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John Guckenheimer
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Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields

With 206 Illustrations

Springer

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To
G. Duffing, E. N. Lorenz and B. van der Pol,
pioneers in a chaotic land

Correspondances

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familiers.

Comme de longs échos qui de loin se confondent
Dans une ténébreuse et profonde unité,
Vaste comme la nuit et comme la clarté,
Les parfums, les couleurs et les sons se répondent.

Il est des parfums frais comme des chairs d'enfants,
Doux comme les hautbois, verts comme les prairies,
— Et d'autres, corrompus, riches et triomphants,

Ayant l'expansion des choses infinies,
Comme l'ambre, le musc, le benjoin et l'encens,
Qui chantent les transports de l'esprit et des sens.

CHARLES BAUDELAIRE
Les Fleurs du Mal, 1857

Preface

Introductory Remarks

Problems in dynamics have fascinated physical scientists (and mankind in general) for thousands of years. Notable among such problems are those of celestial mechanics, especially the study of the motions of the bodies in the solar system. Newton's attempts to understand and model their observed motions incorporated Kepler's laws and led to his development of the calculus. With this the study of models of dynamical problems as differential equations began.

In spite of the great elegance and simplicity of such equations, the solution of specific problems proved remarkably difficult and engaged the minds of many of the greatest mechanicians and mathematicians of the eighteenth and nineteenth centuries. While a relatively complete theory was developed for linear ordinary differential equations, nonlinear systems remained largely inaccessible, apart from successful applications of perturbation methods to weakly nonlinear problems. Once more, the most famous and impressive applications came in celestial mechanics.

Analysis remained the favored tool for the study of dynamical problems until Poincaré's work in the late-nineteenth century showed that perturbation methods might not yield correct results in all cases, because the series used in such calculations diverged. Poincaré then went on to marry analysis and geometry in his development of a qualitative approach to the study of differential equations.

The modern methods of qualitative analysis of differential equations have their origins in this work (Poincaré [1880, 1890, 1899]) and in the work of Birkhoff [1927], Liapunov [1949], and others of the Russian School: Andronov and co-workers [1937, 1966, 1971, 1973] and Arnold [1973, 1978, 1982]. In the past 20 years there has been an explosion of research. Smale, in a classic paper [1967], outlined a number of outstanding problems and stimulated much of this work. However, until the mid-1970s the new tools were largely in the hands of pure mathematicians, although a number of potential applications had been sketched, notably by Ruelle and Takens [1971], who suggested the importance of "strange attractors" in the study of turbulence.

Over the past few years applications in solid and structural mechanics as well as fluid mechanics have appeared, and there is now widespread interest in the engineering and applied science communities in strange attractors, chaos, and dynamical systems theory. We have written this book primarily for the members of this community, who do not generally have the necessary mathematical background to go directly to the research literature. We see the book primarily as a "user's guide" to a rapidly growing field of knowledge. Consequently we have selected for discussion only those results which we feel are applicable to physical problems, and have generally excluded proofs of theorems which we do not feel to be illustrative of the applicability. Nor have we given the sharpest or best results in all cases, hoping rather to provide a background on which readers may build by direct reference to the research literature.

This is far from a complete treatise on dynamical systems. While it may irritate some specialists in this field, it will, we hope, lead them in the direction of important applications, while at the same time leading engineers and physical scientists in the direction of exciting and useful "abstract" results. In writing for a mixed audience, we have tried to maintain a balance in our statement of results between mathematical pedantry and readability for those without formal mathematical training. This is perhaps most noticable in the way we define terms. While major new terms are defined in the traditional mathematical fashion, i.e., in a separate paragraph signalled by the word **Definition**, we have defined many of the more familiar terms as they occur in the body of the text, identifying them by *italics*. Thus we formally define *structural stability* on p. 39, while we define *asymptotic stability* (of a fixed point) on p. 3. For the reader's convenience, the index contains references to the terms defined in both manners.

