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Perturbation Analysis of Optimization Problems



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**This book is dedicated to our families, our wives Viviane and Julia,
and our children Juliette and Antoine, and Benjamin and Daniel.**

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Basic Notation

Basic Sets and Spaces

“ $:=$ ” equal by definition

“ \equiv ” identically equal

\emptyset empty set

$|I|$ cardinality of the set I

$x \mapsto f(x)$ mapping of the point x into $f(x)$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ extended real line

\mathbb{R}^n is n -dimensional Euclidean space

$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ nonnegative orthant

$\mathbb{R}_-^n = -\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \leq 0, i = 1, \dots, n\}$ nonpositive orthant

X, Y are Banach or locally convex topological vector spaces

S^p linear space of $p \times p$ symmetric matrices

S_+^p (S_-^p) cone of $p \times p$ symmetric positive (negative) semidefinite matrices

$\mathcal{W}_r \subset S^p$ set of matrices of rank r

ℓ_2 Hilbert space of sequences $x = (x_i)_{i=1}^\infty$ such that $\sum_{i=1}^\infty x_i^2 < \infty$ and with

$$\|x\| = \left(\sum_{i=1}^\infty x_i^2\right)^{1/2} \text{ and } \langle x, y \rangle = \sum_{i=1}^\infty x_i y_i, x, y \in \ell_2$$

$L_2[0, 1]$ Hilbert space of equivalence classes of real valued, square integrable, functions $\psi(t)$, with $\psi_1 \sim \psi_2$ if $\psi_1(t) = \psi_2(t)$ for all $t \in [0, 1]$ except possibly on a set of Lebesgue measure zero, and $\langle \psi, \phi \rangle = \int_0^1 \psi(t)\phi(t)dt$

$L_p(\Omega, \mathcal{F}, \mu)$ Banach space of \mathcal{F} -measurable functions $\psi : \Omega \rightarrow \mathbb{R}$ having finite norm $\|\psi\|_p := \left(\int_\Omega |\psi(\omega)|^p d\mu(\omega)\right)^{1/p}$

$[L_p(\Omega)]_+ \subset L_p(\Omega, \mathcal{F}, \mu)$ set of almost everywhere nonnegative valued functions

- $C(\Omega)$ Banach space of continuous functions $\psi : \Omega \rightarrow \mathbb{R}$ defined on the compact metric space Ω and equipped with the sup-norm $\|\psi\| = \sup_{\omega \in \Omega} |\psi(\omega)|$
- $C^\ell(\Omega)$ Banach space of ℓ -times continuously differentiable functions $\psi : \Omega \rightarrow \mathbb{R}$, with $\Omega \subset \mathbb{R}^n$
- $C^{1,1}(\Omega)$ space of continuously differentiable functions $\psi : \Omega \rightarrow \mathbb{R}$ and such that $D\psi(\cdot)$ is locally Lipschitz continuous
- $\mathcal{D}(\Omega)$ Set of real valued C^∞ -smooth functions over Ω with compact support
- $C_{00}(\Omega)$ Set of continuous functions with compact support in Ω
- $\mathcal{O}_{K,m}$ Family of barrel sets associated with the topology on $\mathcal{D}(\Omega)$
- \mathcal{O}_K Family of barrel sets associated with the topology on $C_{00}(\Omega)$
- $W^{m,s}(\Omega) = \{\psi \in L_s(\Omega) : D^q \psi \in L_s(\Omega) \text{ if } |q| \leq m\}$ Sobolev space, where $D^q \psi = \partial^{|q|} \psi / \partial x_1^{q_1} \cdots \partial x_\ell^{q_\ell}$ and $|q| = q_1 + \cdots + q_\ell$
- $W_0^{m,s}(\Omega)$ Closure of $\mathcal{D}(\Omega)$ in $W^{m,s}(\Omega)$
- $W^{1,\infty}(\Omega)$ Banach space of Lipschitz continuous functions $\psi : \Omega \rightarrow \mathbb{R}$
- $H^m(\Omega), H^{-1}(\Omega), W^{-1,s'}(\Omega)$ Sobolev space $W^{m,2}(\Omega)$, dual space to $H_0^1(\Omega)$, and dual space to $W_0^{1,s}(\Omega)$, respectively
- $C_+(\Omega)$ set of nonnegative valued functions in the space $C(\Omega)$
- $C_-(\Omega)$ set of nonpositive valued functions in the space $C(\Omega)$
- $\mathcal{L}(X, Y)$ space of linear continuous operators $A : X \rightarrow Y$ equipped with the operator norm $\|A\| = \sup_{x \in B_X} \|Ax\|$
- $X^* = \mathcal{L}(X, \mathbb{R})$ dual space of X
- $B(x, r) = \{x' \in X : \|x' - x\| < r\}$ open ball of radius $r > 0$ centered at x
- $B_X = B_X(0, 1)$ open unit ball in X
- \bar{B}_X closed unit ball in X
- $\|x\| = \{tx : t \in \mathbb{R}\}$ linear space generated by vector x
- 2^X the set of subsets of X
- $\dim(X)$ dimension of the linear space X
- $\mathcal{P}_\Omega = \{\mu \in C(\Omega)^* : \mu(\Omega) = 1, \mu \geq 0\}$ set of probability measures over Ω
- $\text{cap}(A)$ capacity of the set A

