

Applied Mathematical Sciences

Volume 167

Editors

S.S. Antman J.E. Marsden L. Sirovich

Advisors

J.K. Hale P. Holmes J. Keener

J. Keller B.J. Matkowsky A. Mielke

C.S. Peskin K.R. Sreenivasan

For further volumes:
<http://www.springer.com/series/34>

Otmar Scherzer
Harald Grossauer
Frank Lenzen

Markus Grasmair
Markus Haltmeier

Variational Methods in Imaging

With 72 Figures

 Springer

Otmar Scherzer
otmar.scherzer@uibk.ac.at

Markus Grasmair
markus.grasmair@uibk.ac.at

Harald Grossauer
harald.grossauer@uibk.ac.at

Markus Haltmeier
markus.haltmeier@uibk.ac.at

Frank Lenzen
frank.lenzen@uibk.ac.at

All affiliated with:
Department of Mathematics
University of Innsbruck
Techniker Str. 21a/2
6020 Innsbruck
Austria

Editors

S.S. Antman
Department of Mathematics
and
Institute for Physical Science
and Technology
University of Maryland
College Park, MD 20742-4015
USA
ssa@math.umd.edu

J.E. Marsden
Control and Dynamical
Systems, 107-81
California Institute of
Technology
Pasadena, CA 91125
USA
marsden@cds.caltech.edu

L. Sirovich
Laboratory of Applied
Mathematics
Department of
Biomathematical
Sciences
Mount Sinai School
of Medicine
New York, NY 10029-6574
USA
chico@camelot.mssm.edu

ISBN: 978-0-387-30931-6

e-ISBN: 978-0-387-69277-7

DOI: 10.1007/978-0-387-69277-7

Library of Congress Control Number: 2008934867

Mathematics Subject Classification (2000): 68U10

© 2009 Springer Science+Business Media, LLC

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

springer.com

This book is dedicated to *Zuhair Nashed* on the occasion of his 70th birthday. Zuhair has collaborated with Heinz Engl, University of Linz, Austria. Heinz Engl in turn has supervised Otmar Scherzer, who was also supervised afterwards by Zuhair during several long- and-short term visits in the USA. Finally, Markus Grasmair was supervised by Otmar Scherzer during his PhD studies, and the thesis was also evaluated by Zuhair. Three generations of mathematicians in Austria congratulate Zuhair and his family on his 70th birthday.

Otmar Scherzer also dedicates this book to his family: Roswitha, Anna, Simon, Heide, Kurt, Therese, Franz, Paula, and Josef.

Markus Haltmeier dedicates this book to his family.
Frank Lenzen dedicates this book to Bettina, Gisela, Dieter, and Ulli.

Preface

Imaging is an interdisciplinary research area with profound applications in many areas of science, engineering, technology, and medicine. The most primitive form of *imaging* is *visual inspection*, which has dominated the area before the technical and computer revolution era. Today, computer imaging covers various aspects of *data filtering*, *pattern recognition*, *feature extraction*, *computer aided inspection*, and *medical diagnosis*. The above mentioned areas are treated in different scientific communities such as *Imaging*, *Inverse Problems*, *Computer Vision*, *Signal and Image Processing*, . . . , but all share the common thread of recovery of an object or one of its properties.

Nowadays, a core technology for solving imaging problems is *regularization*. The foundations of these approximation methods were laid by Tikhonov in 1943, when he generalized the classical definition of *well-posedness* (this generalization is now commonly referred to as *conditional well-posedness*). The heart of this definition is to specify a *set of correctness* on which it is known *a priori* that the considered problem has a unique solution. In 1963, Tikhonov [371, 372] suggested what is nowadays commonly referred to as Tikhonov (or sometimes also Tikhonov–Phillips) regularization. The abstract setting of regularization methods presented there already contains all of the variational methods that are popular nowadays in imaging. Morozov’s book [277], which is the English translation of the Russian edition from 1974, is now considered the first standard reference on Tikhonov regularization.

