Orthogonal 3-D Graph Drawing *

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Abstract. This paper studies 3-D orthogonal grid drawings for graphs of arbitrary degree, K_n in particular, with vertices drawn as boxes. It establishes an asymptotic lower bound for the volume of the bounding box of such drawings and exhibits a construction that achieves this bound. No edge route in this unconstrained construction bends more than three times.

For drawings constrained to have at most k bends on any edge route, simple constructions are given for $k = 1$ and $k = 2$. The unconstrained construction handles the $k \geq 3$ cases, while for $k = 0$ (no bends), it is proved here that not all graphs can be drawn.

1 Introduction

This paper offers methods for constructing 3-D orthogonal grid drawings for graphs of *arbitrary* degree. It also contributes a lower bound result for the volumes of such drawings, establishing that one of our constructions is in some sense optimal. To state the main results clearly, we explain, following some terminology, the drawing conventions and volume measure used.

A *grid point* is a point in R^3 whose coordinates are all integers. A *grid box* is the set of all points (x, y, z) in R^3 satisfying $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$ and $z_0 \leq z \leq z_1$ for some integers $x_0, x_1, y_0, y_1, z_0, z_1$. A *port* of a box is any grid point of the box that is extremal in at least one direction. A grid box is said to have *dimensions* $a \times b \times c$ whenever $x_1 = x_0 + a - 1$, $y_1 = y_0 + b - 1$, and $z_1 = z_0 + c - 1$. The *volume* of such a box is defined to be the number of grid points it contains, namely abc. For example, a single grid point is a $1 \times 1 \times 1$ box of volume 1. The *volume* of a drawing is the volume of its *bounding box*, which is the smallest volume grid box containing the drawing. Often we refer to the bounding box as an $X \times Y \times Z$-grid.

Throughout this paper, a *3-D orthogonal grid drawing* of a graph $G = (V, E)$ is a drawing that satisfies the following. Distinct vertices of V are represented

* The authors gratefully thank N.S.E.R.C. for financial assistance.

by disjoint grid boxes[4]. An edge $e = (v_1, v_2)$ of E is drawn as a simple path that follows grid lines, possibly turning ("bending") at grid points; the endpoints of the path for e are ports on the boxes representing v_1 and v_2. The intermediate points along the path for an edge do not belong to any vertex box, nor do they belong to any other edge path. See Figure 1. In what follows, graph theoretic terms such as *vertex* are typically used to refer both to the graph theoretic object and to its representation in a drawing.

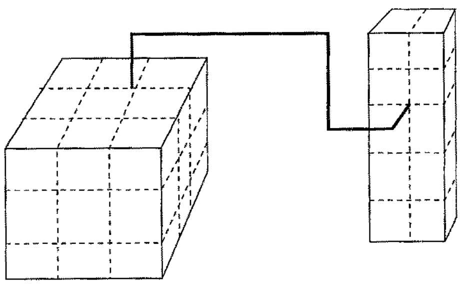

Fig. 1. Two boxes joined by a 4-bend edge.

For graphs drawn orthogonally in the 2-D grid, early research mainly considered graphs of maximum degree 4 and represented vertices as single grid points. More recently, 2-D orthogonal grid drawings of higher degree graphs have been investigated, where vertices have been drawn as rectangular boxes. See for example [FK96], [PT96], [BMT97].

At present, there are few results on 3-D orthogonal grid drawings. Rosenberg showed that any graph of maximum degree 6 can be embedded in a 3-D grid of volume $\mathcal{O}(n^{3/2})$, and that this is asymptotically optimal [Ros83]. No bounds on the number of bends were given. Recently, Eades, Symvonis and Whitesides gave a method for drawing graphs of maximum degree 6 in a grid of side-length $4\sqrt{n}$, with vertices represented by single grid points and each edge having at most 7 bends [ESW97]. They also gave a simple method for drawing such graphs in a grid of side-length $3n$, creating at most 3 bends on each edge. Papakostas and Tollis have proposed a more elaborate method that produces a drawing of volume at most $4.66n^3$ [PT97].

