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Perturbation Analysis of Optimization Problems

Springer

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This book is dedicated to our families, our wives Viviane and Julia, and our children Juliette and Antoine, and Benjamin and Daniel.

Contents

Basic Notation

Basic Sets and Spaces

" := " equal by definition

" ≡ " identically equal

$\emptyset$ empty set

$|I|$ cardinality of the set I

$x \mapsto f(x)$ mapping of the point x into $f(x)$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ extended real line

$\mathbb{R}^n$ is n-dimensional Euclidean space

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$ nonnegative orthant

$\mathbb{R}^n_- = -\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \leq 0,\ i = 1, \ldots, n\}$ nonnpositive orthant

X, Y are Banach or locally convex topological vector spaces

$\mathcal{S}^p$ linear space of $p \times p$ symmetric matrices

$\mathcal{S}^p_+$ ($\mathcal{S}^p_-$) cone of $p \times p$ symmetric positive (negative) semidefinite matrices

$\mathcal{W}_r \subset \mathcal{S}^p$ set of matrices of rank r

ℓ_2 Hilbert space of sequences $x = (x_i)_{i=1}^\infty$ such that $\sum_{i=1}^\infty x_i^2 < \infty$ and with $\|x\| = \left(\sum_{i=1}^\infty x_i^2\right)^{1/2}$ and $\langle x, y\rangle = \sum_{i=1}^\infty x_i y_i$, $x, y \in \ell_2$

$L_2[0, 1]$ Hilbert space of equivalence classes of real valued, square integrable, functions $\psi(t)$, with $\psi_1 \sim \psi_2$ if $\psi_1(t) = \psi_2(t)$ for all $t \in [0, 1]$ except possibly on a set of Lebesgue measure zero, and $\langle \psi, \phi\rangle = \int_0^1 \psi(t)\phi(t)dt$

$L_p(\Omega, \mathcal{F}, \mu)$ Banach space of $\mathcal{F}$-measurable functions $\psi : \Omega \to \mathbb{R}$ having finite norm $\|\psi\|_p := \left(\int_\Omega |\psi(\omega)|^p d\mu(\omega)\right)^{1/p}$

$[L_p(\Omega)]_+ \subset L_p(\Omega, \mathcal{F}, \mu)$ set of almost everywhere nonnegative valued functions

$C(\Omega)$ Banach space of continuous functions $\psi : \Omega \to \mathbb{R}$ defined on the compact metric space Ω and equipped with the sup-norm $\|\psi\| = \sup_{\omega \in \Omega} |\psi(\omega)|$
$C^\ell(\Omega)$ Banach space of ℓ-times continuously differentiable functions $\psi : \Omega \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^n$
$C^{1,1}(\Omega)$ space of continuously differentiable functions $\psi : \Omega \to \mathbb{R}$ and such that $D\psi(\cdot)$ is locally Lipschitz continuous
$\mathcal{D}(\Omega)$ Set of real valued C^∞-smooth functions over Ω with compact support
$C_{00}(\Omega)$ Set of continuous functions with compact support in Ω
$\mathcal{O}_{K,m}$ Family of barrel sets associated with the topology on $\mathcal{D}(\Omega)$
$\mathcal{O}_K$ Family of barrel sets associated with the topology on $C_{00}(\Omega)$
$W^{m,s}(\Omega) = \{\psi \in L_s(\Omega) : D^q\psi \in L_s(\Omega) \text{ if } |q| \le m\}$ Sobolev space, where $D^q\psi = \partial^{|q|}\psi/\partial x_1^{q_1}\cdots\partial x_\ell^{q_\ell}$ and $|q| = q_1 + \cdots + q_\ell$
$W_0^{m,s}(\Omega)$ Closure of $\mathcal{D}(\Omega)$ in $W^{m,s}(\Omega)$
$W^{1,\infty}(\Omega)$ Banach space of Lipschitz continuous functions $\psi : \Omega \to \mathbb{R}$
$H^m(\Omega)$, $H^{-1}(\Omega)$, $W^{-1,s'}(\Omega)$ Sobolev space $W^{m,2}(\Omega)$, dual space to $H_0^1(\Omega)$, and dual space to $W_0^{1,s}(\Omega)$, respectively
$C_+(\Omega)$ set of nonnegative valued functions in the space $C(\Omega)$
$C_-(\Omega)$ set of nonpositive valued functions in the space $C(\Omega)$
$\mathcal{L}(X, Y)$ space of linear continuous operators $A : X \to Y$ equipped with the operator norm $\|A\| = \sup_{x \in B_X} \|Ax\|$
$X^* = \mathcal{L}(X, \mathbb{R})$ dual space of X
$B(x, r) = \{x' \in X : \|x' - x\| < r\}$ open ball of radius $r > 0$ centered at x
$B_X = B_X(0, 1)$ open unit ball in X
$\bar{B}_X$ closed unit ball in X
$[\![x]\!] = \{tx : t \in \mathbb{R}\}$ linear space generated by vector x
2^X the set of subsets of X
$\dim(X)$ dimension of the linear space X
$\mathcal{P}_\Omega = \{\mu \in C(\Omega)^* : \mu(\Omega) = 1,\ \mu \succeq 0\}$ set of probability measures over Ω
$\operatorname{cap}(A)$ capacity of the set A

