# Temporal Graph Classes: A View Through Temporal Separators

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#### Abstract

We investigate for temporal graphs the computational complexity of separating two distinct vertices s and z by vertex deletion. In a temporal graph, the vertex set is fixed but the edges have (discrete) time labels. Since the corresponding TEMPORAL (s, z)-SEP-ARATION problem is NP-complete, it is natural to investigate whether relevant special cases exist that are computationally tractable. To this end, we study restrictions of the underlying (static) graph—there we observe polynomial-time solvability in the case of bounded treewidth—as well as restrictions concerning the "temporal evolution" along the time steps. Systematically studying partially novel concepts in this direction, we identify sharp borders between tractable and intractable cases.

*Keywords:* Temporal Paths, Temporal Restrictions, Unit Interval Graphs, NP-completeness, Fixed-Parameter Tractability, Dynamic Programming

# 1 Introduction

Reachability, connectivity, and robustness in networks depend often on time. For instance, in public transport or human contact networks, available connections or contacts are time-dependent. To model such time-dependent aspects, one turns from static graphs to temporal graphs. Formally, an undirected temporal graph  $G = (V, E, \tau)$  is an ordered triple consisting of a set V of vertices, a set  $E \subseteq \binom{V}{2} \times \{1, 2, \ldots, \tau\}$  of time-edges, and a maximal time label  $\tau \in \mathbb{N}$ . We study the problem of finding a small set of vertices in a temporal graph whose removal disconnects two designated terminals: a classic, polynomial-time solvable problem in (static) graph theory.

TEMPORAL (s, z)-SEPARATION

**Input:** A temporal graph  $G = (V, E, \tau)$ , two distinct vertices  $s, z \in V$ , and  $k \in \mathbb{N}$ . **Question:** Does G admit a temporal (s, z)-separator of size at most k?

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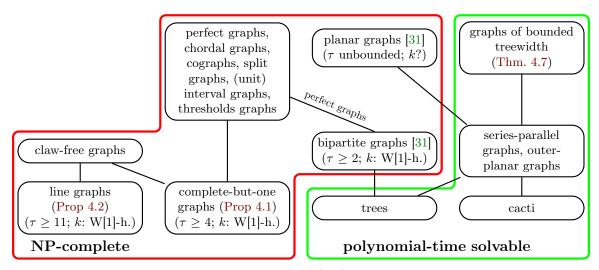


Figure 1: Computational complexity of TEMPORAL (s, z)-SEPARATION for some graph classes of the underlying graph. An edge between two classes indicates containment of the lower in the upper class. For the classes of line, complete-but-one, bipartite, and planar graphs, we provide for which values of the maximum time label  $\tau$  NP-completeness is proven as well as the parameterized complexity of TEMPORAL (s, z)-SEPARATION when parameterized by the solution size k. Note that in the case of planar graphs our NP-hardness proof only holds for unbounded  $\tau$ . Moreover, the parameterized complexity regarding k is unknown.

Herein, a vertex set  $S \subseteq V \setminus \{s, z\}$  is a temporal (s, z)-separator for a given temporal graph  $G = (V, E, \tau)$  with  $s, z \in V$  if there is no temporal (s, z)-path in  $G - S := (V \setminus S, \{(\{v, w\}, t) \in E \mid v, w \in V \setminus S\}, \tau)$ . A temporal (s, z)-path of length  $\ell$  in a temporal graph  $G = (V, E, \tau)$  is a sequence  $P = ((\{v_0, v_1\}, t_1), (\{v_1, v_2\}, t_2), \ldots, (\{v_{\ell-1}, v_\ell\}, t_\ell))$  of time-edges in E, where  $s = v_0, z = v_\ell, v_i \neq v_j$  for all  $i, j \in \{0, 1, \ldots, \ell\}$  with  $i \neq j$ , and  $t_i \leq t_{i+1}$  for all  $i \in \{1, 2, \ldots, \ell - 1\}$ .<sup>1</sup> TEMPORAL (s, z)-SEPARATION is NP-complete [19]. In this work, we study TEMPORAL (s, z)-SEPARATION on restricted classes of temporal graphs with the goal to identify computationally tractable cases.

So far, in the literature one basically finds two different directions concerning the definition of temporal graph classes. One direction is to define temporal graph classes through the underlying graph (that is, essentially, the graph obtained by forgetting about the time labels of the edges) [3, 15, 31]. Herein, one restricts the input temporal graph to have its underlying graph being contained in some specific graph class. The other direction is to consider properties expressible through temporal aspects [11, 16, 22, 27]. Such properties are, for instance, each layer being a subgraph of its succeeding layer, or the temporal graph being periodic, that is, having a subsequence of layers which is repeated in the same order for some periods. In this work, we study TEMPORAL (s, z)-SEPARATION on temporal graph classes from both directions.

**Our contributions.** We show that TEMPORAL (s, z)-SEPARATION remains NP-complete on many restricted temporal graph classes.

• TEMPORAL (s, z)-SEPARATION remains NP-complete on temporal graphs whose underlying graph falls into a class of graphs containing complete-but-one graphs (that is,

<sup>&</sup>lt;sup>1</sup>In the literature, temporal paths are also known as journeys [9]. However, in some work a journey has strictly increasing labels [1, 2, 25, 26].

complete graphs where exactly one edge is missing) or line graphs. However, if the underlying graph has bounded treewidth, then TEMPORAL (s, z)-SEPARATION becomes polynomial-time solvable (see Figure 1 for an overview).

- TEMPORAL (s, z)-SEPARATION remains NP-complete on temporal graphs where each layer contains only one edge (Corollary 3.2). In contrast, if we require each layer to be a unit interval graph and impose suitable restrictions on how the intervals may change over time, then TEMPORAL (s, z)-SEPARATION becomes tractable (Theorem 6.4, Theorem 6.6).
- Regarding temporal graph classes defined solely by restrictions on how the edge sets of the layers may change over time, TEMPORAL (s, z)-SEPARATION becomes solvable in polynomial time on temporal graphs where one layer contains all others (grounded), on graphs where all layers are identical (1-periodic or 0-steady), or when the number of periods is at least the number of vertices. In all other considered cases TEMPORAL (s, z)-SEPARATION remains NP-complete (see Table 1 in Section 5 for an overview).

**Related work.** Kempe et al. [19] proved that TEMPORAL (s, z)-SEPARATION is NP-complete. Zschoche et al. [31] proved that TEMPORAL (s, z)-SEPARATION remains NP-complete on temporal graphs with bipartite or planar underlying graphs. Moreover, TEMPORAL (s, z)-SEPA-RATION is W[1]-hard when parameterized by the separator size k [31].

Casteigts et al. [11] defined twelve different classes of temporal graphs and showed a corresponding inclusion diagram. Among these classes, they define temporal graph classes with recurrence or periodicity of edges. On a slightly different notion of the latter class, Flocchini et al. [16] studied the problem of exploring a temporal graph, that is, asking whether it is possible to visit all vertices of the graph with a temporal walk. Kuhn et al. [22] studied the problem of token dissemination on temporal graphs where for each time-interval of length T, all layers in the interval admit the same spanning tree.