The focus of this paper is on 3-D orthogonal grid drawings of *complete* graphs. Since any simple graph G on n vertices is a subgraph of the complete graph K_n, a drawing of K_n immediately provides a drawing for G, since irrelevant edges may be deleted from the drawing. Complete graphs are also critical for many lower bound arguments.

[4] This paper allows vertices to be represented by degenerate boxes, i.e., by boxes that have dimension 1 with respect to one or more coordinate directions. Such degeneracies can be removed by adding additional grid lines, which increases the volume of the drawing by a multiplicative constant.

convention: From now on, the terms *drawing* and 3-D *orthogonal grid drawing* are used interchangably.

In informal language, the main results of this paper are as follows.

- For all sufficiently large n, K_n has no bend-free drawing.
- Any drawing of K_n has volume $\Omega(n^{2.5})$.
- K_n can be drawn in $O(n^3)$ volume with at most $k = 1$ bend per edge.
- K_n can be drawn in $O(n^3)$ volume with at most $k = 2$ bends per edge.
- K_n can be drawn in $O(n^{2.5})$ volume with at most $k = 3$ bends per edge.

Note that for $k \geq 3$, the upper and lower bounds on the volume match (within a constant factor) when a maximum of k bends per edge is allowed. The constructions of this paper have reasonably small constant factors for the volume. Only for the $k = 1$ and $k = 2$ cases do the bounds not match; in each of these cases we give an $O(n^3)$ volume drawing of K_n and leave as an open problem whether this drawing indeed has asymptotically optimum volume.

The results can be restated more precisely with the following terminology.

definition: Let $vol(n)$ denote the minimum possible volume of any drawing of K_n, and let $vol_k(n)$ denote the minimum possible volume for drawings of K_n that have k or fewer bends on any edge.

In these terms, the main results are that $vol_0(n)$ is undefined for large n, $vol(n)$ is in $\Omega(n^{2.5})$, $vol_1(n)$ and $vol_2(n)$ are in $O(n^3)$, and $vol_k(n)$ is in $\Theta(n^{2.5})$ for $k \geq 3$.

2 No Bends

The main result of this section is that there exist graphs that have no 0-bend 3-D orthogonal drawing. If no bends are permitted in the drawing, then the edges correspond to axis-parallel visibility lines between pairs of boxes. Such visibility representations have been studied in 2-D by Wismath [Wis85] and by Tamassia and Tollis [TT86], and in 3-D with 2-D objects [BEFLMRSW93], [FHW96]. A 3-D orthogonal drawing of a graph with no bends splits the edges into three classes, depending on the direction of visibility. Each class of edges forms a graph that has a visibility representation using only one direction of visibility. Our lower bound result depends on the fact that K_{56} has no such visibility representation, as shown by [FHW96].

The *3-Ramsey number* $R(r, b, g)$ is the smallest number such that any arbitrary colouring of the edges of $K_{R(r,b,g)}$ with colours red, blue and green induces either a red K_r, or a blue K_b, or a green K_g as a subgraph. This number exists and is finite; see for example [GRS80].

Theorem 1. *For all sufficiently large n (e.g., $n \geq R(56, 56, 56)$), K_n has no bend-free 3-D orthogonal grid drawing.*

One consequence of the previous theorem is that $\Omega(n^2)$ bends are required in any 3-D orthogonal grid drawing of K_n. Details are omitted.

3 A Lower Bound on the Volume

Recall that $vol(n)$ is the minimum possible volume for a drawing of K_n. This definition is valid since, as later sections show, every K_n has a drawing if edges are allowed to bend.

A *z-line* is a line that is parallel to the z-axis; *y-lines* and *x-lines* are defined analogously. A *z-plane* is a plane that is orthogonal to the z-axis; *x-planes* and *y-planes* are defined analogously.

Theorem 2. $vol(n) \in \Omega(n^{2.5})$.

Proof. The constants that appear below were chosen for convenience and have no special significance other than that they give a simple proof. We make no attempt here to produce a large constant multiplier for the $n^{2.5}$.

Consider a drawing of K_n in a grid of dimensions $X \times Y \times Z$.

