O(n)$ time using Fact 1. The other steps can be implemented in $O(n)$ time in straightforward manners using radix sort and tree traversal. As discussed in $\widehat{\}}$, the RP mappings are evaluated by radix sort. Once the forms $\bar{A}_{u}$ are obtained, we can in $O(n)$ time radix sort the pairs in all $\bar{A}_{u}$ and then delete all unnecessary pairs to obtain the desired minimal condensed forms.

Lemma 5.12. All the non-rr terms of the first two types for the pairs in $G \cup E \cup B$ can be evaluated in $O(n \log (p+1) \log (q+1))$ time.

Proof. The value of $\mathrm{m}\left(X_{i}, y_{j}\right)$ is that of the point query $([i, i], j)$ for $A_{1}, \cdots, A_{q^{\prime}}$, and the value of $\max _{i \in\left[i_{1}, i_{2}\right]} \mathrm{M}\left(X_{i}, y_{j}\right)$ is that of the interval query $\left(\left[i_{1}, i_{2}\right], j\right)$. By Lemma 5.5( $\Phi$ ), there are $O(n \log (p+1))$ such terms required for the pairs in $G \cup E \cup B$. Given the results of Steps 2 and 3 of One-One, we can determine all such terms and the corresponding queries in $O(n \log (p+1))$ time. By Lemma 5.6(5), only $O(n \log (q+1))$ of these queries are not $P$-regular. By Lemmas 5.11(2) and 5.2(2), we can evaluate these queries in $O(n \log (p+1) \log (q+1))$ time. The terms $\mathrm{m}\left(x_{i}, Y_{j}\right)$ and $\max _{j \in\left[j_{1}, j_{2}\right]} \mathrm{M}\left(x_{i}, Y_{j}\right)$ are similarly evaluated is $O(n \log (p+1) \log (q+1))$ time. The analysis for these terms is easier because $p^{\prime} \geq q^{\prime}$ and it does not involve the notion of $Q$-regularity. $\square$

Lemma 5.13. The non-rr terms of the third and the fourth type for the pairs in $G \cup E \cup B$ can be evaluated within the following time complexity:

1. $O(n d \log d)$ or alternatively $O(n \sqrt{d} \log n)$ for the third type;
2. $O\left(n d^{2} \log d\right)$ or alternatively $O(n d \sqrt{d} \log n)$ for the fourth type.

Proof. To prove Statement 1, we consider the graphs ( $X_{i}, Y_{j}, X_{i} \times Y_{j}$ ) on which the desired terms $\mathrm{m}\left(X_{i}, Y_{j}\right)$ are defined. Let $Z_{i, j}$ be the subgraph of $\left(X_{i}, Y_{j}, X_{i} \times Y_{j}\right)$ constructed by removing all zero-weight edges and all resulting isolated vertices. The edges of $Z_{i, j}$ are computed as follows:

1. For all $u \in K\left(P^{\prime}, S_{1}\right)$, compute $S_{2, u}=S_{2} \| S_{1}^{u}$ and $Q_{u}=Q^{\prime}| | S_{1}^{u}$.
2. For all $S_{2, u}$ is nonempty, do the following steps:
(a) If $Q_{u}$ is nonempty, compute all tuples $(u, v, w)$ where $\hat{v} \in K\left(Q_{u}, S_{2, u}\right)$, $v \in K\left(Q^{\prime}, S_{2}\right), \hat{v} \in S_{2}^{v}$ and $w=\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)(\hat{v})$.
(b) If $Q_{u}$ is empty, compute the tuple $(u, v, w)$ where $\hat{v}$ is the root of $S_{2, u}$, $v \in K\left(Q^{\prime}, S_{2}\right), \hat{v} \in S_{2}^{v}$ and $w=\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)(\hat{v})$.
This algorithm captures all the nonzero-weight $(u, v)$. At Step $2, S_{2, u}^{\hat{v}}=S_{2}^{v}| | S_{1}^{u}$ and by Lemma $3.1 \operatorname{Rr}\left(S_{1}^{u}, S_{2}^{v}\right)=\operatorname{RP}\left(S_{1}^{u}, S_{2}, Q^{\prime}\right)(\hat{v})$. Thus, the first two components of the obtained tuples form the edges of all desired $Z_{i, j}$ and the third components are the weights of these edges. We use Fact 1 to implement Step 1 in $O(n)$ time. We can implement Step 2 in $O(n)$ time using radix sort and tree traversal. Note that Step $Z$ uses radix sort to evaluate RP mappings. With the tuples $(u, v, w)$ obtained, we use radix sort to construct all desired $Z_{i, j}$ in $O(n)$ time. Let $m_{i, j}$ and $n_{i, j}$ be the numbers of edges and vertices in $Z_{i, j}$, respectively. Since an edge weighs at most $n$, we can compute $\mathrm{m}\left(X_{i}, Y_{j}\right)$ in $O\left(n_{i, j} \cdot m_{i, j}+n_{i, j}^{2} \cdot \log n_{i, j}\right)$ and alternatively in
$O\left(m_{i, j} \cdot \sqrt{n_{i, j}} \cdot \log \left(n \cdot n_{i, j}\right)\right)$ time 21, 42]. Statement 1 then follows from the fact that $n_{i, j} \leq 2 d^{\prime}, n_{i, j} \leq 2 m_{i, j}$, and by Lemma 5.5(1) the sum of all $m_{i, j}$ is at most $n$.