The class of temporal graphs with underlying graphs of bounded treewidth are considered in the context of temporal graph exploration [15] and single-source temporal connectivity [3]. Erlebach et al. [15] studied the problem of temporal graph exploration on temporal graphs with underlying graphs being planar and of bounded vertex degree. They also introduced the class of temporal graphs with regularly present edges, where the number of consecutive time steps for which any edge can be absent is lower- and upper-bounded (a similar class without the lower bound is also introduced by Casteigts et al. [11, Class 7]). Michail and Spirakis [27] studied a temporal version of the TRAVELING SALESPERSON PROBLEM on temporal graphs with respect to the smallest number d such that every vertex can reach any other vertex at any time in at most d time steps.

**Organization.** In Section 2 we introduce all necessary notation and terminology concerning graph theory and (parameterized) computational complexity theory. In the next three section, we discuss and investigate three canonical and incomparable ways to restrict temporal graphs: In Section 3 we present our results for TEMPORAL (s, z)-SEPARATION on temporal graph classes that are defined by restricting the layers to be contained in certain graph classes. In Section 4 we present our results for TEMPORAL (s, z)-SEPARATION on temporal graphs with restricted underlying graphs. In Section 5 we discuss some *temporal* restrictions known from the literature that restrict how the edge sets of layers may relate to each other. In Section 6 we introduce a new class of temporal graphs that combines restrictions on the layers with

temporal restrictions and hence does not fit in any of the previous three categories: (almost) order-preserving temporal unit interval graphs and we present our results for TEMPORAL (s, z)-SEPARATION on those temporal graphs. We conclude in Section 7.

# 2 Preliminaries

As a convention,  $\mathbb{N}$  denotes the natural numbers without zero. For  $n \in \mathbb{N}$ , we use  $[n] := [1 : n] := \{1, 2, ..., n\}$ . Analogously, for a sequence  $x_1, x_2, ..., x_n$  and  $a, b \in [n]$ , a < b, we write  $x_{[a:b]}$  for the subsequence  $x_a, x_{a+1}, ..., x_b$ .

**Static graphs.** We use basic notations from (static) graph theory [13]. Let G = (V, E) be an *undirected, simple graph.* V(G) and E(G) denote the set of vertices and set of edges of G, respectively. We denote by  $G-V' := (V \setminus V', \{\{v, w\} \in E \mid v, w \in V \setminus V'\})$  the graph G without the vertices in  $V' \subseteq V$ . For  $V' \subseteq V$ ,  $G[V'] := G - (V \setminus V')$  is the *induced subgraph* of G on the vertices V'. A path of length  $\ell$  is sequence of edges  $P = (\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_\ell, v_{\ell+1}\})$ where  $v_i \neq v_j$  for all  $i, j \in [\ell - 1]$  with  $i \neq j$ . We set  $V(P) := \{v_1, v_2, \dots, v_{\ell+1}\}$ . Path P is an (s, z)-path if  $s = v_1$  and  $z = v_{\ell+1}$ . A set  $S \subseteq V \setminus \{s, z\}$  of vertices is an (s, z)-separator in Gif there is no (s, z)-path in G - S.

A tree decomposition of a graph G is a pair  $\mathcal{T} := (T, (B_i)_{i \in V(T)})$  consisting of a tree T and a family  $(B_i)_{i \in V(T)}$  of bags  $B_i \subseteq V(G)$ , such that

- (i) for all vertices  $v \in V(G)$  the set  $B^{-1}(v) := \{i \in V(T) \mid v \in B_i\}$  is non-empty and induces a subtree of T, and
- (ii) for every edge  $e \in E(G)$  there is an  $i \in V(T)$  with  $e \subseteq B_i$ .

The width of  $\mathcal{T}$  is max{ $|B_i| - 1 | i \in V(T)$ }. The treewidth tw(G) of G is defined as the minimal width over all tree decompositions of G.

**Temporal graphs.** Let  $G = (V, E, \tau)$  be a temporal graph. We call the graph  $G_i(G) = (V, E_i(G))$  the layer *i* of G where  $E_i(G) := \{\{v, w\} | (\{v, w\}, i) \in E\}$ . The underlying graph  $G_{\downarrow}$  of G is defined as  $G_{\downarrow} := (V, E_{\downarrow})$ , where  $E_{\downarrow} := \{e \mid \exists t : (e, t) \in E\}$ . (We drop G in the notations if it is clear from the context.) For  $X \subseteq V$  we define the *induced temporal subgraph* of G by X by  $G[X] := (X, \{(\{v, w\}, t) \in E \mid v, w \in X\}, \tau)$ . We say that a temporal graph G is connected if its underlying graph  $G_{\downarrow}$  is connected. Let  $s, z \in V$ . The departure time (arrival time) of a temporal (s, z)-path  $P = ((e_1, t_1), (e_2, t_2), \ldots, (e_\ell, t_\ell))$  is  $t_1(t_\ell)$ , the traversal time of P is  $t_\ell - t_1$ , and the length of P is  $\ell$ . The vertices visited by P are denoted by  $V(P) := \bigcup_{i=1}^{\ell} e_i$ . Throughout the whole paper we assume that the temporal input graph G is connected and that there is no time-edge between s and z. Furthermore, in accordance with Wu et al. [30] we assume that the time-edge set E is ordered by ascending labels.<sup>2</sup> The concatenation of two temporal graphs  $G_1 = (V, E_1, \tau_1), G_2 = (V, E_2, \tau_2)$  is denoted by  $G_1 \circ G_2 := (V, E_1 \cup \{(e, t + \tau_1) \mid (e, t) \in E_2\}, \tau_1 + \tau_2)$ . Furthermore, we define that  $G_1^1 := G_1$  and  $G_1^r := G_1^{r-1} \circ G_1$  for all integers  $x \ge 2$ .

We begin by noting that one can efficiently find temporal (s, z)-paths by using the *static* expansion of a temporal graph. Intuitively, the static expansion of a temporal graph G is a directed graph consisting of the union of the layers of G where each layer has its own vertex set, and additional edges from one vertex of a layer to the same vertex in the next layer.

<sup>&</sup>lt;sup>2</sup>If this is not the case, then E can be sorted by ascending labels with bucketsort or mergesort in  $\mathcal{O}(\min\{\tau, |E| \log |E|\})$  time.

**Definition 2.1.** For a temporal graph  $G = (V = \{v_1, \ldots, v_{n-2}, s, z\}, E, \tau)$ , the static expansion of (G, s, z) is the directed graph H := (V', A) with

$$V' := \{s, z\} \cup \{u_{t,j} \mid j \in [n-2] \land t \in \phi(v_j)\}$$
  

$$A := A' \cup A_s \cup A_z \cup A_{col}$$
  

$$A' := \{(u_{i,j}, u_{i,j'}), (u_{i,j'}, u_{i,j}) \mid (\{v_j, v_{j'}\}, i) \in \mathbf{E}\}$$
  

$$A_s := \{(s, u_{i,j}) \mid (\{s, v_j\}, i) \in \mathbf{E}\}$$
  

$$A_z := \{(u_{i,j}, z) \mid (\{v_j, z\}, i) \in \mathbf{E}\}$$
  

$$A_{col} := \{(u_{t,j}, u_{t',j}) \mid (t, t') \in \vec{\phi}(v_j) \land j \in [n-2]\},$$

where, for all  $v \in \{v_1, v_2, ..., v_{n-2}\} = V \setminus \{s, z\},\$ 

$$\begin{split} \phi(v) &:= \{t \mid t \in [\tau], \exists w : (\{v, w\}, t) \in \mathbf{E}\} \\ \vec{\phi}(v) &:= \{(t, t') \in \phi(v)^2 \mid t < t' \land \nexists t'' \in \phi(v) : t < t'' < t'\}. \end{split}$$

The set  $A_{col}$  is referred to as the set of *column-edges* of H.