Case 1: A line intersects many vertices

Assume there exists a z-line intersecting at least t vertices, where t is even and $t \geq \frac{1}{16}n$. Let $v_1, \ldots, v_t$ be any t of the vertices intersected by the z-line, listed in order of occurrence along the line, and let P_z be a z-plane (not necessarily with integer z-coordinate) that intersects none of these t vertices and that separates the first half of them from the second half.

Since the $\frac{1}{4}t^2$ edges connecting these two groups must cross the plane P_z, this plane must contain at least $\frac{1}{4}t^2$ points having integer x- and y-coordinates. Hence $XY \geq \frac{1}{4}t^2 > \frac{1}{1024}n^2$. Also, $Z \geq \frac{1}{16}n$ since the z-line intersects at least $\frac{1}{16}n$ vertices. Thus $vol(n) > \frac{1}{16384}n^3$.

Case 2: A plane intersects many vertices

Assume now that no x-line, y-line or z-line intersects as many as $\frac{1}{16}n$ vertices, but that there exists a z-plane P_z intersecting at least $\frac{1}{4}n$ vertices.

A vertex is *left* of an x-plane P_x if all the points in its grid box have x-coordinates less than x. The notion of *right* of P_x is analogous. As P_x is swept from smaller to larger values of x, the y-line determined by its intersection with P_z intersects fewer than $\frac{1}{16}n$ vertices, by assumption. As x increases, an integer $x = x_0$ is encountered where, for the last time, there are fewer than $\frac{1}{16}n$ vertices left of P_x and intersecting P_z.

The number of vertices that intersect P_z and that lie left of P_{x_0+1} is at least $\frac{1}{16}n$ but at most $\frac{2}{16}n - 2$. Thus at least $\frac{1}{16}n$ vertices intersect P_z and lie right of P_{x_0+1}. There are at least $\frac{1}{256}n^2$ edges between the vertices on the left and the vertices on the right, so $YZ \geq \frac{1}{256}n^2$. Apply exactly the same argument in the y-direction to obtain $XZ \geq \frac{1}{256}n^2$. Finally, note that $XY \geq \frac{1}{4}n$, since P_z intersects $\frac{1}{4}n$ vertices. Consequently, $XYZ = \sqrt{YZ \cdot XZ \cdot XY} \geq \frac{1}{512}\, n^{5/2}$.

Case 3: No plane intersects many vertices

Assume now that no plane intersects as many as $\frac{1}{4}n$ vertices. Consider P_x planes in order of increasing x value. By an argument analogous to the one in Case 2, a P_x will be encountered for which at least $\frac{1}{4}n$ vertices lie left of P_x, and at least $\frac{1}{4}n$ vertices lie right of P_x. Consequently, P_x contains at least $\frac{1}{16}n^2$ points with integer y- and z-coordinates, and $YZ \geq \frac{1}{16}n^2$. Since the same argument holds for the other two directions, $XYZ \geq (\frac{1}{16}n^2)^{3/2} = \frac{1}{64}n^3$.

For all sufficiently large n, the bound given by Case 2 is the smallest of the three; hence $vol(n) \in \Omega(n^{5/2})$.

4 Constructions

The lower bound of the previous section provides a volumetric goal for layout strategies. This section presents a construction that achieves this lower bound with a small constant factor. For the $k = 1$ case, two strategies are described and then modified to give a drawing for the $k = 2$ case. A simple construction that realizes the $\Omega(n^{2.5})$ lower bound for volume is described in subsection 4.3. The construction generates at most 3 bends on any edge and hence is valid for each $k \geq 3$. Whether the lower bound is attainable when $k = 1$ or 2 remains an open problem.

In each of the constructions, vertices are first placed as points in a 2-D x, y-plane. Next, all the edges are routed in the same xy-plane, with overlap and crossings of edges temporarily permitted. Then a number Z of z-planes is introduced, and edges are assigned to these planes so that no edges overlap or cross. The vertices are stretched into segments of z-lines.