To prove Statement 2, we similarly process the bipartite graphs on which the desired terms $\overline{\mathrm{M}}\left(X_{i} \cup\left\{x_{i+1}\right\}, Y_{j} \cup\left\{y_{j+1}\right\}, x_{i+1}, y_{j+1}\right)$ are defined. The key difference from the third type is that in addition to some of the edges in $Z_{i, j}$, we need certain nonzero-weight $\left(u, y_{j+1}\right)$ for $u \in X_{i}$ and $\left(x_{i+1}, v\right)$ for $v \in Y_{j}$. Since these edges are required only for intersecting $\left(x_{i}, y_{j}\right)$, by Lemma 5.5(1) $), O(d n)$ such edges are needed. We use Lemma $5.11(1)$ to compute the weights of these edges in $O(d n)$ time. Due to these edges, the total time complexity for the fourth type is $O(d)$ times that for the third type.

The next theorem serves to prove Theorem 4.1 given at the start of $\S \boxed{4}$.
Theorem 5.14. One-One solves Problem $\begin{aligned} & \text { with the following time complexities: }\end{aligned}$

$$
O\left(n d^{2} \log d+n \log (p+1) \log (q+1)\right)
$$

or alternatively

$$
O(n d \sqrt{d} \log n+n \log (p+1) \log (q+1))
$$

Proof. The correctness of One-One follows from Lemma 5.3 and $\S 5.3$. 5.5 . As for the time complexity, Step 1 is obvious and takes $O(n)$ time. By computing $\Psi$, we can compute the sets $E$ and $I\left(\stackrel{\rightharpoonup}{P}\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)$. Since the leaf labels of $S_{1}$ and $S_{2}$ are from $[1, O(n)]$, each level of $\Psi$ can be computed in $O(n)$ time. Since $\Psi$ has $O(\log (p+1))$ levels, $E$ and $I\left(P\left[1, p^{\prime}-1\right], Q\left[1, q^{\prime}-1\right]\right)$ can be computed in $O(n \log (p+1))$ time. With these two sets obtained, we can compute all the desired sets in $O(n \log (p+1))$ time. Thus, Step 2 takes $O(n \log (p+1))$ time. Step 3 takes $O(n \log (p+1))$ time using radix sort. The time complexity of Step dominates that of One-One. This step uses Lemmas 5.12 and 5.13 and takes $O\left(n \log (p+1) \log (q+1)+n d^{2} \log d\right)$ time or alternatively $O(n \log (p+1) \log (q+1)+n d \sqrt{d} \log n)$ time. Step 5 spends $O(n \log (p+1))$ time using radix sort to create pointers from the pairs in $G \cup E \cup B$ to appropriate predecessors. Step 5 then takes $O(1)$ time per pair in $G \cup E \cup B$ and $O(n \log (p+1))$ time in total. Step 6 takes $O(n \log (p+1))$ time. It uses radix sort to access the desired RR values and evaluate the input mappings. It also uses Fact 1 to compute all $T_{1} \| T_{2}^{v}$ and $T_{2} \| T_{1}^{u}$.
6. Discussions. We answer the main problem of this paper with the following theorem and conclude with an open problem.