Matrices and Vectors

$\langle \alpha, x \rangle$ value of the linear functional $\alpha \in X^*$ on $x \in X$
$x \cdot y = \sum_{i=1}^n x_i y_i$ scalar product of two finite dimensional vectors $x, y \in \mathbb{R}^n$
A^T transpose of the matrix A
$\operatorname{rank}(A)$ rank of the matrix A
$\operatorname{vec}(A)$ vector obtained by stacking columns of the matrix A
$A^\dagger$ Moore-Penrose pseudoinverse of the matrix A
$\operatorname{trace} A = \sum_{i=1}^p a_{ii}$ trace of the $p \times p$ matrix A
$A \bullet B = \operatorname{trace}(AB)$ scalar product of two symmetric matrices $A, B \in \mathcal{S}^p$
$A \otimes B$ Kronecker product of matrices A and B
$\lambda_{\max}(A)$ largest eigenvalue of the symmetric matrix $A \in \mathcal{S}^p$
$A \succeq 0$ $(A \preceq 0)$ means that the matrix $A \in \mathcal{S}^p$ is positive (negative) semidefinite
I_p the $p \times p$ identity matrix

Operations on Sets

$\mathrm{Sp}(S) = I\!R_+(S - S)$ linear space generated by the set $S \subset X$
$I\!R_+(S) = \{tx : x \in S,\ t \geq 0\}$ cone generated by the set $S \subset X$
$\mathrm{cl}(S)$ topological closure of the set $S \subset X$, if X is a Banach space, closure is taken with respect to the norm (i.e., strong) topology
$\mathrm{int}(S) = \{x \in S :$ there is a neighborhood V of x such that $V \subset S\}$ interior of the set S
$\mathrm{bdr}(S)$ (also denoted ∂S) $= \mathrm{cl}(S) \setminus \mathrm{int}(S)$ boundary of the set S
$\mathrm{ri}(S) = \{x \in S :$ there is a neighborhood V of x such that $V \cap (x + L) \subset S\}$ (where $L := \mathrm{cl}[\mathrm{Sp}(S)]$) relative interior of the convex set S
$\mathrm{core}(S) = \{x \in S : \forall x' \in X,\ \exists \varepsilon > 0,\ \forall t \in [-\varepsilon, \varepsilon],\ x + tx' \in S\}$
$\mathrm{dist}(x, S) = \inf_{z \in S} \|x - z\|$ distance from the point $x \in X$ to set $S \subset X$
$\mathrm{Haus}(S, T) = \max\left\{\sup_{x \in S} \mathrm{dist}(x, T), \sup_{x \in T} \mathrm{dist}(x, S)\right\}$ Hausdorff distance between the sets S and T
$S^{\perp} = \{\alpha \in X^* : \langle \alpha, x \rangle = 0,\ \forall x \in S\}$ orthogonal complement of the set $S \subset X$
$S^{\infty} = \{h \in X : \exists x \in S,\ \forall t \geq 0,\ x + th \in S\}$ recession cone of the convex set S
$\sigma(\alpha, S) = \sup_{x \in S} \langle \alpha, x \rangle$ support function of the set S
$I_S(\cdot)$ indicator function of the set S
$\mathrm{conv}(S)$ convex hull of the set S
$\mathrm{diam}(S) = \sup_{x, x' \in S} \|x - x'\|$ diameter of the set S
$C^- = \{\alpha \in X^* : \langle \alpha, x \rangle \leq 0,\ \forall x \in C\}$ polar (negative dual) of the cone $C \subset X$, where X and X^* are paired spaces
$\mathrm{lin}(C)$ lineality subspace of the convex cone C
$a \preceq_C b$ order relation imposed by the cone C, i.e., $b - a \in C$
$a \vee b$ the least upper bound of a and b
$a \wedge b$ the greatest lower bound of a and b
$[a, b]_C = \{x : a \preceq_C x \preceq_C b\}$ interval with respect to the order relation " $\preceq_C$ "
$G \overline{\pitchfork}_x W$ mapping G intersects manifold W transversally at the point x