4.1 Drawings of $O(n^3)$ volume for $k = 1$

In this section, we describe two strategies to draw K_n with at most $k = 1$ bends on any edge. The first layout scheme draws K_n in an $n \times n \times n$-grid. The second scheme then makes two drawings of $K_{n/2}$ (without recursion) using the first scheme; then it positions these drawings in an $\frac{n}{2} \times n \times \frac{n}{2}$-grid and supplies the edges between the two parts. For simplicity, assume below that n is divisible by 4.

Drawing K_n in an $n \times n \times n$-grid for $k = 1$ Enumerate the vertices as $v_1, \ldots, v_n$. Place vertex v_i at (i, i). Route edge $e = (v_i, v_j)$, where $i < j$, with one bend via $(i, i), (i, j), (j, j)$. Note that no vertex or part of an edge is placed at a point (x, y) with $y < x$.

Now partition the edges of K_n into edge sets E_i^a, E_i^b, $i = 1, \ldots, \frac{n}{2}$, defined as $E_i^a = \{(v_{i-l+1}, v_{i+l}) | l = 1, \ldots, \frac{n}{2}\}$ and $E_i^b = \{(v_{i-l}, v_{i+l}) | l = 1, \ldots, \frac{n}{2} - 1\}$ (all additions are modulo n). It is easy to check that these sets indeed partition the edges of K_n, and that no crossings or overlaps occur among edges in E_i^a nor among edges in E_i^b. Hence only n z-planes are needed. See Fig. 2. This gives the following lemma.

Lemma 3. *There exists a drawing of K_n in an $n \times n \times n$-grid with one bend per edge such that the points $\{(x, y, z) : y < x\}$ are unused.*

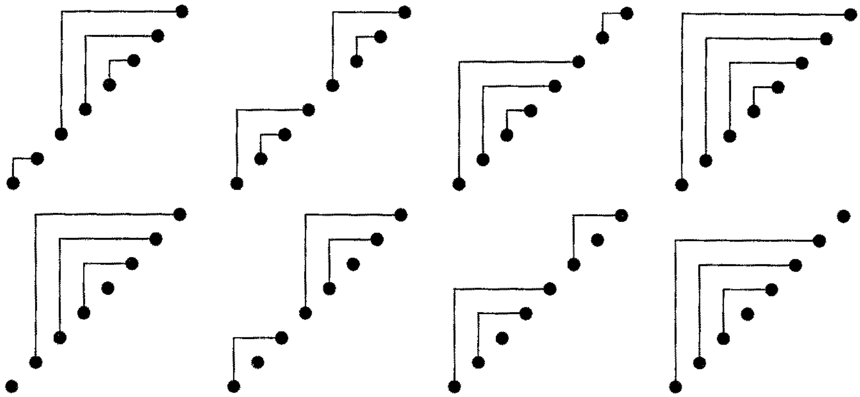

Fig. 2. The sets $E_b^1, \dots, E_b^4$ for K_8.

Remark: Note that E_i^a and E_i^b can be drawn in the same plane by reflecting the edges of E_i^a with respect to the diagonal line through the vertices. This yields a drawing of K_n in an $n \times n \times \frac{n}{2}$-grid. This strategy is closely related to the *pagenumber* of a graph and in fact, may prove a useful idea for drawing sparse graphs. This idea yields, for example, a method for drawing planar graphs in $O(n^2)$ volume in an $n \times n \times 4$-grid, since it is known that planar graphs have pagenumber equal to 4 (see [Yan89]).

Drawing K_n in an $\frac{n}{2} \times n \times \frac{n}{2}$-grid for $k = 1$ Let K^1 and K^2 denote two drawings of $K_{n/2}$ as described in the previous lemma. Thus each drawing has an $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$ bounding box. Reflect the points in the box for K^2 through the $(y = 0)$-plane, so that all points in the reflected K^2 have negative y-coordinate. Then rotate this reflected K^2 so that vertex v_j of the rotated, reflected K^2 overlaps the points $(x, -j, j)$, where $1 \leq x \leq \frac{n}{2}$. See Fig. 3.