ThEOREM 6.1. Let $T_{1}$ and $T_{2}$ be two evolutionary trees with $n$ leaves each. Let $d$ be their maximum degree. Given $T_{1}$ and $T_{2}$, a maximum agreement subtree of $T_{1}$ and $T_{2}$ can be computed in $O\left(n d^{2} \log d \log ^{2} n\right)$ time or alternatively in $O\left(n d \sqrt{d} \log ^{3} n\right)$ time. Thus, if $d$ is bounded by a constant, a maximum agreement subtree can be computed in $O\left(n \log ^{2} n\right)$ time.

Proof. By Theorem4.5, the algorithms in $\S\left({ }^{4} 5\right.$ compute $\operatorname{RR}\left(T_{1}, T_{2}\right)$ within the desired time bounds. With straightforward modifications, these algorithms can compute a maximum agreement subtree within the same time bounds. $\quad$ a

The next lemma establishes a reduction from the longest common subsequence problem to that of computing a maximum agreement subtree.

Lemma 6.2. Let $M_{1}=x_{1}, \ldots, x_{n}$ and $M_{2}=y_{1}, \ldots, y_{n}$ be two sequences. Assume that the symbols $x_{i}$ are all distinct and so are the symbols $y_{j}$. Then, the problem of computing a longest common subsequence of $M_{1}$ and $M_{2}$ can be reduced in linear time to that of computing a maximum agreement subtree of two binary evolutionary trees.

Proof. Given $M_{1}$ and $M_{2}$, we construct two binary evolutionary trees $T_{1}$ and $T_{2}$ as follows. Let $z_{1}$ and $z_{2}$ be two distinct symbols different from all $x_{i}$ and $y_{i}$. Next, we construct two paths $P_{1}=u_{1}, \ldots, u_{n+1}$ and $P_{2}=v_{1}, \ldots, v_{n+1} . T_{1}$ is formed by making $u_{1}$ the root, attaching $x_{i}$ to $u_{i}$ as a leaf, and attaching $z_{1}$ and $z_{2}$ to $u_{n+1}$ as leaves. Symmetrically, $T_{2}$ is formed by making $v_{1}$ the root, attaching $y_{i}$ to $v_{i}$, and attaching $z_{1}$ and $z_{2}$ to $v_{n+1}$. The lemma follows from the straightforward one-to-one onto correspondence between the longest common subsequences of $M_{1}$ and $M_{2}$ and the maximum agreement subtrees of $T_{1}$ and $T_{2}$.

We can use Lemma 6.2 to derive lower complexity bounds for computing a maximum agreement subtree from known bounds for the longest common subsequence problem in various models of computation [3, 6, 23, 29, 30, 32, 50]. This paper assumes a comparison model where two labels $x$ and $y$ can be compared to determine whether $x$ is smaller than $y$ or $x=y$ or $x$ is greater than $y$. Since the longest common subsequence problem in Lemma 6.2 requires $\Omega(n \log n)$ time in this model [31], the same bound holds for the problem of computing a maximum agreement subtree of two evolutionary trees where $d$ is bounded by a constant. It would be significant to close the gap between this lower bound and the upper bound of $O\left(n \log ^{2} n\right)$ stated in Theorem 6.1. Recently, Farach, Przytycka and Thorup [13] independently developed an algorithm that runs in $O\left(n \sqrt{d} \log ^{3} n\right)$ time. For binary trees, Cole and Hariharan (8) gave an $O(n \log n)$-time algorithm. It may be possible to close the gap by incorporating ideas used in those two results and this paper.

Acknowledgments. The author is deeply appreciative for the extremely thorough and useful suggestions given by the anonymous referee. The author thanks Joseph Cheriyan, Harold Gabow, Andrew Goldberg, Dan Gusfield, Dan Hirschberg, Phil Klein, Phil Long, K. Subramani, Bob Tarjan, Tandy Warnow for helpful comments, discussions and references.