The approach to dynamical systems which we adopt is a geometric one. A quick glance will reveal that this book is liberally sprinkled with illustrations—around 200 of them! Throughout we stress the geometrical and topological properties of solutions of differential equations and iterated maps. However, since we also wish to convey the important analytical underpinning of these illustrations, we feel that the numerous exercises, many of which require nontrivial algebraic manipulations and even computer work, are an essential part of the book. Especially in Chapter 2, the direct experience of watching graphical displays of numerical solutions to the systems of differential equations introduced there is extraordinarily valuable in developing an intuitive feeling for their properties. To help the reader along, we have tried to indicate which exercises are fairly routine applications of theory and which require more substantial effort. However, we warn the reader that, towards the end of the book, and especially in Chapter 7, some of our exercises become reasonable material for Ph.D. theses.

We have chosen to concentrate on applications in nonlinear oscillations for three reasons:

(1) There are many important and interesting problems in that field.
(2) It is a fairly mature subject with many texts available on the classical methods for analysis of such problems: the books of Stoker [1950], Minorsky [1962], Hale [1962], Hayashi [1964], or Nayfeh and Mook [1979] are good representatives. The geometrical analysis of two-dimensional systems (free oscillations) is also well established in the books of Lefschetz [1957] and Andronov and co-workers [1966, 1971, 1973].
(3) Most abstract mathematical examples known in dynamical systems theory occur "naturally" in nonlinear oscillator problems.

In this context, the present book should be seen as an attempt to extend the work of Andronov *et al.* [1966] by one dimension. This aim is not as modest as it might seem: as we shall see, the apparently innocent addition of a (small) periodic forcing term $f(t) = f(t + T)$ to a single degree of freedom nonlinear oscillator,

$$\ddot{x} + g(x, \dot{x}) = 0,$$

to yield the three-dimensional system

$$\ddot{x} + g(x, \dot{x}) = f(t),$$

or

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x, y) + f(\theta), \\ \dot{\theta} &= 1, \end{aligned}$$

can introduce an uncountably infinite set of new phenomena, in addition to the fixed points and limit cycles familiar from the planar theory of nonlinear oscillations. This book is devoted to a partial description and understanding of these phenomena.

A somewhat glib observation, which, however, contains some truth, is that the pure mathematician tends to think of some nice (or nasty) property and then construct a dynamical system whose solutions exhibit it. In contrast, the traditional rôle of the applied mathematician or engineer is to take a given system (perhaps a model that he or she has constructed) and find out what its properties are. We mainly adopt the second viewpoint, but our exposition may sometimes seem schizophrenic, since we are applying ideas of the former group to the problems of the latter group. Moreover, we feel strongly that the properties of specific systems cannot be discovered unless one knows what the possibilities are, and these are often revealed only by the general abstract theory. Practice and theory progress best hand-in-hand.

The Contents of This Book

This book is concerned with the application of methods from dynamical systems and bifurcation theories to the study of nonlinear oscillations. The mathematical models we consider are (fairly small) sets of ordinary differential equations and mappings. Many of the results discussed in this book can be extended to infinite-dimensional evolution systems arising from partial differential equations. However, the main ideas are most easily seen in the finite-dimensional context, and it is here that we shall remain. Almost all the methods we describe also generalize to dynamical systems whose phase spaces are differentiable manifolds, but once more, so as not to burden the reader with technicalities, we restrict our exposition to systems with Euclidean phase spaces. However, in the final section of the last chapter we add a few remarks on partial differential equations.