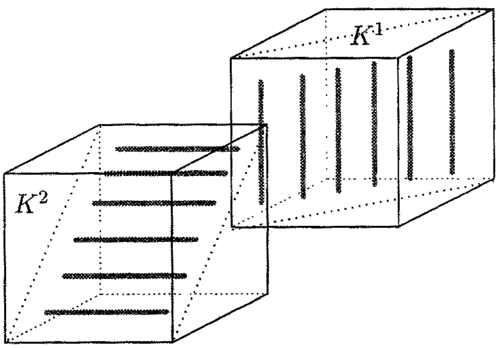

Fig. 3. K_1 and K_2

Each vertex v_i in K^1 sees each vertex v_j in the rotated, reflected K^2 along the y-line segment $[(i,i,j),(i,-j,j)]$. Therefore, these edges can be drawn as straight line segments, thus producing a drawing of K_n. Delete the unused y-plane of y-coordinate 0 to obtain a drawing with dimensions $X = Z = \frac{n}{2}$ and $Y = n$. There are $n^2/4$ edges drawn without a bend, and all other edges have one bend, so the total number of bends is $n^2/4 - n/2$.

Theorem 4. *K_n can be drawn in a $\frac{n}{2} \times n \times \frac{n}{2}$-grid with at most one bend per edge and a total number of bends equal to $n^2/4 - n/2$.*

4.2 A smaller $O(n^3)$ volume drawing for $k = 2$

A similar strategy can be applied when a maximum of $k = 2$ bends on an edge is allowed. In this section, K_n is drawn with at most two bends per edge by first making two copies of a drawing for $K_{\frac{n}{2}}$ and then placing them in a grid of side-length $\frac{n}{2}$ and supplying the edges connecting the two parts.

Drawing in an $n \times \frac{n}{2} \times n$-grid Enumerate the vertices as $\{v_1, \ldots, v_n\}$ and place v_i at $(x,y) = (i,1)$ in a 2-D (x,y)-plane. To route edge $e = (v_i, v_j)$, where $i < j$, let $y = \lceil \frac{j-i}{2} \rceil$ and route e via the points $(i,1),(i,y),(j,y),(j,1)$, creating two bends if $y > 1$ and no bends if $y = 1$.

Define the edge sets E_i^a and E_i^b as above. Again there are no crossings nor overlaps among edges in the same set and so n z-planes suffice. Since the largest y-coordinate is $\lceil \frac{n-1}{2} \rceil$, the bounding box has dimensions $n \times \frac{n}{2} \times n$. The edges (v_i, v_{i+1}) for $i = 1, \ldots, n-1$ are drawn straight; all other edges have two bends, so the total number of bends is $n^2 - 3n + 2$.

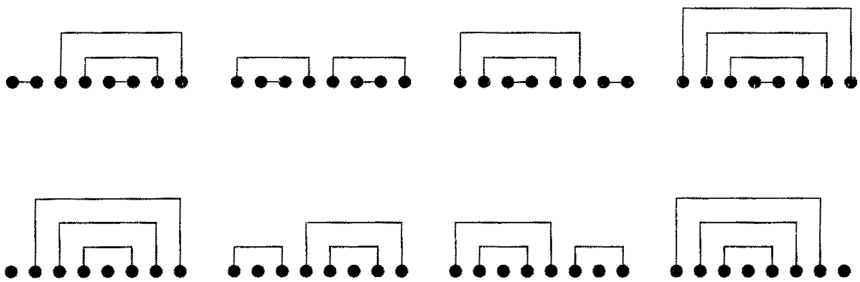

Fig. 4. The edge sets of K_8 drawn with at most two bends per edge.

Lemma 5. *The graph K_n can be drawn in an $n \times \frac{n}{2} \times n$-grid, with a total of $n^2 - 3n + 2$ bends and at most two bends per edge, such that vertex v_i overlaps the points $(i, 1, z)$, where $1 \le z \le n$.*

Drawing in an $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$-grid Let K^1 and K^2 denote two $K_{\frac{n}{2}}$'s drawn as described above. Thus each drawing has a bounding box of dimensions $\frac{n}{2} \times \frac{n}{4} \times \frac{n}{2}$. Reflect K^2 through the $(y = 0)$-plane, so that all points in the reflected K^2 have negative y-coordinate. Then rotate the reflected K^2 so that vertex v_j of the rotated, reflected K^2 now overlaps the points $(x, -1, j)$, where $1 \leq x \leq \frac{n}{2}$.