**Lemma 2.2.** Given a temporal graph  $G = (V, E, \tau)$  and two distinct vertices s and z, a temporal (s, z)-path can be computed in  $\mathcal{O}(|E|)$  time.

Proof. Let  $\mathbf{G} = (V, \mathbf{E}, \tau)$  be a temporal graph with vertex set  $V := \{v_1, v_2, \ldots, v_{n-2}\} \cup \{s, z\}$ and let H be the static expansion of  $\mathbf{G}$ . Observe that each temporal (s, z)-path in  $\mathbf{G}$  has a one-to-one correspondence to some (s, z)-path in H and that H can be computed in  $\mathcal{O}(|\mathbf{E}|)$ time [31]. Thus we can find a temporal (s, z)-path in  $\mathbf{G}$ , using a breadth-first search on the static expansion of  $(\mathbf{G}, s, z)$ . This gives an overall running time of  $\mathcal{O}(|\mathbf{E}|)$ .

**Parameterized complexity.** We use standard notation and terminology from parameterized complexity [12, 14, 17, 29] and give here a brief overview of the most important concepts. A parameterized problem is a language  $L \subseteq \Sigma^* \times \mathbb{N}$ , where  $\Sigma$  is a finite alphabet. We call the second component the parameter of the problem. A parameterized problem is in the complexity class XP if there is an algorithm that solves each instance (I, r) in  $|I|^{f(r)}$  time, for some computable function f. It is *fixed-parameter tractable* (in the complexity class FPT) if there is an algorithm that solves each instance (I, r) in  $f(r) \cdot |I|^{\mathcal{O}(1)}$  time, for some computable function f. There is the W-hierarchy of complexity classes for parameterized problems, of which the most basic one is called W[1]. All parameterized complexity classes discussed here are closed under parameterized reductions, which may run in FPT-time and additionally set the new parameter to a value that only depends on the old parameter. If a parameterized problem is W[1]-hard, then it is (presumably) not in FPT.

# 3 Layer-wise Restrictions for Temporal Graphs

Two approaches to define temporal graph classes derive from restricting either (i) each layer or (ii) the underlying graph to be contained in some specific graph class. Notably, these restrictions are both independent of the order of the layers and hence appear to not fully capture the temporal characteristics of a given temporal graph. This section considers case (i), i.e. restrictions on the layers of a temporal graph. Restricting the layers to fall into a specific graph class neither captures any temporal aspect of the temporal graph nor the full picture drawn by all layers together. In fact, we show that such restrictions alone are not helpful: TEMPORAL (s, z)-SEPARATION is already NP-complete when each layer consists of at most one edge.

**Lemma 3.1.** There is a polynomial-time many-one reduction that maps any instance ( $G = (V, E, \tau), s, z, k$ ) of TEMPORAL (s, z)-SEPARATION to an equivalent instance ( $G' = (V, E', \tau'), s, z, k$ ) such that each layer in G' has at most one edge and  $\tau' \leq \tau \cdot |V|^4$ .

*Proof.* Let  $\mathbf{G} = (V, \mathbf{E}, \tau)$  be a temporal graph. We construct  $\mathbf{G}' := (V, \mathbf{E}', \tau')$  by concatenating for each layer i of  $\mathbf{G}$  a temporal graph  $\mathbf{G}_i^{|E_i|}$  such that there is a temporal path in  $\mathbf{G}_i^{|E_i|}$  if and only if there is a path in layer i of  $\mathbf{G}$ .

For each layer *i* of  $\mathbf{G}$  we construct a temporal graph  $\mathbf{G}_i := (V, \mathbf{E}_i, \tau_i)$  by fixing an arbitrary total order on the edge set  $E_i = \{e_1, e_2, \ldots, e_m\}$  of layer *i* in  $\mathbf{G}$  and setting the time-edge set of layer *j* of  $\mathbf{G}_i$  to be  $\{(e_j, j)\}$ . Now, we build  $\mathbf{G}' := \mathbf{G}_1^{|E_1|} \circ \mathbf{G}_2^{|E_2|} \circ \cdots \circ \mathbf{G}_{\tau}^{|E_{\tau}|}$ , where  $|E_i|$  is the number of edges in layer *i* of  $\mathbf{G}$  for all  $i \in [\tau]$ . This is obviously a polynomial-time construction. Since, for all  $i \in [\tau]$ ,  $|E_i| \leq |V|^2$  and each  $\mathbf{G}_i$  has  $|E_i|$  many layers, we know that  $\tau' \leq \tau \cdot |V|^4$ .

Let  $i \in [\tau]$  and  $v, w \in V$ . Observe that  $G_i(G)$  is the underlying graph of both,  $G_i$ and  $G_i^{|E_i|}$ . Since every temporal path is also a path in the underlying graph, it is easy to see that for each temporal (v, w)-path in  $G_i^{|E_i|}$  there is a (v, w)-path in layer i of G which visits the vertices in the same order. We claim that for each (v, w)-path P of length  $\ell$  in layer iof G there is a temporal (v, w)-path in  $G_i^{\ell}$  which visits the vertices in the same order. Let  $V(P) =: \{v = v_0, v_1, \ldots, v_{\ell+1} = w\}$  such that  $v_j$  is visited before  $v_{j+1}$ , for all  $j \in [0:\ell]$ . We prove the claim by induction on  $\ell$ . If  $\ell = 1$ , then we know that there is a time-edge between v and w in  $G_1$ . For the induction step we observe that there is a time-edge between  $v = v_0$ and  $v_1$  in  $G_i$  and, by the induction hypothesis, there is a temporal  $(v_1, w)$ -path of length  $\ell - 1$ in  $G_i^{\ell-1}$  which visits the vertices in the same order as P. Since  $\ell \leq |E_i|$ , we have that for each (v, w)-path in layer i of G there is a temporal (v, w)-path in  $G_i^{|E_i|}$  which visits the vertices in the same order, where  $v, w \in V$  and  $i \in [\tau]$ . If follows that a vertex set  $S \subseteq V \setminus \{s, z\}$  is a temporal (s, z)-separator in G if and only if S is a temporal (s, z)-separator in G', because in the construction of G' we replaced layer i of G with  $G_i^{|E_i|}$ .

Lemma 3.1 together with known hardness reductions for TEMPORAL (s, z)-SEPARATION [19, 31] implies the following.

**Corollary 3.2.** TEMPORAL (s, z)-SEPARATION is NP-complete and W[1]-hard when parameterized by the separator size k even if each layer has at most one edge.

Now we consider a scenario in which the temporal graphs have a certain geometric interpretation. For example in data sets where vertices are individuals and edges model physical proximity (see e.g. [4]), it is a plausible assumption that the individual layers are disc intersection graphs (assuming the individuals only move in the plane). We investigate the restriction to (unit) interval graphs, which constitute the one-dimensional equivalent, meant as a starting point for further research.

