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Variational Methods in Imaging

With 72 Figures

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This book is dedicated to *Zuhair Nashed* on the occasion of his 70th birthday. Zuhair has collaborated with Heinz Engl, University of Linz, Austria. Heinz Engl in turn has supervised Otmar Scherzer, who was also supervised afterwards by Zuhair during several long- and-short term visits in the USA. Finally, Markus Grasmair was supervised by Otmar Scherzer during his PhD studies, and the thesis was also evaluated by Zuhair. Three generations of mathematicians in Austria congratulate Zuhair and his family on his 70th birthday.

Otmar Scherzer also dedicates this book to his family: Roswitha, Anna, Simon, Heide, Kurt, Therese, Franz, Paula, and Josef.
Markus Haltmeier dedicates this book to his family.
Frank Lenzen dedicates this book to Bettina, Gisela, Dieter, and Ulli.

Preface

Imaging is an interdisciplinary research area with profound applications in many areas of science, engineering, technology, and medicine. The most primitive form of *imaging* is *visual inspection*, which has dominated the area before the technical and computer revolution era. Today, computer imaging covers various aspects of *data filtering*, *pattern recognition*, *feature extraction*, *computer aided inspection*, and *medical diagnosis*. The above mentioned areas are treated in different scientific communities such as *Imaging*, *Inverse Problems*, *Computer Vision*, *Signal* and *Image Processing*, ..., but all share the common thread of recovery of an object or one of its properties.

Nowadays, a core technology for solving imaging problems is *regularization*. The foundations of these approximation methods were laid by Tikhonov in 1943, when he generalized the classical definition of *well-posedness* (this generalization is now commonly referred to as *conditional well-posedness*). The heart of this definition is to specify a *set of correctness* on which it is known *a priori* that the considered problem has a unique solution. In 1963, Tikhonov [371, 372] suggested what is nowadays commonly referred to as Tikhonov (or sometimes also Tikhonov–Phillips) regularization. The abstract setting of regularization methods presented there already contains all of the variational methods that are popular nowadays in imaging. Morozov's book [277], which is the English translation of the Russian edition from 1974, is now considered the first standard reference on Tikhonov regularization.

In the early days of regularization methods, they were analyzed mostly theoretically (see, for instance, [191, 277, 278, 371–373]), whereas later on numerics, efficient solutions (see, for instance, the monographs [111, 204, 207, 378]), and applications of regularization methods became important (see, for instance, [49, 112–114]).

Particular applications (such as, for instance, segmentation) led to the development of specific variational methods. Probably the most prominent among them is the Mumford–Shah model [276, 284], which had an enormous impact on the analysis of regularization methods and revealed challenges for the efficient numerical solution (see, e.g., [86, 88]). However, it is notable that the Mumford–Shah method also reveals the common features of the abstract form of Tikhonov regularization. In 1992, Rudin, Osher, and Fatemi published *total variation regularization* [339]. This paper had an enormous impact on theoretical mathematics and applied sciences. From an analytical point of view, properties of the solution of regularization functionals have been analyzed (see, for instance, [22]), and efficient numerical algorithms (see [90, 133, 304]) have been developed.

Another stimulus for regularization methods has come from the development of non-linear parabolic partial differential equations for *image denoising* and *image analysis*. Here we are interested in two types of evolution equations: *parabolic subdifferential inclusion* equations and *morphological* equations (see [8, 9, 194]). Subdifferential inclusion equations can be associated in a natural way with Tikhonov regularization functionals. This for instance applies to *anisotropic diffusion filtering* (see the monograph by Weickert [385]). As we show in this book, we can associate *non-convex* regularization functionals with morphological equations.

Originally, Tikhonov type regularization methods were developed with the emphasis on the stable solution of *inverse problems*, such as tomographical problems. These inverse problems are quite challenging to analyze and to solve numerically in an efficient way. In this area, mainly simple (quadratic) Tikhonov type regularization models have been used for a long time. In contrast, the underlying physical model in image analysis is simple (for instance, in denoising, the identity operator is inverted), but sophisticated regularization techniques are used. This discrepancy between the different scientific areas led to a split.

The abstract formulation of Tikhonov regularization can be considered in *finite dimensional* space setting as well as in *infinite dimensional function space* setting, or in a combined *finite-infinite* dimensional space setting. The latter is frequently used in spline and wavelet theory. Moreover, we mention that Tikhonov regularization can be considered in a *deterministic* setting as well as in a *stochastic* setting (see, for instance, [85, 231]).

This book attempts to bridge the gap between the two research areas of image analysis and imaging problems in inverse problems and to find a common language. However, we also emphasize that our research is biased toward *deterministic* regularization and, although we use statistics to motivate regularization methods, we do not make the attempt to give a stochastic analysis.

For applications of imaging, we have chosen examples from our own research experience, which are *denoising*, *telescope imaging*, *thermoacoustic imaging*, and *schlieren tomography*. We do not claim that these applications are most representative for imaging. Certainly, there are many other active research areas and applications that are not touched in this book.

Of course, this book is not the only one in the field of *Mathematical Imaging*. We refer for instance to [26, 98]. Imaging from an inverse problems point of view is treated in [49]. There exists also a vast number of proceedings and edited volumes that are concerned with mathematical imaging; we do not provide detailed references on these volumes. Another branch of imaging is mathematical methods in tomography, where also a vast amount of literature exists. We mention exemplarily the books [232, 288, 289].

The objective of this book certainly is to bridge the gap between regularization theory in image analysis and in inverse problems, noting that both areas have developed relatively independently for some time.

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Markus Grasmair, Harald Grossauer, Markus Haltmeier, Frank Lenzen, Otmar Scherzer

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