In Chapter 1 we provide a *review* of basic results in the theory of dynamical systems, covering both ordinary differential equations (flows) and discrete mappings. (We concentrate on diffeomorphisms: smooth invertible maps.) We discuss the connection between diffeomorphisms and flows obtained by their Poincaré maps and end with a review of the relatively complete theory of two-dimensional differential equations. Our discussion moves quickly and is quite cursory in places. However, the bulk of this material has been treated in greater detail from the dynamical systems viewpoint in the books of Hirsch and Smale [1974], Irwin [1980], and Palis and de Melo [1982], and from the oscillations viewpoint in the books of Andronov and his co-workers [1966, 1971, 1973] and we refer the reader to these texts for more details. Here the situation is fairly straightforward and solutions generally behave nicely.

Chapter 2 presents four examples from nonlinear oscillations: the famous oscillators of van der Pol [1927] and Duffing [1918], the Lorenz equations [1963], and a bouncing ball problem. We show that the solutions of these problems can be markedly chaotic and that they seem to possess strange attractors: attracting motions which are neither periodic nor even quasiperiodic. The development of this chapter is not systematic, but it provides a preview of the theory developed in the remainder of the book. We recommend that either the reader skim this chapter to gain a general impression before going on to our systematic development of the theory in later chapters, or read it with a microcomputer at hand, so that he can simulate solutions of the model problems we discuss.

We then retreat from the chaos of these examples to muster our forces. Chapter 3 contains a discussion of the methods of local bifurcation theory for flows and maps, including center manifolds and normal forms. Rather different, less geometrical, and more analytical discussions of local bifurcations can be found in the recent books by Iooss and Joseph [1981] and Chow and Hale [1982].

In Chapter 4 we develop the analytical methods of averaging and perturbation theory for the study of periodically forced nonlinear oscillators, and show that they can yield surprising global results. We end this chapter with a brief discussion of chaos and nonintegrability in Hamiltonian systems and the Kolmogorov–Arnold–Moser theory. More complete introductions to this area can be found in Arnold [1978], Lichtenberg and Lieberman [1982], or, for the more mathematically inclined, Abraham and Marsden [1978].

In Chapter 5 we return to chaos, or rather to the close analysis of geometrically defined two-dimensional maps with complicated invariant sets. The famous horseshoe map of Smale is discussed at length, and the method of symbolic dynamics is described and illustrated. A section on one-dimensional (noninvertible) maps is included, and we return to the specific examples of Chapter 2 to provide additional information and illustrate the analytical methods. We end this chapter with a brief discussion of Liapunov exponents and invariant measures for strange attractors.

In Chapter 6 we discuss global homoclinic and heteroclinic bifurcations, bifurcations of one-dimensional maps, and once more illustrate our results with the examples of Chapter 2. Finally, in our discussion of global bifurcations of two-dimensional maps and wild hyperbolic sets, we arrive squarely at one of the present frontiers of the field. We argue that, while the one-dimensional theory is relatively complete (cf. Collet and Eckmann [1980]), the behavior of two-dimensional diffeomorphisms appears to be considerably more complex and is still incompletely understood. We are consequently unable to complete our analysis of the nonlinear oscillators of van der Pol and Duffing, but we are able to give a clear account of much of their behavior and to show precisely what presently obstructs further analysis.

In the final chapter we show how the global bifurcations, discussed previously, reappear in degenerate local bifurcations, and we end with several more models of physical problems which display these rich and beautiful behaviors.

Throughout the book we continually return to specific examples, and we have tried to illustrate even the most abstract results. In our Appendix we give suggestions for further reading. We make no claims for the completeness of our bibliography. We have, however, tried to include references to the bulk of the papers, monographs, lecture notes, and books which have proved useful to ourselves and our colleagues, but we recognize that our biases probably make this a rather eclectic selection.

We have included a glossary of the more important terminology for the convenience of those readers lacking a formal mathematical training.

Finally, we would especially like to acknowledge the encouragement, advice, and gentle criticisms of Bill Langford, Clark Robinson and David Rod, whose careful readings of the manuscript enabled us to make many corrections and improvements.