Next, we introduce temporal interval graphs. We call a temporal graph  $\boldsymbol{G} = (V, \boldsymbol{E}, \tau)$ a temporal interval graph if every layer  $G_i$  is an interval graph. We say that a temporal graph  $\boldsymbol{G} = (V, \boldsymbol{E}, \tau)$  is a temporal unit interval graph if every layer  $G_i$  is a unit interval graph. By Lemma 3.1, TEMPORAL (s, z)-SEPARATION on temporal unit interval graph is NP-complete. Furthermore the problem remains NP-complete even if  $\tau$  is constant:

**Proposition 3.3.** TEMPORAL (s, z)-SEPARATION on temporal unit interval graphs is NPcomplete for any fixed  $\tau \geq 6$ .

*Proof.* Zschoche et al. showed in [31, Thm. 3.1] by reduction from VERTEX COVER that TEMPORAL (s, z)-SEPARATION is NP-complete for fixed  $\tau \geq 2$ . We modify that proof to ensure that each layer of the resulting temporal graph is a unit interval graph.

#### VERTEX COVER

**Input:** An undirected graph G = (V, E) and an integer  $k \in \mathbb{N}$ .

**Question:** Is there a subset  $V' \subseteq V$  of size at most k such that for all  $\{v, w\} \in E$  it holds  $\{v, w\} \cap V' \neq \emptyset$ ?

The basic idea behind the reduction is to create a gadget for each vertex such that one can use two types of vertex sets to separate s from z in this gadget: a small one and large one. Then, for each edge in the VERTEX COVER instance, we connect the corresponding gadgets in such a way, that at least in one of the gadgets it is necessary to take the large vertex set. Hence, taking the large vertex set from a gadget into the temporal (s, z)-separator corresponds to taking the vertex into the vertex cover.

Let  $\mathcal{I} := (G = (V, E), k)$  be a VERTEX COVER instance and n := |V|. We construct a TEMPORAL (s, z)-SEPARATION instance  $\mathcal{I}' := (\mathbf{G}' := (V', \mathbf{E}', 6), s, z, n + k)$  by setting

$$V' := \{x, v, x_v, x'_v, x_{vw} \mid v, w \in V, x \in \{s, z\}\}$$

and

$$\mathbf{E}' := (E_{\alpha}(s) \times \{1\}) \cup (E_{\alpha}(z) \times \{6\}) \\
\cup (E_{\beta}(s) \times \{2\}) \cup (E_{\beta}(z) \times \{5\}) \\
\cup (E_{\gamma}(s, z) \times \{4\}) \cup (E_{\gamma}(z, s) \times \{3\}) \\
\cup (E_{\delta} \times \{3\})$$

where we define, for any  $x, y \in \{s, z\}$ , the following four edge classes

$$E_{\alpha}(x) := \begin{pmatrix} \{x, x_{v}, x'_{v} : v \in V\} \\ 2 \end{pmatrix},$$

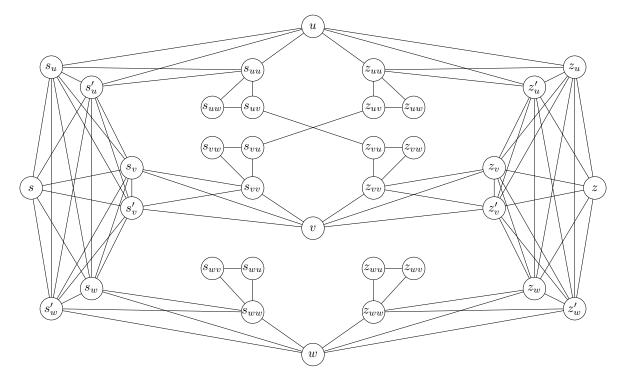
$$E_{\beta}(x) := \bigcup_{v \in V} \begin{pmatrix} \{x_{v}, x'_{v}, x_{vv}\} \\ 2 \end{pmatrix} \cup \begin{pmatrix} \{x_{vw} : w \in V\} \\ 2 \end{pmatrix},$$

$$E_{\gamma}(x, y) := \bigcup_{v \in V} \{\{x_{v}, x'_{v}\}, \{v, x_{v}\}, \{v, x'_{v}\}, \{v, y_{vv}\}\},$$

$$E_{\delta} := \{\{s_{vw}, z_{wv}\}, \{s_{wv}, z_{vw}\} \mid \{v, w\} \in E\}$$

(compare also Figure 2).

Observe that no temporal path can use more than one edge from  $E_{\delta}$  as it would need to use an edge from  $E_{\beta}$  in between. Consequently we may assume that any minimum temporal (s, z)-separator only contains vertices from the set  $\{v, s_{vv}, z_{vv} | v \in V\}$  as we could exchange any other vertex for one of these. After these observations the rest of the proof works in complete analogy to the proof of Zschoche et al. [31, Prop. 3.2].



**Figure 2:** Underlying graph of the TEMPORAL (s, z)-SEPARATION instance resulting from a VERTEX COVER instance G = (V, E) on three vertices  $V = \{u, v, w\}$  and one edge  $E = \{\{u, v\}\}$ .

To see that each layer of G' is in fact a unit interval graph, first observe that  $E_{\gamma}(z, s)$  and  $E_{\delta}$  are vertex-disjoint and thus each connected component of each layer is taken from a single edge class. Furthermore, for any choice  $x, y \in \{s, z\}$ ,

- $E_{\alpha}(x)$  forms a clique of size 2n + 1;
- each connected component of  $E_{\beta}(x)$  consists of a triangle and a size *n* clique that share exactly one vertex;
- each connected component of  $E_{\gamma}(x, y)$  is the union of a triangle and a single edge, joined on a common vertex;
- $E_{\delta}$  is a disjoint union of edges.

In summary, each connected component of each layer is either a clique or a union of two cliques sharing a single vertex and thus an interval graph.  $\Box$ 

# 4 Restrictions of the Underlying Graph

After having investigated layer-wise restrictions, we now turn to case (ii), i.e. the study of temporal graphs whose underlying graph is contained in some graph class. See Figure 1 for an overview of the results.

One such class is that of *complete-but-one* graphs, in which all but one possible edges are present. We show that TEMPORAL (s, z)-SEPARATION is NP-hard even if the underlying graph of the temporal input graph is complete-but-one. The main idea that we can reduce the general problem to that on temporal graphs with a complete-but-one underlying graph by saturating the instance with "useless" edges, that do not create any new temporal (s, z)-paths. **Proposition 4.1.** There is a polynomial-time many-one reduction that maps any instance  $(\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  of TEMPORAL (s, z)-SEPARATION to an equivalent instance  $(\mathbf{G}' = (V, \mathbf{E}', \tau'), s, z, k)$  such that  $E(\mathbf{G}'_{\downarrow}) = {V \choose 2} \setminus \{s, t\}.$ 

*Proof.* We construct G' as  $(V, E', \tau + 2)$  where

$$\begin{split} \boldsymbol{E}' &:= \{(e,t+1) \,|\, (e,t) \in \boldsymbol{E}\} \\ &\cup \left\{ (\{v,w\},1) \,\middle|\, \{v,w\} \in \binom{V \setminus \{s\}}{2} \setminus E(\boldsymbol{G}_{\downarrow}) \right\} \\ &\cup \{(\{s,v\},\tau+2) \,|\, v \in V \setminus \{z\} \wedge \{s,v\} \notin E(\boldsymbol{G}_{\downarrow}) \} \end{split}$$

The one-to-one correspondence of the temporal (s, z)-separators in G and G' is immediate.  $\Box$ 

Proposition 4.1 implies that TEMPORAL (s, z)-SEPARATION remains NP-complete on all temporal graphs where the underlying graph falls into a graph class containing all complete-butone graphs, for instance the classes of unit interval or threshold graphs (see Brandstadt et al. [8] for definitions). We refer to Figure 1 in Section 1 for an overview.