In the early days of regularization methods, they were analyzed mostly theoretically (see, for instance, [191, 277, 278, 371–373]), whereas later on numerics, efficient solutions (see, for instance, the monographs [111, 204, 207, 378]), and applications of regularization methods became important (see, for instance, [49, 112–114]).

Particular applications (such as, for instance, segmentation) led to the development of specific variational methods. Probably the most prominent among them is the Mumford–Shah model [276, 284], which had an enormous impact on the analysis of regularization methods and revealed challenges for the efficient numerical solution (see, e.g., [86, 88]). However, it is

notable that the Mumford–Shah method also reveals the common features of the abstract form of Tikhonov regularization. In 1992, Rudin, Osher, and Fatemi published *total variation regularization* [339]. This paper had an enormous impact on theoretical mathematics and applied sciences. From an analytical point of view, properties of the solution of regularization functionals have been analyzed (see, for instance, [22]), and efficient numerical algorithms (see [90, 133, 304]) have been developed.

Another stimulus for regularization methods has come from the development of non-linear parabolic partial differential equations for *image denoising* and *image analysis*. Here we are interested in two types of evolution equations: *parabolic subdifferential inclusion* equations and *morphological* equations (see [8, 9, 194]). Subdifferential inclusion equations can be associated in a natural way with Tikhonov regularization functionals. This for instance applies to *anisotropic diffusion filtering* (see the monograph by Weickert [385]). As we show in this book, we can associate *non-convex* regularization functionals with morphological equations.

Originally, Tikhonov type regularization methods were developed with the emphasis on the stable solution of *inverse problems*, such as tomographical problems. These inverse problems are quite challenging to analyze and to solve numerically in an efficient way. In this area, mainly simple (quadratic) Tikhonov type regularization models have been used for a long time. In contrast, the underlying physical model in image analysis is simple (for instance, in denoising, the identity operator is inverted), but sophisticated regularization techniques are used. This discrepancy between the different scientific areas led to a split.

The abstract formulation of Tikhonov regularization can be considered in *finite dimensional* space setting as well as in *infinite dimensional function space* setting, or in a combined *finite-infinite* dimensional space setting. The latter is frequently used in spline and wavelet theory. Moreover, we mention that Tikhonov regularization can be considered in a *deterministic* setting as well as in a *stochastic* setting (see, for instance, [85, 231]).

This book attempts to bridge the gap between the two research areas of image analysis and imaging problems in inverse problems and to find a common language. However, we also emphasize that our research is biased toward *deterministic* regularization and, although we use statistics to motivate regularization methods, we do not make the attempt to give a stochastic analysis.

For applications of imaging, we have chosen examples from our own research experience, which are *denoising*, *telescope imaging*, *thermoacoustic imaging*, and *schlieren tomography*. We do not claim that these applications are most representative for imaging. Certainly, there are many other active research areas and applications that are not touched in this book.

Of course, this book is not the only one in the field of *Mathematical Imaging*. We refer for instance to [26, 98]. Imaging from an inverse problems point of view is treated in [49]. There exists also a vast number of proceedings and

edited volumes that are concerned with mathematical imaging; we do not provide detailed references on these volumes. Another branch of imaging is mathematical methods in tomography, where also a vast amount of literature exists. We mention exemplarily the books [232, 288, 289].

The objective of this book certainly is to bridge the gap between regularization theory in image analysis and in inverse problems, noting that both areas have developed relatively independently for some time.

Acknowledgments

The authors are grateful for the support of the Austrian Science Foundation (FWF), which supported the authors during writing of the book. The relevant supporting grants are Y-123 INF, FSP 92030, 92070, P18172-N02, S10505.

Moreover, Otmar Scherzer is grateful to the Radon Institute in Linz and the available research possibilities there.

The authors thank the Infmath group in Innsbruck and the Imaging group in Linz for their proofreading. We are grateful to many researchers that stimulated our research and spared much time for discussion.