Tangent Sets

$T_S(x) = \limsup_{t \downarrow 0} (S - x)/t$ contingent (Bouligand) cone to the set S at the point $x \in S$
$T_S^i(x) = \liminf_{t \downarrow 0} (S - x)/t = \{h \in X : \mathrm{dist}(x + th, S) = o(t),\ t \geq 0\}$ inner tangent cone to the set S at the point $x \in S$
$T_S^c(x)$ Clarke tangent cone to the set S at the point $x \in S$
$\mathcal{R}_S(x) = \{h \in X : \exists t > 0,\ x + th \in S\}$ radial cone to the convex set S at the point $x \in S$
$T_S(x) = \mathrm{cl}[\mathcal{R}_S(x)] = T_S^i(x)$ tangent cone to the convex set S at the point $x \in S$
$T_S^2(x, h) = \limsup_{t \downarrow 0} (S - x - th)/(\frac{1}{2}t^2)$ outer second order tangent set to the set S at the point $x \in S$ in the direction h
$T_S^{i,2}(x, h) = \liminf_{t \downarrow 0} (S - x - th)/(\frac{1}{2}t^2)$ inner second order tangent set to the set S at the point $x \in S$ in the direction h

$T_S^{i,2,\sigma}(x,h) = \liminf_{n\to\infty}(S - x - t_n h)/(\frac{1}{2}t_n^2)$ sequential second order tangent set associated with the sequence $\sigma = \{t_n\}$ such that $t_n \downarrow 0$
Σ set of sequences $\sigma = \{t_n\}$ of positive numbers converging to zero
$N_S(x) = [T_S(x)]^-$ normal cone to the set $S \subset X$ at the point $x \in S$
$N_S(x) = \{\alpha \in X^* : \langle \alpha, z - x\rangle \le 0,\ \forall z \in S\}$ normal cone to the convex set S
$PN_S(x)$ set of proximal normals to S at x
$PN_S^\delta(x)$ set of δ-proximal normals to S at x