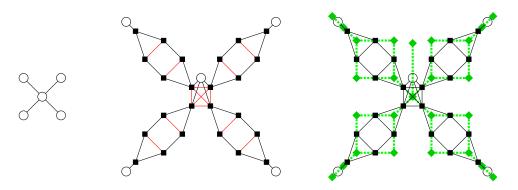
Note that complete-but-one graphs are not line graphs (see Brandstadt et al. [8] for line graphs), as each complete-but-one graph (with at least five vertices) contains the complete-but-one graph on five vertices as an induced subgraph (see Beineke [5, Graph  $G_3$ ]). Hence, we next study TEMPORAL (s, z)-SEPARATION on temporal graphs where the underlying graph is a line graph.

**Proposition 4.2.** TEMPORAL (s, z)-SEPARATION on temporal graphs where the underlying graph is a line graph is NP-complete.

Proof. A temporal (s, z)-path  $P = ((\{s = v_0, v_1\}, t_1), (\{v_1, v_2\}, t_2), \ldots, (\{v_{\ell-1}, v_{\ell} = z\}, t_{\ell}))$  is called *strict* if  $t_i < t_{i+1}$  for all  $i \in [\ell-1]$ . A vertex set S is a *strict temporal* (s, z)-separator if there is no strict temporal (s, z)-path in the temporal graph G - S. The STRICT TEMPORAL (s, z)-SEPARATION problem is the "strict" variant of TEMPORAL (s, z)-SEPARATION and asks for a strict temporal (s, z)-separator instead of a temporal (s, z)-separator.

We reduce from the NP-complete STRICT TEMPORAL (s, z)-SEPARATION where each layer is equal and there is no vertex in the underlying graph of degree at most one [31]. Our reduction is similar to the reduction from STRICT TEMPORAL (s, z)-SEPARATION to TEMPORAL (s, z)-SEPARATION due to Zschoche et al. [31]. Let  $(\boldsymbol{G} = (V, \boldsymbol{E}, \tau), s, z, k)$  be an instance of STRICT TEMPORAL (s, z)-SEPARATION with  $G_i(\boldsymbol{G}) = G_j(\boldsymbol{G})$  for all  $i, j \in [\tau]$ . We construct an instance  $(\boldsymbol{G}' = (V', \boldsymbol{E}', \tau'), s^*, z^*, k)$  of TEMPORAL (s, z)-SEPARATION, where  $\boldsymbol{G}'_{\downarrow}$  is a line graph, as follows.

Let  $G = (V, E) := \mathbf{G}_{\downarrow}$ . We construct a graph G' = (V', E') which will be the underlying graph of  $\mathbf{G}'$  (refer to Figure 3 for an illustration). Let G' be initially a copy of G. As a first step, iteratively replace each vertex  $v \in V(G)$  by a set  $W_v$  of  $\deg(v) + 1$  vertices such that each edge incident with v is incident with exactly one vertex from  $W_v$  and every vertex in  $W_v$ is of degree at most one, where  $\deg(v)$  denotes the degree of v. Note that there is exactly one vertex in  $W_v$  not being incident with an edge, and we call this vertex  $v^*$ . Denote the edge set of G' after the first step by E''. Next, replace each edge  $\{x, y\} \in E'$  by two paths of length three. Denote by  $e^x_{(x,y)}$ ,  $e^y_{(x,y)}$  and by  $e^x_{(y,x)}$ ,  $e^y_{(y,x)}$  the inner vertices of each path respectively, where  $e^x_{(x,y)}$ ,  $e^x_{(y,x)}$  are neighbors of x and  $e^y_{(x,y)}$ ,  $e^y_{(y,x)}$  are neighbors of y. Next, connect the



**Figure 3:** The underlying graph  $G_{\downarrow}$  on the left-hand side, the graph G' in the middle, and the graph H (dotted/green) on the right-hand side. Red edges (stilts) are the only edges present in layer 1.

neighbors of x on both paths by an edge, and connect the neighbors of y on both paths by an edge (we refer to these edges as *path stilts*). Finally, for each  $v \in V$ , turn  $W_v$  into a clique (and refer to all edges in the clique not incident with  $v^*$  as *clique stilts*). This finishes the construction of G'. It is not hard to see that G' is indeed a line graph (see Figure 3 for the graph H for which holds  $\mathcal{L}(H) = G'$ ).

We construct G' with vertex set V' and underlying graph G' as follows. Add the set  $\{(e, 1) \mid e \in E' \text{ is a stilt}\}$ . For each  $2 \leq t \leq 2\tau + 1$ , add the set  $\{(\{v^*, w\}, t) \mid w^* \in W_v \setminus \{v^*\}\}$ . For each  $1 \leq t \leq \tau$ , add the set of temporal edges  $\{(\{x, e^x_{(x,y)}\}, 2t), (\{e^x_{(x,y)}, e^y_{(x,y)}\}, 2t), (\{y, e^y_{(y,x)}\}, 2t), (\{e^x_{(y,x)}, e^y_{(y,x)}\}, 2t) \mid \{x, y\} \in E''\}$  and  $\{(\{x, e^x_{(y,x)}\}, 2t + 1), (\{y, e^y_{(y,x)}\}, 2t + 1) \mid \{x, y\} \in E''\}$ . This finishes the construction of G'. It is not difficult to see that  $G'_{\downarrow} = G'$ .

For the correctness, it is enough to observe the following. There is no temporal  $(s^*, z^*)$ -path starting at time step one. It holds that  $\{v, w\} \in E$  if and only if there is a temporal  $(v^*, w^*)$ path starting at t and ending at t + 1 for every  $2 \leq t \leq 2\tau$  that does not contain any  $u^*$ except for  $v^*$  and  $w^*$ . We can assume a minimum temporal  $(s^*, z^*)$ -separator in G' to only contain vertices in  $\{v^* \mid v \in V\}$ . Hence, the following is immediate: if  $S \subseteq V$  is a strict temporal (s, z)-separator in G, then  $\{v^* \mid v \in S\}$  is a temporal  $(s^*, z^*)$ -separator in G', and vice versa.

An alternative way to classify an instance of a graph-theoretic problem is through its (graph) parameters. We study TEMPORAL (s, z)-SEPARATION according to some parameterizations. In the following we show that any upper bound on the maximum length of a temporal (s, z)-path leads to a straightforward search-tree algorithm. This gives us a tool to solve TEMPORAL (s, z)-SEPARATION on temporal graphs where the underlying graph is restricted in a way that allows us to upper-bound the length of any temporal path.

**Lemma 4.3.** TEMPORAL (s, z)-SEPARATION is solvable in  $\mathcal{O}(\ell^k \cdot |\mathbf{E}|)$  time, and thus is fixedparameter tractable when parameterized by  $k + \ell$ , where k is the solution size and  $\ell$  is the maximum length of a temporal (s, z)-path.

