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Integration and Modern Analysis

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To my wife, Catherine

Moim Rodzicom

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Preface

This book is both a text and a paean to twentieth-century real variables, measure theory, and integration theory. As a text, the book is aimed at graduate students. As an exposition, extolling this area of analysis, the book is necessarily limited in scope and perhaps unnecessarily unlimited in idiosyncrasy.

More than half of this book is a fundamental graduate real variables course as we now teach it. Since there are excellent textbooks that generally cover the course material herein, part of this Preface renders an apologia for our content, presentation, and existence. The following section presents our syllabus properly liberated from too many demands. Subsequent sections deal with outline, theme, features, and the roles of Fourier analysis and Vitali, respectively.

Mathematics is a creative adventure driven by beauty, structure, intrinsic mathematical problems, extrinsic problems from engineering and the sciences, and serendipity. This book treats integration theory and its fascinating creation through the past century.

What about the rest of our title?

"Analysis" is many subjects to many mathematicians. "Modern analysis" is hardly a constraint for a single volume such as ours; one can argue the opposite. Differentiation and integration are still the essence of analysis, and, along with "integration", the title could very well have included the word "differentiation" because of our emphasis on it. Guided by the creativity of mathematics, our title is meant to assert that the technology we have recorded is a basis for many of the analytic adventures of our time.

Syllabus

We shall outline the material we have used in teaching a first-year graduate course in real analysis. Sometimes a student will take only the first semester of this two-semester sequence.

- Chapter 1. Sections 1.1, 1.2, 1.3.1, 1.3.2, 1.3.3.
- Chapter 2. Sections 2.1, 2.2, 2.4, 2.5, with emphasis on Sections 2.2 and 2.5.

- Chapter 3. Sections 3.1, 3.2, 3.3, 3.7.
- Chapter 4. Sections 4.1, 4.2, 4.3, 4.4, 4.5.
- Chapter 5. Sections 5.1, 5.2, 5.3, 5.4, 5.5.

Exercises are listed in the problem sections of the first five chapters. The problems range in difficulty from the routine to the challenging; and they are computational, theoretical, and often provide perspective. There are suggested problems listed at the beginning of each of these five chapters. These problems are appropriate for the course. There are no problem sections in the remaining chapters, which can be considered as providing special topics material in a full-year course.

Typically, finer points will come later in a section than basic, essential theory. For example, in recommending Section 3.3 in the course syllabus, we note that the items are numbered 3.3.1–3.3.15 in Section 3.3, whereas items 3.3.1–3.3.6 are sufficient for the course. This caveat to the recommended sections listed above is crucial for presenting the essential topics for a graduate course in real analysis.

Chapters 1, 2, 3 and parts of Chapter 4 and Appendix A are the usual content of the first-semester graduate course in real analysis. Chapter 5 is essential for the second-semester graduate course in real analysis along with other parts of Chapter 4 and Appendix A. There will usually be some time remaining in the second semester, and the authors have presented Fourier analysis (Appendix B) or some of the material in Chapters 6–9. Naturally, instructors should present their favorite applications of the theory in this remaining time.

Students are encouraged to read the Potpourri and Titillation sections at the ends of chapters. These sections are meant to be educational and fun, providing some breadth without depth.

The book may be used not only for a graduate course in real analysis, but also for advanced independent study, or as a source for student projects.

Outline

The basic text, Chapters 1 through 5, begins with the fundamental classical problems that led to Lebesgue's definition of the integral, and develops the theory of integration and the structure of measures in a measure-theoretical format. Chapter 1 is meant to be a pointed account of classical real variables theory. We have inserted details for some of the elementary material; but there is a bloc of more advanced matter, with details omitted, which should be read for perspective. Of course, the instructor may choose to develop this material more fully at any time.

Chapters 2 through 5 form the core of the course, and they are presented in full detail. Chapters 2 and 3 are Lebesgue's theory of measure and integration. Chapter 4 presents the fundamental theorem of calculus

(FTC), and Chapter 5 deals with spaces of measures and the Radon–Nikodym theorem (R–N). The motivation for Lebesgue's theory viewed from the FTC is based on Volterra's example in Chapter 1 of a differentiable function whose derivative is not Riemann integrable.

The remaining chapters, Chapters 6 through 9, are also systematically presented, but could be viewed as eclectic topics by some readers. We view them as essential to our intellectual vision of twentieth-century real analysis.