Matrices and Vectors

- $\langle \alpha, x \rangle$ value of the linear functional $\alpha \in X^*$ on $x \in X$
- $x \cdot y = \sum_{i=1}^n x_i y_i$ scalar product of two finite dimensional vectors $x, y \in \mathbb{R}^n$
- A^T transpose of the matrix A
- $\text{rank}(A)$ rank of the matrix A
- $\text{vec}(A)$ vector obtained by stacking columns of the matrix A
- A^\dagger Moore-Penrose pseudoinverse of the matrix A
- $\text{trace} A = \sum_{i=1}^p a_{ii}$ trace of the $p \times p$ matrix A
- $A \bullet B = \text{trace}(AB)$ scalar product of two symmetric matrices $A, B \in \mathcal{S}^p$
- $A \otimes B$ Kronecker product of matrices A and B
- $\lambda_{\max}(A)$ largest eigenvalue of the symmetric matrix $A \in \mathcal{S}^p$
- $A \geq 0$ ($A \leq 0$) means that the matrix $A \in \mathcal{S}^p$ is positive (negative) semidefinite
- I_p the $p \times p$ identity matrix

Operations on Sets

$\text{Sp}(S) = \mathbb{R}_+(S - S)$ linear space generated by the set $S \subset X$

$\mathbb{R}_+(S) = \{tx : x \in S, t \geq 0\}$ cone generated by the set $S \subset X$

$\text{cl}(S)$ topological closure of the set $S \subset X$, if X is a Banach space, closure is taken with respect to the norm (i.e., strong) topology

$\text{int}(S) = \{x \in S : \text{there is a neighborhood } V \text{ of } x \text{ such that } V \subset S\}$ interior of the set S

$\text{bdr}(S)$ (also denoted $\partial S = \text{cl}(S) \setminus \text{int}(S)$) boundary of the set S

$\text{ri}(S) = \{x \in S : \text{there is a neighborhood } V \text{ of } x \text{ such that } V \cap (x + L) \subset S\}$ (where $L := \text{cl}[\text{Sp}(S)]$) relative interior of the convex set S

$\text{core}(S) = \{x \in S : \forall x' \in X, \exists \varepsilon > 0, \forall t \in [-\varepsilon, \varepsilon], x + tx' \in S\}$

$\text{dist}(x, S) = \inf_{z \in S} \|x - z\|$ distance from the point $x \in X$ to set $S \subset X$

$\text{Haus}(S, T) = \max \left\{ \sup_{x \in S} \text{dist}(x, T), \sup_{x \in T} \text{dist}(x, S) \right\}$ Hausdorff distance between the sets S and T

$S^\perp = \{\alpha \in X^* : \langle \alpha, x \rangle = 0, \forall x \in S\}$ orthogonal complement of the set $S \subset X$

$S^\infty = \{h \in X : \exists x \in S, \forall t \geq 0, x + th \in S\}$ recession cone of the convex set S

$\sigma(\alpha, S) = \sup_{x \in S} \langle \alpha, x \rangle$ support function of the set S