*Proof.* We present a depth-first search algorithm (see Algorithm 1) to show fixed-parameter tractability. Let  $\mathcal{I} := (\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  be an instance of TEMPORAL (s, z)-SEPARA-TION. The basic idea of our algorithm is simple: at least one vertex of each temporal (s, z)-path must be in the temporal (s, z)-separator. Thus, we compute an arbitrary temporal (s, z)-path (Line 4) and branch over all visited vertices of that temporal (s, z)-path (Line 9) until we

Algorithm 1: The algorithm behind Lemma 4.3. **Input:** A temporal graph  $\boldsymbol{G} = (V, \boldsymbol{E}, \tau)$ , two distinct vertices  $s, z \in V$ , and an integer  $k \in \mathbb{N}$ . **Output:** Whether G admits a temporal (s, z)-separator of size at most k. 1 getSeparator( $\emptyset$ , k); 2 output no; 3 function getSeparator(S, k) compute temporal (s, z)-path P in G - S;  $\mathbf{4}$ if there is no temporal (s, z)-path in G - S then  $\mathbf{5}$ 6 output yes; exit; 7 else if k > 0 then 8 for  $v \in V(P) \setminus \{s, z\}$  do 9 getSeparator( $S \cup \{v\}$ , k-1); 10 11 end end 12

cannot find a temporal (s, z)-path in G - S, in which case the algorithm outputs yes, or until we already picked k vertices to be in the temporal (s, z)-separator, in which case the algorithm outputs no. Hence, if the algorithm outputs yes, then S is a temporal (s, z)-separator.

It remains to show that if there is a temporal (s, z)-separator in G, then the algorithm outputs yes. We call a tuple (S', k') a partial solution if there is a temporal (s, z)-separator S of size k such that  $S' \subseteq S$  and  $k' \geq k - |S'|$ . Note that  $(\emptyset, k)$  is a trivial partial solution. Now assume getSeparator is called with a partial solution (S', k'), then we have that either S' is already a temporal (s, z)-separator in which case the algorithm outputs yes, or there is a temporal (s, z)-path P in G - S' and a temporal (s, z)-separator S such that  $S' \subseteq S$ . It is clear that  $S \cap V(P) \neq \emptyset$ , let  $v \in S \cap V(P)$ . At some point the algorithm chooses the vertex vin the for-loop in Line 9 and thus invokes a recursive call with  $(S' \cup \{v\}, k'-1)$ . It is clear that  $(S' \cup \{v\}) \subseteq S$ , we additionally have that  $k' - 1 \geq k - |S' \cup \{v\}|$  since  $v \notin S'$ . Hence, we have that  $(S' \cup \{v\}, k' - 1)$  is a partial solution. Furthermore, we have that  $|S'| < |S' \cup \{v\}|$ . It is easy to see that if there is a partial solution  $(S^*, k^*)$  with  $|S^*| = k$ , then  $S^*$  is a temporal (s, z)separator. This implies that the algorithm eventually finds a temporal (s, z)-separator if one exists and hence is correct.

From Lemma 2.2, we know that we can perform the computation in Line 4 in  $\mathcal{O}(|\mathbf{E}|)$  time. Now, we upper-bound the size of the search tree in which each node is a call of the getSeparator() function. We can upper-bound the maximum depth of the search tree by k as in each recursive call we decrease k by one, until k = 0. Furthermore, a temporal (s, z)-path of length at most  $\ell$  visits at most  $\ell - 1$  vertices different from s and z. Thus we can upper-bound the running time of Algorithm 1 by  $\mathcal{O}(\ell^k \cdot |\mathbf{E}|)$ .

From Lemma 4.3 we can derive that TEMPORAL (s, z)-SEPARATION is linear-time solvable on temporal graph classes where the underlying graph has a constant vertex cover number.<sup>3</sup>

 $<sup>^{3}</sup>$ The vertex cover number of a graph is the size of the smallest vertex subset that intersects all edges of the graph.

**Corollary 4.4.** TEMPORAL (s, z)-SEPARATION can be solved in  $\mathcal{O}((2 \cdot vc)^{vc} \cdot |\boldsymbol{E}|)$  time, and thus is fixed-parameter tractable when parameterized by the vertex cover number vc of the underlying graph.

*Proof.* Let  $\mathcal{I} := (\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  be an instance of TEMPORAL (s, z)-SEPARATION and vc be the vertex cover number of the underlying graph. We prove this in two steps. We first show that the maximum length  $\ell$  of a temporal (s, z)-path is upper-bounded by  $2 \cdot vc$ , and then we show that k can be upper-bounded by vc.

Since at least one endpoint of each edge of the underlying graph  $G_{\downarrow} = (V, E_{\downarrow})$  must be in the vertex cover, the maximum length of a path in  $G_{\downarrow}$ , and hence the maximum length of a temporal (s, z)-path, is at most  $2 \cdot vc$ .

Without loss of generality we assume that there is no temporal (s, z)-path P of length two, because the vertex  $v \in V(P) \setminus \{s, z\}$  must be contained in every temporal (s, z)-separator. We can find such a temporal (s, z)-path by restricting the breadth-first search of Lemma 2.2 such that it explores only vertices which are reachable by a path which contains at most two non-column edges in the static expansion. Let  $V' \subseteq V$  be a vertex cover of size at most vc for  $G_{\downarrow}$ . The graph  $G_{\downarrow} - (V' \setminus \{s, z\})$  does not contain any (s, z)-paths of length greater than two because all remaining edges are incident with s or z. By our assumption, we know that neither of these (s, z)-paths correspond to a temporal (s, z)-path in G. Hence, k < vc or  $\mathcal{I}$ is a yes-instance. It is well-known that if  $G_{\downarrow}$  admits a vertex cover of size vc, then we can compute one in  $\mathcal{O}(2^{vc} \cdot |E_{\downarrow}|)$  time [12]. The application of Lemma 4.3 completes the proof.  $\Box$ 

Another graph parameter which upper-bounds the maximum length of an (s, z)-path in the underlying graph is the *tree-depth* of the underlying graph. First, we provide a formal definition of tree-depth. For more details, we refer to Nešetřil and de Mendez [28].

**Definition 4.5.** The *tree-depth* for graph G with connected components  $G_1, G_2, \ldots, G_p$  is recursively defined by:

$$\operatorname{td}(G) := \begin{cases} 1 & \text{if } G \text{ has only one vertex,} \\ \max_{i \in [p]} \operatorname{td}(G_i) & \text{if } G \text{ is not connected, and} \\ 1 + \min_{v \in V(G)} \operatorname{td}(G - \{v\}) & \text{if } G \text{ is connected.} \end{cases}$$

**Corollary 4.6.** TEMPORAL (s, z)-SEPARATION is solvable in  $\mathcal{O}(2^{\operatorname{td}(\mathbf{G}_{\downarrow})\cdot k} \cdot |\mathbf{E}|)$  time, and thus is fixed-parameter tractable when parameterized by  $k + \operatorname{td}(\mathbf{G}_{\downarrow})$ , where k is the solution size.

Proof of Corollary 4.6. The tree-depth of a graph G is bounded by  $\log_2(h) \leq \operatorname{td}(G)$  [28, Lemma 17.2], where h denotes the height of a depth-first search tree of G. It follows that  $h \leq 2^{\operatorname{td}(G)}$  and hence, all paths in G are of length at most  $2^{\operatorname{td}(G)}$ . Then, application of Lemma 4.3 completes the proof.