Otmar Scherzer acknowledges the possibility to teach preliminary parts of the book in summer schools in Vancouver (thanks to Ian Frigaard), in Jyväskylä (thanks to Kirsi Majava), and at CMLA, Paris (thanks to Mila Nikolova).

The authors are grateful to GE Medical Systems Kretz Ultrasound AG for providing the ultrasound data frequently used in the book as test data. Moreover, the authors thank Vaishali Damle and Marcia Bunda of Springer New York for their constant support during the preparation of the book.

Innsbruck,
2008

*Markus Grasmair, Harald Grossauer, Markus Haltmeier,
Frank Lenzen, Otmar Scherzer*

Contents

Part I Fundamentals of Imaging

1	Case Examples of Imaging	3
1.1	Denoising	3
1.2	Chopping and Nodding	6
1.3	Image Inpainting	8
1.4	X-ray-Based Computerized Tomography	10
1.5	Thermoacoustic Computerized Tomography	13
1.6	Schlieren Tomography	24
2	Image and Noise Models	27
2.1	Basic Concepts of Statistics	27
2.2	Digitized (Discrete) Images	31
2.3	Noise Models	33
2.4	Priors for Images	36
2.5	Maximum A Posteriori Estimation	43
2.6	MAP Estimation for Noisy Images	46

Part II Regularization

3	Variational Regularization Methods for the Solution of Inverse Problems	53
3.1	Quadratic Tikhonov Regularization in Hilbert Spaces	54
3.2	Variational Regularization Methods in Banach Spaces	60
3.3	Regularization with Sparsity Constraints	79
3.4	Linear Inverse Problems with Convex Constraints	89
3.5	Schlieren Tomography	109
3.6	Further Literature on Regularization Methods for Inverse Problems	112

4	Convex Regularization Methods for Denoising	115
4.1	The \ast -Number	120
4.2	Characterization of Minimizers	125
4.3	One-dimensional Results	131
4.4	Taut String Algorithm	137
4.5	Mumford–Shah Regularization	151
4.6	Recent Topics on Denoising with Variational Methods	155
5	Variational Calculus for Non-convex Regularization	159
5.1	Direct Methods	160
5.2	Relaxation on Sobolev Spaces	162
5.3	Relaxation on BV	167
5.4	Applications in Non-convex Regularization	172
5.5	One-dimensional Results	178
5.6	Examples	180
6	Semi-group Theory and Scale Spaces	185
6.1	Linear Semi-group Theory	186
6.2	Non-linear Semi-groups in Hilbert Spaces	190
6.3	Non-linear Semi-groups in Banach Spaces	193
6.4	Axiomatic Approach to Scale Spaces	197
6.5	Evolution by Non-convex Energy Functionals	200
6.6	Enhancing	202
7	Inverse Scale Spaces	205
7.1	Iterative Tikhonov Regularization	206
7.2	Iterative Regularization with Bregman Distances	209
7.3	Recent Topics on Evolutionary Equations for Inverse Problems	217

Part III Mathematical Foundations

8	Functional Analysis	221
8.1	General Topology	221
8.2	Locally Convex Spaces	224
8.3	Bounded Linear Operators and Functionals	227
8.4	Linear Operators in Hilbert Spaces	231
8.5	Weak and Weak \ast Topologies	234
8.6	Spaces of Differentiable Functions	237
9	Weakly Differentiable Functions	239
9.1	Measure and Integration Theory	239
9.2	Distributions and Distributional Derivatives	248
9.3	Geometrical Properties of Functions and Domains	250
9.4	Sobolev Spaces	254

9.5	Convolution	261
9.6	Sobolev Spaces of Fractional Order	262
9.7	Bochner Spaces	263
9.8	Functions of Bounded Variation	265
10	Convex Analysis and Calculus of Variations	273
10.1	Convex and Lower Semi-continuous Functionals	274
10.2	Fenchel Duality and Subdifferentiability	276
10.3	Duality Mappings	280
10.4	Differentiability of Functionals and Operators	281
10.5	Derivatives of Integral Functionals on $L^p(\Omega)$	284
References		287
Nomenclature		309
Index		315