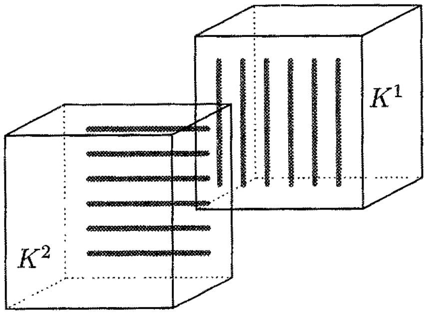

Fig. 5. Two $K_{\frac{n}{2}}$'s, with K^2 reflected and rotated

Each vertex v_i in K^1 sees each vertex v_j in the rotated, reflected K^2 along the y-line segment $[(i, 1, j), (i, -1, j)]$. Therefore, these edges can be drawn as straight lines, thus producing a drawing of K_n. Removing the unused y-plane of y-coordinate 0 yields a drawing of dimensions $X = Y = Z = \frac{n}{2}$. The total number of bends is $2(n^2/4 - \frac{3}{2}n + 2) = n^2/2 - 3n + 4$.

Theorem 6. *K_n can be drawn in an $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$-grid with a total of $n^2/2 - 3n + 4$ bends and at most two bends per edge.*

4.3 An $O(n^{2.5})$ Volume drawing for $k = 3$

In this section, we draw K_n with at most $k = 3$ bends on any edge and with volume $O(n^{2.5})$. Case 2 of the lower bound proof suggests what general form such a drawing might take. For simplicity, assume below that $n = N^2$ for some integer N. Enumerate the vertices as ordered pairs (i, j), where $1 \leq i \leq N$, $1 \leq j \leq N$, and place vertex (i, j) at $(2i, 2j)$ in the 2-D x, y-plane. Suppose edge e joins vertex (i_1, j_1) and vertex (i_2, j_2). After possible renaming, we may assume that $i_1 \leq i_2$, and that if $i_1 = i_2$, then $j_1 > j_2$. Call e an *L-edge* if $j_1 > j_2$ and a *Γ-edge* otherwise.

Initially route each L-edge via the points $(2i_1, 2j_1), (2i_1+1, 2j_1), (2i_1+1, 2j_2+1), (2i_2, 2j_2+1), (2i_2, 2j_2)$, thus with three bends. Route each Γ-edge via points $(2i_1, 2j_1), (2i_1+1, 2j_1), (2i_1+1, 2j_2-1), (2i_2, 2j_2-1), (2i_2, 2j_2)$.

Split the L-edges into $N(N-1)$ groups E_{d_x, d_y}, with $0 \leq d_x \leq N-1$ and $1 \leq d_y \leq N-1$. Each group E_{d_x, d_y} consists of those edges $((i_1, j_1), (i_2, j_2))$ for which $i_2 = i_1 + d_x$ and $j_2 = j_1 - d_y$. These groups cover all L-edges since $i_1 \leq i_2$ and $j_1 > j_2$ for any L-edge.

Now split each group E_{d_x,d_y} into at most $d_x + d_y$ sets of edges as follows. For $p = 0, \ldots, d_x + d_y - 1$, let $E^p_{d_x,d_y}$ be the edges in E_{d_x,d_y} for which $j_2 - i_1 = p$ modulo (d_x+d_y). In other words, the lower left "corners" of the L-edges in $E^p_{d_x,d_y}$ lie on diagonals that intersect the y-axis at the value $2p$ modulo $(2d_x + 2d_y)$. See Fig. 6. It is easy to check that no two edges in $E^p_{d_x,d_y}$ overlap or intersect since the corners of the L's are placed on a sequence of diagonals having a vertical spacing of $2(d_x + d_y)$ between adjacent diagonals. Also, note that $E^p_{d_x,d_y}$ is non-empty only if $p \leq 2N - d_x - d_y$.[5]