## REFERENCES

[1] K. Abrahamson, N. Dadoun, D. G. Kirkpatrick, and T. Przytycka, A simple tree contraction algorithm, Journal of Algorithms, 10 (1989), pp. 287-302.
[2] R. Agarwala and D. Fernández-Baca, A polynomial-time algorithm for the perfect phylogeny problem when the number of character states is fixed, SIAM Journal on Computing, 23 (1994), pp. 1216-1224.
[3] A. V. Aho, D. S. Hirschberg, and J. D. Ullman, Bounds on the complexity of the longest common subsequence problem, Journal of the ACM, 23 (1976), pp. 1-12.
[4] A. V. Aho, J. E. Hopcroft, and J. D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Massachusetts, 1974.
[5] A. V. Aho, Y. Savig, T. G. Szymanski, and J. D. Ullman, Inferring a tree from the lowest common ancestors with an application to the optimization of relational expressions, SIAM Journal on Computing, 10 (1981), pp. 405-421.
[6] A. Apostolico and C. Guerra, The longest common subsequence problem revisited, Algorithmica, 2 (1987), pp. 315-336.
[7] H. L. Bodlaender, M. R. Fellows, and T. J. Warnow, Two strikes against perfect phylogeny, in Lecture Notes in Computer Science 623: Proceedings of the 19th International Colloquium on Automata, Languages, and Programming, Springer-Verlag, New York, NY, 1992, pp. 273-283.
[8] R. Cole and R. Hariharan, An $O(n \log n)$ algorithm for the maximum agreement subtree problem for binary trees, in Proceedings of the 7 th Annual ACM-SIAM Symposium on Discrete Algorithms, 1996, pp. 323-332.
[9] T. H. Cormen, C. L. Leiserson, and R. L. Rivest, Introduction to Algorithms, MIT Press, Cambridge, MA, 1991.
[10] W. H. E. DAy and D. Sankoff, Computational complexity of inferring phylogenies from chromosome inversion data, Journal of Theoretical Biology, 124 (1987), pp. 213-218.
[11] S. Dress and M. Steel, Convex tree realizations of partitions, Applied Mathematics Letters, 5 (1992), pp. 3-6.
[12] M. Farach, S. Kannan, and T. Warnow, A robust model for finding optimal evolutionary trees, Algorithmica, 13 (1995), pp. 155-179.
[13] M. Farach, T. M. Przytycka, and M. Thorup, Computing the agreement of trees with bounded degrees, in Lecture Notes in Computer Science 979: Proceedings of the Third Annual European Symposium on Algorithms, P. Spirakis, ed., Springer-Verlag, New York, NY, 1995, pp. 381-393.
[14] M. Farach and M. Thorup, Fast comparison of evolutionary trees (extended abstract), in Proceedings of the 5th Annual ACM-SIAM Symposium on Discrete Algorithms, 1994, pp. 481-488.
[15] - Optimal evolutionary tree comparison by sparse dynamic programming (extended abstract), in Proceedings of the 35th Annual IEEE Symposium on the Foundations of Computer Science, 1994, pp. 770-779.
[16] J. Felsenstein, Numerical methods for inferring evolutionary trees, The Quarterly Review of Biology, 57 (1982), pp. 379-404.
[17] ——, Inferring evolutionary trees from DNA sequences, in Statistical Analysis of DNA Sequence Data, B. Weir, ed., Dekker, 1983, pp. 133-150.
[18] —— Phylogenies from molecular sequences: Inference and reliability, Annual Review of Genetics, 22 (1988), pp. 521-565.
[19] C. R. Finden and A. D. Gordon, Obtaining common pruned trees, Journal of Classification, 2 (1985), pp. 255-276.
[20] A. Friday, Quantitative aspects of the estimation of evolutionary trees, Folia Primatologica, 53 (1989), pp. 221-234.
[21] H. N. Gabow and R. E. TarJan, Faster scaling algorithms for network problems, SIAM Journal on Computing, 18 (1989), pp. 1013-1036.
[22] H. Gazit, G. L. Miller, and S. H. Teng, Optimal tree contraction in the EREW model, in Concurrent Computations: Algorithms, Architecture, and Technology, S. Tewksbury, B. Dickinson, and S. Schwartz, eds., Plenum, New York, 1988, pp. 139-156.
[23] R. N. Goldberg, Minimal string difference encodings, Journal of Algorithms, 3 (1982), pp. 147-156.
[24] D. GUSFIELD, Efficient algorithms for inferring evolutionary trees, Networks, 21 (1991), pp. 1928.
[25] D. Harel and R. E. Tarjan, Fast algorithms for finding nearest common ancestors, SIAM Journal on Computing, 13 (1984), pp. 338-355.