One of the tools from the repertoire for designing fixed-parameter algorithms for (static) graph problems are tree decompositions [12, 14, 17, 29]. A tree decomposition is a mapping of a graph into a related tree-like structure. For many graph problems this tree-like structure can be used to formulate a bottom-up dynamic program that starts at the leaves and ends at the root of the tree decomposition. Indeed, if we parameterize by the treewidth of the underlying graph tw( $G_{\downarrow}$ ), then we obtain an XP-algorithm by dynamic programming. Furthermore, if we add the maximum label  $\tau$  to the parameter, then we obtain fixed-parameter tractability when parameterized by tw( $G_{\downarrow}$ ) +  $\tau$ .

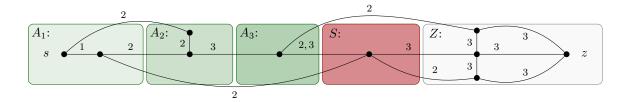


Figure 4: The idea for the dynamic program from Theorem 4.7 for a temporal graph G. Vertices in S form the temporal (s, z)-separator, vertices in Z are not reachable from s in G - S, and vertices in  $A_t$  are not reachable from s in G - S before time t.

**Theorem 4.7.** For a given tree decomposition of the underlying graph, one can solve TEMPO-RAL (s, z)-SEPARATION in  $\mathcal{O}((\tau+2)^{\operatorname{tw}(\mathbf{G}_{\downarrow})+2} \cdot \operatorname{tw}(\mathbf{G}_{\downarrow}) \cdot |V| \cdot |\mathbf{E}|)$  time, where  $\tau$  is the maximum time label.

Theorem 4.7 is proved by constructing a dynamic program which is based on the fact that for each vertex  $v \in V \setminus \{s\}$  in a temporal graph  $\mathbf{G} = (V, \mathbf{E}, \tau)$  there is a point of time  $t \in [\tau]$  such that v cannot be reached from  $s \in V$  before time t. In particular, we guess a partition V = $A_1 \uplus A_2 \uplus \ldots \uplus A_\tau \uplus S \uplus Z$  such that (i) S is the temporal (s, z)-separator, (ii) in  $\mathbf{G} - S$  no vertex contained in Z is reachable from s, and (iii) no vertex  $v \in A_t$  can be reached from sbefore time step t, where  $t \in [\tau]$ . See Figure 4 for an illustrative example. Due to its length, the formal proof of Theorem 4.7 is deferred to  $\mathbf{A}$ .

Note that this result implies that TEMPORAL (s, z)-SEPARATION is fixed-parameter tractable when parameterized by tw $(\mathbf{G}_{\perp}) + \tau$ .

It remains open whether TEMPORAL (s, z)-SEPARATION is fixed-parameter tractable when parameterized by tw $(\mathbf{G}_{\downarrow})$  or by  $k + \text{tw}(\mathbf{G}_{\downarrow})$ .

### 5 Temporal Restrictions

In Sections 3 and 4 we considered restrictions on the layers and the underlying graph. Importantly, these restrictions do not cover essential temporal aspects of a temporal graph, that is, any reordering of the layers yields a different temporal graph obeying the same restrictions. In this section, we study temporal graph classes whose definitions do rely on the order of the layers. Herein, we study *monotone*, *periodic*, *consecutively connected*, and *steady* temporal graphs.

Note that the properties *monotone*, *periodic*, and *consecutively connected* yield quite specific temporal graph classes [11]. Unfortunately, even on these specific temporal graph classes, except for trivial cases, we obtain hardness by straight-forward arguments. We refer to Table 1 for an overview on our results.

Monotone temporal graphs. Intuitively, a temporal graph is p-monotone if it can be decomposed into p time intervals in each of which the layers are ordered by inclusion.

**Definition 5.1.** A temporal graph  $G = (V, E, \tau)$  is *p*-monotone if  $p \in \mathbb{N}$  is the smallest number such that there are  $1 = i_1 < i_2 < \ldots < i_{p+1} = \tau$  such that for all  $\ell \in [p]$ 

- $E_j \subseteq E_{j+1}$  for all  $i_{\ell} \leq j < i_{\ell+1}$ , or
- $E_j \supseteq E_{j+1}$  for all  $i_{\ell} \le j < i_{\ell+1}$

	Temporal $(s, z)$ -Se	Temporal $(s, z)$ -Separation	
	polynomial-time	NP-hard	
<i>p</i> -monotone temporal graphs	p = 1	$p \ge 2$	
p-periodic temporal graphs	$p = 1$ , or $r \ge n$	$p \ge 2$	
T-interval connected temporal graphs		$T \ge 1$	
$\lambda$ -steady temporal graphs	$\lambda = 0$ or $(\lambda, \tau \text{ const.})$	$\lambda \geq 1$	

**Table 1:** Summary of the results of Section 5, where  $\tau$  denotes the maximum time label and r the number of periods in G.

holds.

Khodaverdian et al. [21] call a temporal graph monotone if whenever an edge is contained in a layer, this edge is contained in all succeeding layers. Their motivation is based on temporal graphs that model activation of proteins or, more generally, activation by connected components. Note that their definition of monotone temporal graphs is equivalent to our definition of 1-monotone temporal graphs where each layer is a subgraph of its successor.

If a temporal graph G has a layer  $G_i = G_{\downarrow}$ , then TEMPORAL (s, z)-SEPARATION can trivially be solved by finding an (s, z)-separator in  $G_i$ . In that case we call G grounded. Therefore, a straightforward application of the folklore Ford-Fulkerson algorithm gives the following:

**Observation 5.2.** TEMPORAL (s, z)-SEPARATION is solvable in  $\mathcal{O}(k \cdot |\mathbf{E}|)$  time on grounded temporal graphs, where k is the solution size and  $|\mathbf{E}|$  the number of time-edges.

Note that 1-monotone temporal graphs are always grounded. However, the situation changes already when the temporal graph is 2-monotone but not grounded. To see that, first note that one can make every temporal graph  $\tau$ -monotone by simply adding edge-free layers between any two consecutive layers, formally:

**Observation 5.3.** There is a polynomial-time many-one reduction that maps any instance  $(\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  of TEMPORAL (s, z)-SEPARATION to an equivalent instance  $(\mathbf{G}' := (V, \mathbf{E}', 2\tau - 1), s, z, k)$  such that for all  $i \in [\tau]$  it holds that  $E_{2i-1}(\mathbf{G}') = E_i(\mathbf{G})$  and for all  $i \in [\tau - 1]$  it holds that  $E_{2i}(\mathbf{G}') = \emptyset$ .

As TEMPORAL (s, z)-SEPARATION is already NP-complete for  $\tau = 2$  [31], this yields the following.

**Observation 5.4.** For all  $p \ge 2$ , TEMPORAL (s, z)-SEPARATION on p-monotone temporal graphs is NP-complete.

**Periodic temporal graphs.** In several real-world scenarios one observes periodicity; indeed, whenever one observes mobile entities with periodic movements [11], such as satellites or (scheduled) public transport, over longer time periods, periodic patterns appear. Such models motivate the following class of temporal graphs.