$I_S(\cdot)$ indicator function of the set S

$\text{conv}(S)$ convex hull of the set S

$\text{diam}(S) = \sup_{x, x' \in S} \|x - x'\|$ diameter of the set S

$C^- = \{\alpha \in X^* : \langle \alpha, x \rangle \leq 0, \forall x \in C\}$ polar (negative dual) of the cone $C \subset X$, where X and X^* are paired spaces

$\text{lin}(C)$ lineality subspace of the convex cone C

$a \preceq_C b$ order relation imposed by the cone C , i.e., $b - a \in C$

$a \vee b$ the least upper bound of a and b

$a \wedge b$ the greatest lower bound of a and b

$[a, b]_C = \{x : a \preceq_C x \preceq_C b\}$ interval with respect to the order relation “ \preceq_C ”

$G \pitchfork_x W$ mapping G intersects manifold W transversally at the point x

Tangent Sets

$T_S(x) = \limsup_{t \downarrow 0} (S - x)/t$ contingent (Bouligand) cone to the set S at the point $x \in S$

$T_S^i(x) = \liminf_{t \downarrow 0} (S - x)/t = \{h \in X : \text{dist}(x + th, S) = o(t), t \geq 0\}$ inner tangent cone to the set S at the point $x \in S$

$T_S^c(x)$ Clarke tangent cone to the set S at the point $x \in S$

$\mathcal{R}_S(x) = \{h \in X : \exists t > 0, x + th \in S\}$ radial cone to the convex set S at the point $x \in S$

$T_S(x) = \text{cl}[\mathcal{R}_S(x)] = T_S^i(x)$ tangent cone to the convex set S at the point $x \in S$

$T_S^2(x, h) = \limsup_{t \downarrow 0} (S - x - th)/(\frac{1}{2}t^2)$ outer second order tangent set to the set S at the point $x \in S$ in the direction h

$T_S^{i,2}(x, h) = \liminf_{t \downarrow 0} (S - x - th)/(\frac{1}{2}t^2)$ inner second order tangent set to the set S at the point $x \in S$ in the direction h

$T_S^{i,2,\sigma}(x, h) = \liminf_{n \rightarrow \infty} (S - x - t_n h) / (\frac{1}{2} t_n^2)$ sequential second order tangent set associated with the sequence $\sigma = \{t_n\}$ such that $t_n \downarrow 0$
 Σ set of sequences $\sigma = \{t_n\}$ of positive numbers converging to zero
 $N_S(x) = [T_S(x)]^\circ$ normal cone to the set $S \subset X$ at the point $x \in S$
 $N_S(x) = \{\alpha \in X^* : \langle \alpha, z - x \rangle \leq 0, \forall z \in S\}$ normal cone to the convex set S
 $PN_S(x)$ set of proximal normals to S at x
 $PN_S^\delta(x)$ set of δ -proximal normals to S at x

Functions and Operators

$f : X \rightarrow \overline{\mathbb{R}}$ extended real valued function
 $\text{dom } f = \{x \in X : f(x) < +\infty\}$ domain of the function f
 $\text{gph } f = \{(x, f(x)) : x \in X\} \subset X \times \mathbb{R}$ graph of the function f
 $\text{epi } f = \{(x, \alpha) : \alpha \geq f(x), x \in X\} \subset X \times \mathbb{R}$ epigraph of the function f
 $\text{lsc } f(x) = \min\{f(x), \liminf_{x' \rightarrow x} f(x')\}$ lower semicontinuous hull of f
 $\text{cl } f(x)$ closure of the function f
 $\text{conv } f$ convex hull of the function f
 $\text{lev}_\alpha f = \{x \in X : f(x) \leq \alpha\}$ level set of the function f
 $f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$ conjugate of the function f
 $\hat{f}_\varepsilon(\cdot)$ Moreau-Yosida regularization of the function f
 $f \diamond g(u) = \inf_{x \in X} \{f(u - x) + g(x)\}$ infimal convolution of the extended real valued functions $[f, g : X \rightarrow \overline{\mathbb{R}}]$
 $f \circ g$ composition of a mapping $g : X \rightarrow Y$ and a mapping (extended real valued function) $f : Y \rightarrow \overline{\mathbb{R}}$, i.e., $(f \circ g)(x) = f(g(x))$
 $\lambda^\perp = \text{Ker } \lambda = \{y \in Y : \langle \lambda, y \rangle = 0\}$ null space of $\lambda \in Y^*$
 $\mathcal{N}(Q) = \{x \in X : Q(x) = 0\}$ null space of the quadratic form $Q : X \rightarrow \mathbb{R}$
 $A^* : Y^* \rightarrow X^*$ adjoint operator of the continuous linear operator $A : X \rightarrow Y$, i.e., $\langle A^* \lambda, x \rangle = \langle \lambda, Ax \rangle$, for all $x \in X$ and $\lambda \in Y^*$
 $\Pi_S(x) = \arg \min_{z \in S} \|x - z\|$ set-valued metric projection of the point x onto S
 $P_S(x) \in \Pi_S(x)$ a metric projection of the point x onto S
 $\Delta y = \sum_{i=1}^n \partial^2 y / \partial \omega_i^2$ Laplace operator
 $\delta(\omega)$ measure of mass one at the point ω (Dirac measure)
 $\mu \geq 0$ means that the measure μ is nonnegative valued
 $\text{supp}(\mu)$ support of the measure μ
 $|\mu|$ total variation of the measure μ
 $[a]_+ = \max\{0, a\}$, for $a \in \mathbb{R}$
 \forall for all, \exists exists