Chapter 6 is devoted to deep results by Vitali, Nikodym, Hahn–Saks, Dieudonné, Dunford–Pettis, and Grothendieck as they relate to interchanging limits and integration, and to the characterization of weak sequential convergence of measures. Our goal has been to highlight the importance of this material and to show how it fits centrally into integration theory.

We prove the Riesz representation theorem (RRT) in Chapter 7 in the setting of locally compact Hausdorff spaces. We begin historically with Riesz' original proof and are led to Radon measures and Laurent Schwartz' theory of distributions. The RRT establishes the equivalence of the set-theoretic measure theory of the previous chapters with the theory of Radon measures considered functional-analytically as continuous linear functionals. It is striking that on $\mathbb{R}$, the RRT asserts that a Schwartz distribution is a bounded Radon measure if and only if it is the first distributional derivative of a function of bounded variation. We view this material as a quantitative approach to apply measure theory in harmonic analysis, partial differential equations, and distribution theory. In Schwartz' obituary (Notices Amer. Math. Soc. 50 (2003), 1072–1084), it is noted that Schwartz became disenchanted with Bourbaki's presentation of measure theory in the setting of Radon measures on locally compact Hausdorff spaces, even though he was on the Bourbaki writing group for this material. His main objection was concerned with its inadequacy in dealing with the research of probabilists, e.g., Paul Lévy and Joseph L. Doob, establishing measure theory on infinite-dimensional spaces such as $C([0,1])$. Notwithstanding our awe of Schwartz, we have presented the Bourbakist point of view. We think that it is spectacularly beautiful and unifying as far as it goes. Further, as illustrated in Chapter 7, its capacity to transmute the Riesz representation theorem from theorem to definition is stunning. It is also a natural setting for developing Schwartz' own theory of distributions.

Chapter 8 develops differentiation theory on Euclidean space $\mathbb{R}^d$, and proves the d-dimensional version of the Lebesgue differentiation theorem on $\mathbb{R}$, which itself was proved in Chapter 4. Substantial technology is required, which includes the notion of bounded variation on $\mathbb{R}^d$, Vitali and Besicovich covering lemmas, maximal functions, rearrangement inequalities, and a semimartingale maximal theorem. This material has also had significant impact on other areas of mathematics. We close in Chapter 9 by analyzing self-similar sets and fractals. In a sense, this material comes full circle from Chapter 1 by presenting a modern treatment and generalization of our analysis of Cantor sets, which were vital in classical real variables and the development of measure theory.

We have included two appendices: functional analysis (Appendix A) and Fourier analysis (Appendix B). Functional analysis is a significant area of twentieth-century analysis, and it is an integral part of the structure and language of modern analysis. Appendix A provides those topics that we use throughout Chapters 1–9, and it is also a standalone outline of a broadly useful part of functional analysis. Appendix B is more of a luxury, meant to provide the background for some of the Fourier-analytic examples we give, as well as a prelude for harmonic analysis, establishing its dependence on the material in the book.

Theme

One of our themes is the notion of *absolute continuity* and its role as the unifying concept for the major results of the theory, viz., the fundamental theorem of calculus (FTC) (Chapter 4), the Lebesgue dominated convergence theorem (LDC) (Chapter 3), and the Radon–Nikodym theorem (R–N) (Chapter 5). The main mathematical reason that we have written this book is that none of the other texts in the area stresses this issue to the extent that we think it should be stressed.

Let us be more specific.

The problem of taking limits under the integral sign, that is, "switching limits", is in a very real sense the fundamental problem in real analysis. Lebesgue's axiomatization that formulates and proves LDC in an optimal way yields the most important general technique for examining such problems. This material is developed in Chapter 3. Shortly after Lebesgue's initial work, Vitali gave necessary and sufficient conditions for interchanging limits in terms of uniform absolute continuity. Vitali's result led to research that has culminated in the Vitali–Hahn–Saks theorem and in Grothendieck's study of weak convergence of measures. This material, found in Chapter 6, is usually not included in most texts; in particular, its relationship to LDC is not emphasized.

Knowledge of the structure of measures provides an important tool in potential theory, harmonic analysis, probability theory, and nonlinear dynamics, and it plays a role in a host of other subjects from number theory to representation theory. Its scope of application ranges from establishing a mathematical model for the continuous spectrum of white light to formulating the action of the stock market as Brownian motion in terms of the Wiener measure.