[26] J. J. Hein, An optimal algorithm to reconstruct trees from additive distance data, Bulletin of Mathematical Biology, 51 (1989), pp. 597-603.
[27] M. D. HEndy, The relationship between between simple evolutionary tree models and observable sequence data, Systematic Zoology, 38 (1989), pp. 310-321.
[28] M. D. Hendy and D. Penny, Branch and bound algorithms to determine minimal evolutionary trees, Mathematical Biosciences, 59 (1982), pp. 277-290.
[29] D. S. Hirschberg, A linear space algorithm for computing maximal common subsequences, Communications of the ACM, 18 (1975), pp. 341-343.
[30] , Algorithms for the longest common subsequence problem, Journal of the ACM, 24 (1977), pp. 664-675.
[31] , An information theoretic lower bound for the longest common subsequence problem, Information Processing Letters, 7 (1978), pp. 40-41.
[32] J. W. Hunt and T. G. Szymanski, A fast algorithm for computing longest common subsequences, Communications of the ACM, 20 (1977), pp. 350-353.
[33] T. Jiang, E. L. LaWler, and L. Wang, Aligning sequences via an evolutionary tree: complexity and approximation, in Proceedings of the 26 th Annual ACM Symposium on Theory of Computing, 1994, pp. 760-769.
[34] S. K. Kannan, E. L. Lawler, and T. J. Warnow, Determining the evolutionary tree using experiments, Journal of Algorithms, 21 (1996), pp. 26-50.
[35] S. K. Kannan and T. J. Warnow, Inferring evolutionary history from DNA sequences, SIAM Journal on Computing, 23 (1994), pp. 713-737.
[36] D. Keselman and A. Amir, Maximum agreement subtree in a set of evolutionary trees metrics and efficient algorithms, in Proceedings of the 35th Annual IEEE Symposium on the Foundations of Computer Science, 1994, pp. 758-769. To appear in SIAM Journal on Computing.
[37] L. C. Klotz and R. L. Blanken, A practical method for calculating evolutionary trees from sequence data, Journal of Theoretical Biology, 91 (1981), pp. 261-272.
[38] S. R. Kosaraju and A. L. Delcher, Optimal parallel evaluation of tree-structured computations by raking, in Lecture Notes in Computer Science 319: Proceedings of the 3rd Aegean Workshop on Computing, J. H. Reif, ed., Springer-Verlag, New York, NY, 1988, pp. 101110.
[39] E. Kubicka, G. Kubicki, and F. McMorris, An algorithm to find agreement subtrees, Journal of Classification, 12 (1995), pp. 91-99.
[40] G. L. Miller and J. H. Reif, Parallel tree contraction, part 1: Fundamentals, in Advances in Computing Research: Randomness and Computation, S. Micali, ed., vol. 5, JAI Press, Greenwich, CT, 1989, pp. 47-72.
[41] ——, Parallel tree contraction part 2: Further applications, SIAM Journal on Computing, 20 (1991), pp. 1128-1147.
[42] J. B. Orlin and R. K. Ahuja, New scaling algorithms for the assignment and minimum mean cycle problems, Mathematical Programming, 54 (1992), pp. 41-56.
[43] D. Penny and M. Hendy, Estimating the reliability of evolutionary trees, Molecular Biology and Evolution, 3 (1986), pp. 403-417.
[44] A. Rzhetsky and M. Nei, A simple method for estimating and testing minimum-evolution trees, Molecular Biology and Evolution, 9 (1992), pp. 945-967.
[45] B. Schieber and U. Vishkin, On finding lowest common ancestors: Simplification and parallelization, SIAM Journal on Computing, 17 (1988), pp. 1253-1262.
[46] M. Steel, The complexity of reconstructing trees from qualitative characters and subtrees, Journal of Classification, 9 (1992), pp. 91-116.
[47] M. Steel and T. Warnow, Kaikoura tree theorems: Computing the maximum agreement subtree, Information Processing Letters, 48 (1993), pp. 77-82.
[48] L. Wang, T. Jiang, and E. LaWler, Approximation algorithms for tree alignment with a given phylogeny, Algorithmica, 16 (1996), pp. 302-315.
[49] T. J. Warnow, Tree compatibility and inferring evolutionary history, Journal of Algorithms, 16 (1994), pp. 388-407.
[50] C. K. Wong and A. K. Chandra, Bounds for the string editing problem, Journal of the ACM, 23 (1976), pp. 13-16.


[^0]:    * Department of Computer Science, Yale University, New Haven, CT 06520 (kao-mingyang@cs.yale.edu). This research was supported in part by NSF grant CCR-9531028.