Functions and Operators

$f : X \to \overline{I\!R}$ extended real valued function
$\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ domain of the function f
$\operatorname{gph} f = \{(x, f(x)) : x \in X\} \subset X \times I\!R$ graph of the function f
$\operatorname{epi} f = \{(x,\alpha) : \alpha \ge f(x),\ x \in X\} \subset X \times I\!R$ epigraph of the function f
$\operatorname{lsc} f(x) = \min\{f(x), \liminf_{x'\to x} f(x')\}$ lower semicontinuous hull of f
$\operatorname{cl} f(x)$ closure of the function f
$\operatorname{conv} f$ convex hull of the function f
$\operatorname{lev}_\alpha f = \{x \in X : f(x) \le \alpha\}$ level set of the function f
$f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}$ conjugate of the function f
$\hat{f}_\varepsilon(\cdot)$ Moreau-Yosida regularization of the function f
$f \diamond g(u) = \inf_{x\in X}\{f(u-x) + g(x)\}$ infimal convolution of the extended real valued functions $[f, g : X \to \overline{I\!R}]$
$f \circ g$ composition of a mapping $g : X \to Y$ and a mapping (extended real valued function) $f : Y \to Z$, i.e., $(f \circ g)(x) = f(g(x))$
$\lambda^\perp = \operatorname{Ker}\lambda = \{y \in Y : \langle \lambda, y\rangle = 0\}$ null space of $\lambda \in Y^*$
$\mathcal{N}(Q) = \{x \in X : Q(x) = 0\}$ null space of the quadratic form $Q : X \to I\!R$
$A^* : Y^* \to X^*$ adjoint operator of the continuous linear operator $A : X \to Y$, i.e., $\langle A^*\lambda, x\rangle = \langle \lambda, Ax\rangle$, for all $x \in X$ and $\lambda \in Y^*$
$\Pi_S(x) = \arg\min_{z\in S}\|x - z\|$ set-valued metric projection of the point x onto S
$P_S(x) \in \Pi_S(x)$ a metric projection of the point x onto S
$\Delta y = \sum_{i=1}^n \partial^2 y/\partial\omega_i^2$ Laplace operator
$\delta(\omega)$ measure of mass one at the point ω (Dirac measure)
$\mu \succeq 0$ means that the measure μ is nonnegative valued
$\operatorname{supp}(\mu)$ support of the measure μ
$|\mu|$ total variation of the measure μ
$[a]_+ = \max\{0, a\}$, for $a \in I\!R$
$\forall$ for all, $\exists$ exists

Multifunctions

$\Psi : X \to 2^Y$ multifunction (point-to-set mapping), which maps X into the set of subsets of Y
$\operatorname{dom}(\Psi) = \{x \in X : \Psi(x) \ne \emptyset\}$ domain of Ψ
$\operatorname{range}(\Psi) = \Psi(X) = \{y \in Y : y \in \Psi(x),\ x \in X\}$ range of Ψ
$\operatorname{gph}(\Psi) = \{(x, y) \in X \times Y : y \in \Psi(x),\ x \in X\}$ graph of Ψ

$\Psi^{-1}(y) = \{x \in X : y \in \Psi(x)\}$ inverse multifunction of Ψ

$\limsup_{x \to x_0} \Psi(x) = \{y \in Y : \liminf_{x \to x_0} [\mathrm{dist}(y, \Psi(x))] = 0\}$ upper set limit of the multifunction Ψ at the point x

$\liminf_{x \to x_0} \Psi(x) = \{y \in Y : \limsup_{x \to x_0} [\mathrm{dist}(y, \Psi(x))] = 0\}$ lower set limit of the multifunction Ψ at the point x

Limits and Derivatives

$r(h) = o(h)$ means that $r(h)/\|h\| \to 0$ as $h \to 0$

$r(h) = O(h)$ means that $r(h)/\|h\|$ is bounded for all h in a neighborhood of $0 \in X$

$\nabla f(x) = (\partial f(x)/\partial x_1, \dots, \partial f(x)/\partial x_n)$ gradient of the function $f : I\!R^n \to I\!R$ at the point $x \in I\!R^n$

$\nabla^2 f(x) = [\partial^2 f(x)/\partial x_i \partial x_j]_{i,j=1}^n$ Hessian matrix of second order partial derivatives of the function $f : I\!R^n \to I\!R$ at the point $x \in I\!R^n$

$Dg(x) : X \to Y$ derivative (Gâteaux, Hadamard, or Fréchet, depending on the context) of the mapping $g : X \to Y$ at the point $x \in X$

$D^2 g(x) : X \to \mathcal{L}(X, Y)$ second-order derivative of the mapping g at the point x

$D^2 g(x)(h, h) = [D^2 g(x)h]h$ quadratic form corresponding to $D^2 g(x)$

$D_x g(x, u)$ partial derivative of the mapping $g : X \times U \to Y$

$g'(x, d) = \lim_{t \downarrow 0} [g(x + td) - g(x)]/t$ directional derivative of the mapping $g : X \to Y$ at the point x in the direction d

$f'_+(x, d) = \limsup_{t \downarrow 0} [f(x + td) - f(x)]/t$ upper directional derivative of the function $f : X \to \overline{I\!R}$

$f'_-(x, d) = \liminf_{t \downarrow 0} [f(x + td) - f(x)]/t$ lower directional derivative of the function $f : X \to \overline{I\!R}$