Multifunctions

$\Psi : X \rightarrow 2^Y$ multifunction (point-to-set mapping), which maps X into the set of subsets of Y
 $\text{dom}(\Psi) = \{x \in X : \Psi(x) \neq \emptyset\}$ domain of Ψ
 $\text{range}(\Psi) = \Psi(X) = \{y \in Y : y \in \Psi(x), x \in X\}$ range of Ψ
 $\text{gph}(\Psi) = \{(x, y) \in X \times Y : y \in \Psi(x), x \in X\}$ graph of Ψ

$\Psi^{-1}(y) = \{x \in X : y \in \Psi(x)\}$ inverse multifunction of Ψ

$\limsup_{x \rightarrow x_0} \Psi(x) = \{y \in Y : \liminf_{x \rightarrow x_0} [\text{dist}(y, \Psi(x))] = 0\}$ upper set limit of the multifunction Ψ at the point x

$\liminf_{x \rightarrow x_0} \Psi(x) = \{y \in Y : \limsup_{x \rightarrow x_0} [\text{dist}(y, \Psi(x))] = 0\}$ lower set limit of the multifunction Ψ at the point x

Limits and Derivatives

$r(h) = o(h)$ means that $r(h)/\|h\| \rightarrow 0$ as $h \rightarrow 0$

$r(h) = O(h)$ means that $r(h)/\|h\|$ is bounded for all h in a neighborhood of $0 \in X$

$\nabla f(x) = (\partial f(x)/\partial x_1, \dots, \partial f(x)/\partial x_n)$ gradient of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at the point $x \in \mathbb{R}^n$

$\nabla^2 f(x) = [\partial^2 f(x)/\partial x_i \partial x_j]_{i,j=1}^n$ Hessian matrix of second order partial derivatives of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at the point $x \in \mathbb{R}^n$

$Dg(x) : X \rightarrow Y$ derivative (Gâteaux, Hadamard, or Fréchet, depending on the context) of the mapping $g : X \rightarrow Y$ at the point $x \in X$

$D^2g(x) : X \rightarrow \mathcal{L}(X, Y)$ second-order derivative of the mapping g at the point x

$D^2g(x)(h, h) = [D^2g(x)h]h$ quadratic form corresponding to $D^2g(x)$

$D_x g(x, u)$ partial derivative of the mapping $g : X \times U \rightarrow Y$

$g'(x, d) = \lim_{t \downarrow 0} [g(x + td) - g(x)]/t$ directional derivative of the mapping $g : X \rightarrow Y$ at the point x in the direction d

$f'_+(x, d) = \limsup_{t \downarrow 0} [f(x + td) - f(x)]/t$ upper directional derivative of the function $f : X \rightarrow \overline{\mathbb{R}}$

$f'_-(x, d) = \liminf_{t \downarrow 0} [f(x + td) - f(x)]/t$ lower directional derivative of the function $f : X \rightarrow \overline{\mathbb{R}}$