Nessen MacGiolla Mhuiris, Xuehai Li, Lloyd Sakazata, Rakesh, Kumarswamy Hebbale, and Pat Hollis suffered through the preparation of this manuscript as students in TAM 776 at Cornell, and pointed out many errors almost as quickly as they were made. Edgar Knobloch, Steve Shaw, and David Whitley also read and commented on the manuscript. The comments of these and many other people have helped us to improve this book, and it only remains for each of us to lay the blame for any remaining errors and omissions squarely on the shoulders of the other.

Barbara Boettcher prepared the illustrations from our rough notes and Dolores Pendell deserves more thanks than we can give for her patient typing and retyping of our almost illegible manuscripts.

Finally, we thank our wives and children for their understanding and patience during the production of this addition to our families.

JOHN GUCKENHEIMER
Santa Cruz, Spring 1983

PHILIP HOLMES
Ithaca, Spring 1983

Preface to the Second Printing

The reprinting of this book some $2\frac{1}{2}$ years after its publication has provided us with the opportunity of correcting many minor typographical errors and a few errors of substance. In particular, errors in Section 6.5 in the study of the Šilnikov return map have been corrected, and we have rewritten parts of Sections 7.4 and 7.5 fairly extensively in the light of recent work by Carr, Chow, Cushman, Hale, Sanders, Zholondek, and others on the number of limit cycles and bifurcations in these unfoldings. In the former case the main result is unaffected, but in the latter case some of our intuitions (as well as the incorrect calculations with which we supported them) have proved wrong. We take some comfort in the fact that our naive assertions stimulated some of the work which disproved them.

Although progress in some areas of applied dynamical systems has been rapid, and significant new developments have occurred since the first printing, we have not seen fit to undertake major revisions of the book at this stage, although we have briefly noted some of the developments which bear directly on topics discussed in the book. These comments appear at the end of the book, directly after the Appendix. A complete revision will perhaps be appropriate 5 or 10 years from now. (Anyone wishing to perform it, please contact us!) In the same spirit, we have not attempted to bring the bibliography up to date, although we have added about 75 references, including those mentioned above. References that were in preprint form at the first printing have been updated in cases where the journal of publication is known. In cases in which the publication date of the journal differs from that of the preprint, the journal date is given at the end of the reference. We note that a useful bibliography due to Shiraiwa [1981] has recently been updated (Shiraiwa [1985]); it contains over 4,400 items.

In preparing the revisions we have benefited from the advice and corrections supplied by many readers, including Marty Golubitsky, Kevin Hockett, Fuhua Ling, Wei-Min Liu, Clark Robinson, Jan Sanders, Steven Shaw, Ed Zehnder, and Zhaoxuan Zhu. Professor Ling, of the Shanghai Jiao Tong University, with the help of his students and of Professor Zhu, of Peking University, has prepared a Chinese translation of this book.

JOHN GUCKENHEIMER
PHILIP HOLMES
Ithaca, Fall 1985

Preface to the Fifth Printing

When it first appeared in 1983, this book was (almost) unique. Thirteen years later, there are dozens of texts, at various levels, that bridge the gap between the mathematical theory of dynamical systems and the "practical" computational tools necessary for applications to problems in the sciences and engineering. In this context, we have been asked several times to revise the book, but, while we might now treat some of the topics differently and add others, we feel that there is little here which should be cut or significantly changed. Indeed, the host of newer books, some of which we note in the Postscript, makes a revision *less* appealing. Most of the details missing here can now be found in one or another of those texts. Including them would make our book larger, more unwieldly, and more expensive. We believe that the topics we originally chose continue to provide a good basis on which more detailed studies of background, technical details, or applications may be built.

In preparing this printing, as with the third and fourth, we have continued to correct errors and oversights. We particularly thank Ralf Wittenberg, Jinqiao Duan, and Mark Johnson for finding many of these.

John Guckenheimer
Ithaca, Fall 1996

Philip Holmes
Princeton, Fall 1996

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