$f''(x; d, w) = \lim_{t \downarrow 0} [f(x + td + \frac{1}{2}t^2 w) - f(x) - tf'(x, d)]/(\frac{1}{2}t^2)$ second-order directional derivative of the function f

$f^{\downarrow}_-(x, h) = \text{e-}\liminf_{t \downarrow 0} [f(x + t\,h) - f(x)]/t$ lower directional epiderivative

$f^{\downarrow}_+(x, h) = \text{e-}\limsup_{t \downarrow 0} [f(x + t\,h) - f(x)]/t$ upper directional epiderivative

$f^{\downarrow\downarrow}_-(x; h, w) = \text{e-}\liminf_{t \downarrow 0} [f(x + th + \frac{1}{2}t^2 w) - f(x) - tf^{\downarrow}_-(x, h)]/(\frac{1}{2}t^2)$ lower second order directional epiderivative

$f^{\downarrow\downarrow}_+(x; h, w) = \text{e-}\limsup_{t \downarrow 0} [f(x + th + \frac{1}{2}t^2 w) - f(x) - tf^{\downarrow}_+(x, h)]/(\frac{1}{2}t^2)$ upper second order directional epiderivative

$d^2 f(x|\alpha)(h) := \liminf_{\substack{t \downarrow 0 \\ h' \to h}} [f(x + th') - f(x) - t\langle \alpha, h' \rangle]/(\frac{1}{2}t^2)$ second order subderivative of the function f at the point x with respect to $\alpha \in X^*$

$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle,\ \forall y \in X\}$ subdifferential of the function $f : X \to \overline{I\!R}$

Optimization Problems

$\mathrm{val}(P)$ optimal value of the problem (P)

Φ feasible set of the problem (P)

$\mathcal{S}(P)$ set of optimal solutions of the problem (P)

$L(x, \lambda) = f(x) + \langle \lambda, G(x) \rangle$ Lagrangian function of the problem (P)
$L^g(x, \alpha, \lambda) = \alpha f(x) + \langle \lambda, G(x) \rangle$ generalized Lagrangian function
$L^s(x, \lambda) = \langle \lambda, G(x) \rangle$ singular Lagrangian function
$\Lambda(x)$ set of Lagrange multipliers at the point x
$\Lambda^g(x)$ set of generalized Lagrange multipliers at the point x
$\Lambda^s(x)$ set of singular Lagrange multipliers at the point x
$\Lambda_N^g(x) = \{(\alpha, \lambda) \in \Lambda^g(x) : \alpha + \|\lambda\| = 1\}$ set of normalized generalized Lagrange multipliers at the point x
$I(x) = \{i : g_i(x) = 0,\ i = q+1, \dots, p\}$ set of active at x inequality constraints
$I_+(x, \lambda) = \{i \in I(x) : \lambda_i > 0\}$
$I_0(x, \lambda) = \{i \in I(x) : \lambda_i = 0\}$
$\Delta(x) = \{\omega \in \Omega : g(x, \omega) = 0\}$ set of active at x constraints of $g(x, \omega) \leq 0, \omega \in \Omega$
$C(x)$ set of critical directions (critical cone) at the point x
$C_\eta(x)$ approximate critical cone at the point x
(P_u) parameterized by $u \in U$ optimization problem
$\Phi(u)$ feasible set of the parameterized problem (P_u)
$v(u) = \mathrm{val}(P_u) = \inf_{x \in \Phi(u)} f(x, u)$ optimal value (marginal) function of (P_u)
$\mathcal{S}(u) = \mathcal{S}(P_u) = \arg\min_{x \in \Phi(u)} f(x, u)$ set of optimal solutions of (P_u)
$\bar{x}(u) \in \mathcal{S}(u)$ an optimal (ε-optimal) solution of (P_u)
$L(x, \lambda, u)$, $L^g(x, \alpha, \lambda, u)$ and $L^s(x, \lambda, u)$ Lagrangian, generalized Lagrangian and singular Lagrangian functions, respectively, of (P_u)
$\Lambda(x, u)$ and $\Lambda^g(x, u)$ sets of Lagrange and generalized Lagrange multipliers, respectively, of (P_u) at the point (x, u)