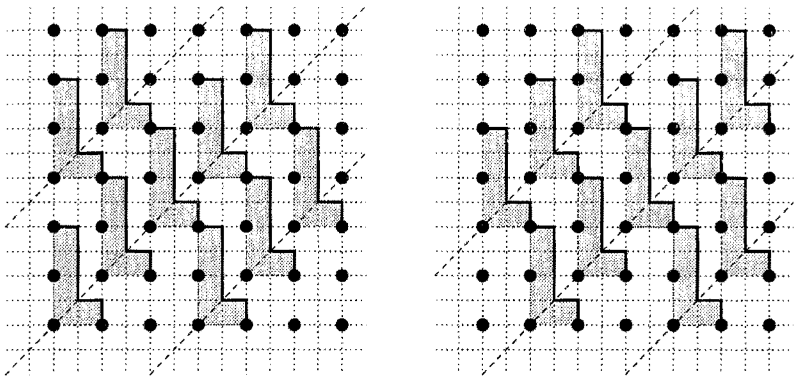

Fig. 6. The edge sets $E^0_{1,2}$ and $E^2_{1,2}$.

Assign a z-plane to each set $E^p_{d_x,d_y}$ to obtain a legal drawing of the L-edges. Route the Γ edges in an analogous fashion. This doubles the number of z-planes, yielding a drawing of K_n in a grid with $X = Y = 2N = 2\sqrt{n}$. The Z dimension is given by

$$2 \sum_{d_x=0}^{N-1} \sum_{d_y=1}^{N-1} \min\{d_x + d_y, 2N - d_x - d_y\}.$$

Some analysis shows that this sum is at most

$$2 \left[\sum_{k=1}^{N-1} k(2k-1) + (N-1)N \right] = \frac{2(N-1)N(2N-1)}{3} - (N-1)N + 2(N-1)N$$

which is less than $\frac{4}{3}N^3$. Every edge has three bends. However, the $2N(N-1) = 2n - 2\sqrt{n}$ edges where $d_x = 0$ and $d_y = 1$, or $d_x = 1$ and $d_y = 0$ can be drawn without a bend. So the total number of bends is $3(n^2/2 - n/2) - 3(2n - 2\sqrt{n}) = \frac{3}{2}n^2 - \frac{15}{2}n + 6\sqrt{n}$.

[5] A java applet demonstrating the sets and their routings for K_{100} can be found at `http://www.cs.uleth.ca/~wismath/ortho.html`.

Theorem 7. *If $n = N^2$ is a square, then K_n can be drawn in a $2N \times 2N \times \frac{4}{3}N^3$-grid (so volume $\frac{16}{3}n^{2.5}$) with $\frac{3}{2}n^2 - \frac{15}{2}n + 6\sqrt{n}$ bends and at most three bends per edge.*

5 Conclusions

This paper is one of the first to address volume and bend considerations for 3-D orthogonal grid drawings of graphs. The focus has been on K_n, since it is the most difficult graph on n vertices to draw in small volume or with restrictions on bends. In particular, we have

- provided a method for drawing K_n with volume that is provably within a constant factor (same constant for all n) of best possible in the case that at most k bends per edge are allowed, where $k \geq 3$;
- proved the non-existence of drawings of K_n for large n in the $k = 0$ case, where no bends are permitted;
- proved a lower bound of $\Omega(n^{2.5})$ and an upper bound of $O(n^3)$ on the volume of drawings of K_n when $k = 1$ and $k = 2$.

An open problem is to close the gap between the upper and lower bounds in the $k = 1$ and $k = 2$ cases, where at most 1 and at most 2 bends on each edge are permitted, respectively. The ideas and methods presented here may serve as a useful starting point for constructing drawings with good constant factors for volume and bends.

6 Acknowledgments

Thanks to Michael Kaufmann for discussions on orthogonal drawings. The joint results of this paper have also appeared as part of the PhD thesis of T. Biedl at Rutgers University.

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