A key theorem in this milieu is R–N, and the major results involve decompositions of a given measure into various parts with specific properties. R–N can be considered as a generalization of the FTC for the case of functions defined on the real line or of the Green and Stokes theorems in Euclidean space. Of course, the FTC, which is the basic and amazing relationship by which integration and differentiation are formulated as inverse operations, is characterized in terms of absolute continuity. We have dwelled on these issues in Chapters 4 and 5. We give the classical point function results, study the abstract setting, examine their relation, and spend a good deal of time with examples. The Fubini–Tonelli theorem (Section 3.7), one of the most important theorems in analysis and a classical case of interchanging operations, can be related to R–N from the point of view of conditional probabilities; and LDC is used in its proof.

Features

Besides the theme of absolute continuity there are features of a more secular nature. We have included some extensive historical and motivational passages. Integration theory did not develop in a vacuum, and we have presented information on the development of Fourier series because of its close relation with many of the notions from real analysis; see the following section. Our problem sets include certain types of problems that abound in the folklore (e.g., the Amer. Math. Monthy), but which are generally omitted from a full-year real analysis text whose purpose is to present systematically n topics. As mentioned in describing the syllabus, some of the problems are quite difficult and will probably challenge even the most mathochistic student.

We hope that the historical remarks, the sections on Potpourri and Titillation, and the problems (especially the harder ones) are *read*, since we believe that they provide relevant perspective. A list of these and other features follows.

- Historical approach and technical perspective on the development of real variables
- Role of classical topics in modern analysis
- Extensive array of problems, ranging from the routine to the challenging, with background, related issues, and hints
- Brief, illuminating biographies
- Many examples, from straightforward to profound
- Treatment of symmetric perfect sets, used as a foundation for the introduction of fractal analysis
- Important role of Vitali for original results and modern proofs given 100 years ago
- Applications to Fourier analysis and fractal geometry
- Comprehensive appendix on elementary functional analysis
- Introductory appendix on Fourier analysis
- Complete indexes of terms, names, and notation
- Substantial selection of references, including original works by the founders of real analysis as well as relevant modern publications

With regard to references, there are a disproportionate number of references to the first-named author, mostly because of perspectives he has written on various topics. These references should generally be viewed as literature sources as opposed to research results.

Topics

- Distributional formulation of the Riesz representation theorem
- A unified theory of measure and integral
- Haar measure
- Functional-analytic equivalence of spaces of measures
- Vitali–Nikodym–Hahn–Saks characterization for interchanging limits
- Weak convergence of measures
- Hausdorff measure
- Maximal functions and Lebesgue differentiation theorem on $\mathbb{R}^d$

Fourier Analysis

In discussing Grothendieck's idea of bringing certain cohomological concepts into algebraic geometry, Fields' medalist David Mumford wrote, "It completely turned the field upside down. It's like analysis before and after Fourier. Once you get Fourier techniques, suddenly you have this whole deep insight into a way of looking at a function" (Notices Amer. Math. Soc. 51 (2001), 1052). It is for this reason, and going beyond the notion of a function to differentiation and integration, where Fourier analysis was also a driving force, that we have included as much Fourier analysis in the text as we have. Of course, this intellectual rationale has to be coupled with the authors' ignorance of so many other aspects of modern analysis, such as partial differential equations, vector measures, stochastic integration, ergodic theory, geometric measure theory, and probability theory. (For measure-theoretic motivation, we do, however, outline some elementary probability theory in the Potpourri and Titillation sections of several chapters.)

Historically, Fourier series were developed in the nineteenth-century for the analysis of some of the classical partial differential equations (PDEs) of mathematical physics; and these series were used to solve such equations. In order to understand Fourier series and what sorts of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concepts of "function" and "bounded variation". The simplest linear PDEs were eventually understood to be convolutions, which could be Fourier transformed into algebraic equations. When these latter equations could be solved, an inverse Fourier transform was required to write the solution of the original PDE. Uniqueness questions naturally arose. Further, since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was developed because of such uniqueness questions.

Nineteenth-century real variables and integration theory went hand in hand with Fourier analysis into the twentieth-century, along with new stimulation from probability theory and statistical mechanics. Lebesgue created his magnificent theory, which is the centerpiece of this book, and which has become the setting for harmonic analysis, as well as the inspiration and language for so much analysis.

Vitali

The catalyst for writing this book occurred many years ago, during the academic year 1970–1971, when one of the authors was a guest at the Scuola Normale Superiore in Pisa. At that time he discovered Vitali's work at a different level from what he had previously known. Vitali is responsible for the notion of absolute continuity, the first nonmeasurable set, the first necessary and sufficient conditions for LDC, the first statement and proof of Lusin's theorem, "modern" proofs given 100 years ago, and more, as you will read in the text.

We have decided to engage in some advertising for this most important figure in integration theory.

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