$f''(x; d, w) = \lim_{t \downarrow 0} [f(x + td + \frac{1}{2}t^2w) - f(x) - tf'(x, d)]/(\frac{1}{2}t^2)$ second-order directional derivative of the function f

$f_-^\downarrow(x, h) = \text{e-lim inf}_{t \downarrow 0} [f(x + th) - f(x)]/t$ lower directional epiderivative

$f_+^\downarrow(x, h) = \text{e-lim sup}_{t \downarrow 0} [f(x + th) - f(x)]/t$ upper directional epiderivative

$f_-^{\downarrow\downarrow}(x; h, w) = \text{e-lim inf}_{t \downarrow 0} [f(x + th + \frac{1}{2}t^2w) - f(x) - tf_-^\downarrow(x, h)]/(\frac{1}{2}t^2)$ lower second order directional epiderivative

$f_+^{\downarrow\downarrow}(x; h, w) = \text{e-lim sup}_{t \downarrow 0} [f(x + th + \frac{1}{2}t^2w) - f(x) - tf_+^\downarrow(x, h)]/(\frac{1}{2}t^2)$ upper second order directional epiderivative

$d^2f(x|\alpha)(h) := \liminf_{\substack{t \downarrow 0 \\ h' \rightarrow h}} [f(x + th') - f(x) - t\langle \alpha, h' \rangle]/(\frac{1}{2}t^2)$ second order subderivative of the function f at the point x with respect to $\alpha \in X^*$

$\partial f(x) = \{x^* \in X^* : \frac{f(y) - f(x)}{\|y - x\|} \geq \langle x^*, y - x \rangle, \forall y \in X\}$ subdifferential of the function $f : X \rightarrow \overline{\mathbb{R}}$

Optimization Problems

$\text{val}(P)$ optimal value of the problem (P)

Φ feasible set of the problem (P)

$\mathcal{S}(P)$ set of optimal solutions of the problem (P)

$L(x, \lambda) = f(x) + \langle \lambda, G(x) \rangle$ Lagrangian function of the problem (P)

$L^g(x, \alpha, \lambda) = \alpha f(x) + \langle \lambda, G(x) \rangle$ generalized Lagrangian function

$L^s(x, \lambda) = \langle \lambda, G(x) \rangle$ singular Lagrangian function

$\Lambda(x)$ set of Lagrange multipliers at the point x

$\Lambda^g(x)$ set of generalized Lagrange multipliers at the point x

$\Lambda^s(x)$ set of singular Lagrange multipliers at the point x

$\Lambda_N^g(x) = \{(\alpha, \lambda) \in \Lambda^g(x) : \alpha + \|\lambda\| = 1\}$ set of normalized generalized Lagrange multipliers at the point x

$I(x) = \{i : g_i(x) = 0, i = q + 1, \dots, p\}$ set of active at x inequality constraints

$I_+(x, \lambda) = \{i \in I(x) : \lambda_i > 0\}$

$I_0(x, \lambda) = \{i \in I(x) : \lambda_i = 0\}$

$\Delta(x) = \{\omega \in \Omega : g(x, \omega) = 0\}$ set of active at x constraints of $g(x, \omega) \leq 0, \omega \in \Omega$

$C(x)$ set of critical directions (critical cone) at the point x

$C_\eta(x)$ approximate critical cone at the point x

(P_u) parameterized by $u \in U$ optimization problem

$\Phi(u)$ feasible set of the parameterized problem (P_u)

$v(u) = \text{val}(P_u) = \inf_{x \in \Phi(u)} f(x, u)$ optimal value (marginal) function of (P_u)

$S(u) = \mathcal{S}(P_u) = \arg \min_{x \in \Phi(u)} f(x, u)$ set of optimal solutions of (P_u)

$\bar{x}(u) \in S(u)$ an optimal (ε -optimal) solution of (P_u)

$L(x, \lambda, u)$, $L^g(x, \alpha, \lambda, u)$ and $L^s(x, \lambda, u)$ Lagrangian, generalized Lagrangian and singular Lagrangian functions, respectively, of (P_u)

$\Lambda(x, u)$ and $\Lambda^g(x, u)$ sets of Lagrange and generalized Lagrange multipliers, respectively, of (P_u) at the point (x, u)