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Medial Representations

Mathematics, Algorithms and Applications

Edited by

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This book is dedicated to Harry Blum, who started it all.

Preface

The last half century has seen the development of many biological or physical theories that have explicitly or implicitly involved medial descriptions of objects and other spatial entities in our world. Simultaneously mathematicians have studied the properties of these skeletal descriptions of shape, and, stimulated by the many areas where medial models are useful, computer scientists and engineers have developed numerous algorithms for computing and using these models. We bring this knowledge and experience together into this book in order to make medial technology more widely understood and used.

The book consists of an introductory chapter, two chapters on the major mathematical results on medial representations, five chapters on algorithms for extracting medial models from boundary or binary image descriptions of objects, and three chapters on applications in image analysis and other areas of study and design. We hope that this book will serve the science and engineering communities using medial models and will provide learning material for students entering this field.

We are fortunate to have recruited many of the world leaders in medial theory, algorithms, and applications to write chapters in this book. We thank them for their significant effort in preparing their contributions. We have edited these chapters and have combined them with the five chapters that we have written to produce an integrated whole.

We are very grateful to Vrinda Narain and Lyn Pizer, both for their generous support and for hosting each of us as we met on several occasions in Montréal and Chapel Hill to write this book. We are grateful to one another for the many ideas we have shared, discussed and refined since this project began. We have learned much from this collaboration. We are indebted to the UNC graduate students and visitors who took the course on Medial Representations in the fall of 2006; their input in identifying needed improvements in a draft of this book was invaluable. We thank Delphine Bull and Kim Jones for their technical help with preparing this manuscript.

Montréal and Chapel Hill — *Kaleem Siddiqi*
July 2007 — *Stephen M. Pizer*

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