**Definition 5.5.** A temporal graph  $G = (V, E, \tau)$  is *p*-periodic if  $p \in \mathbb{N}$  is the smallest number such that  $G = G'^r$  for some G' = (V, E', p) and r is called the number of periods.

Different notions of periodic temporal graphs exist in the literature. Flocchini et al. [16] consider periodic temporal graphs obtained from "carriers", that is, a set of strict temporal paths define a network. Liu and Wu [23] consider delay-tolerant networks where vertices have some cyclic movement pattern and get connected when they are in reach: if the time steps are large enough, then periodicity is observed. In both cases, the smallest common multiple of the time spans of the entities define the length of a period. Casteigts et al. [11, Class 8] define periodic temporal graphs by periodicity of edges, that is, for all edges e, time steps t, and  $c \in \mathbb{N}$ , edge e is present at time step t if and only if e is present at time step  $t + c \cdot p$ , where p is the periodicity. They require the underlying graph to be connected, but they do not require minimality on the periodicity.

We know that TEMPORAL (s, z)-SEPARATION is NP-complete on 2-periodic temporal graphs [31]. Contrarily, on 1-periodic temporal graphs, TEMPORAL (s, z)-SEPARATION collapses to (s, z)-SEPARATION in the underlying graph. Surprisingly, if the number of periods is large enough, then the problem becomes polynomial-time solvable.

Let P be an (s, z)-path of length  $\ell$  in the underlying graph  $G_{\downarrow}$  of the temporal graph  $G = (V, E, \tau)$ . A sequence  $P' = ((e_1, t_1), (e_2, t_2), \dots, (e_{\ell}, t_{\ell}))$  of  $\ell$  time-edges from E is a realization of  $P(P' \simeq P)$  if  $(e_1, e_2, \dots, e_{\ell})$  is P. Note, that the sequence of labels of P' is not necessarily non-decreasing. Intuitively, we want measure how many non-decreasing points a realization of P must have. The distance to temporality of P in G is  $\min_{P' \simeq P} |f_{P'}| - 1$ , where  $|f_{P'}|$  is the number of monotonically increasing intervals of the function  $f_{P'} : [\ell] \to [\tau], f_{P'}(x) := t_x$  where  $t_x$  is the label of the x-th time-edge of P'. Furthermore, the distance to temporality from s to z in G is the maximum distance to temporality over all (s, z)-paths in  $G_{\downarrow}$ .

**Lemma 5.6.** Let  $G = G'^r$  be a *p*-periodic temporal graph such that the number of periods *r* is at least the distance to temporality from *s* to *z* in G'. Then TEMPORAL (s, z)-SEPARATION is solvable in  $\mathcal{O}(k \cdot |\mathbf{E}|)$  time, where *k* is the solution size and  $|\mathbf{E}|$  the number of time-edges.

*Proof.* Let  $\mathbf{G} = \mathbf{G'}^r$  be a *p*-periodic temporal graph such that the number of periods *r* is at least the distance to temporality from *s* to *z* in  $\mathbf{G'}$ . Then every (s, z)-path in  $\mathbf{G}_{\downarrow}$  forms a temporal (s, z)-path in  $\mathbf{G}$ . Hence, we can compute a minimum (s, z)-separator in  $\mathbf{G}_{\downarrow}$ , by *k* rounds of the Ford-Fulkerson algorithm, to solve TEMPORAL (s, z)-SEPARATION.

Observe that the distance to temporality from s to z is two in the temporal graph from the reduction of Zschoche et al. [31] for maximum label  $\tau = 2$ . Thus TEMPORAL (s, z)-SEP-ARATION is NP-complete, even if the input temporal graph  $G = G'^r$  is p-periodic and the number of periods r is one less than the distance to temporality from s to z in G'.

However, the distance to temporality is clearly upper-bounded by the number of vertices. Hence, we obtain the following.

**Corollary 5.7.** Let  $G = (V, E, \tau)$  be a *p*-periodic temporal graph. If the number of periods  $r \ge |V|$ , then TEMPORAL (s, z)-SEPARATION is solvable in  $\mathcal{O}(k \cdot |E|)$  time, where k is the solution size and |E| the number of time-edges.

**Interval-connected temporal graphs.** Kuhn et al. [22, Definition 2.1] introduced the following class of temporal graphs.

**Definition 5.8.** A temporal graph  $G = (V, E, \tau)$  is *T*-interval connected for  $T \ge 1$  if for every  $t \in [\tau - T + 1]$  the static graph  $G := (V, \bigcap_{i=t}^{t+T-1} E_i(G))$  is connected.

Kuhn et al. [22] studied T-interval connected temporal graphs in the context of counting and token dissemination. Note that temporal graphs where each layer is connected are 1interval connected temporal graphs, but are not necessarily T-interval connected for some  $T \ge$ 2. On the contrary, for every T-interval connected temporal graph it holds that each layer is connected.

**Observation 5.9.** There is a polynomial-time many-one reduction that maps any instance  $(\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  of TEMPORAL (s, z)-SEPARATION to an equivalent instance  $(\mathbf{G}' = (V', \mathbf{E}', \tau), s, z, k+1)$  such that  $\mathbf{G}'$  is T-interval connected for every  $T \geq 1$ .

Proof. Let instance  $\mathcal{I} = (\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  of TEMPORAL (s, z)-SEPARATION be given. Obtain the temporal graph  $\mathbf{G}'$  from  $\mathbf{G}$  by adding a vertex v to  $\mathbf{G}$  and connect v to all other vertices in V in each layer of  $\mathbf{G}$ . Clearly, every temporal (s, z)-separator in  $\mathbf{G}'$  contains vertex v. As  $\mathbf{G} = \mathbf{G}' - v$ , instance  $(\mathbf{G}', s, z, k+1)$  is equivalent to  $\mathcal{I}$ . Moreover, for any  $T \ge 1$  and  $t \in [\tau - T + 1]$  the graph  $G := (V, \bigcap_{i=t}^{t+T-1} E_i(\mathbf{G}))$  is a supergraph of the star graph with center v and set V of leaves.

**Steady temporal graphs.** When observing a network over time with high resolution, we expect evolutionary instead of revolutionary changes in each time step. For instance, observing any contact network per second, we do not expect many contacts to appear in the same second. More generally, in several real-world scenarios we do not expect big changes from one time step to the other. This assumption motivates the following class of temporal graphs.

**Definition 5.10.** A temporal graph  $G = (V, E, \tau)$  is  $\lambda$ -steady if  $\lambda \in \mathbb{N}$  is the smallest number such that for each point in time  $t \in [\tau - 1]$  the size of the symmetric difference of two consecutive edge sets  $|E_t \triangle E_{t+1}|$  is at most  $\lambda$ .

To the best of our knowledge, this class has not been considered in the literature.

The following shows that many hardness results for temporal graphs are also valid for  $\lambda$ -steady temporal graphs, even if  $\lambda = 1$ .

**Proposition 5.11.** There is a polynomial-time many-one reduction that maps any instance  $(\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  of TEMPORAL (s, z)-SEPARATION to an equivalent instance  $(\mathbf{G}' = (V', \mathbf{E}', \tau'), s, z, k)$  such that  $\mathbf{G}'$  is 1-steady.

Proof. Let  $\mathcal{I} = (\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  be an instance of TEMPORAL (s, z)-SEPARATION. We define  $\mathbf{G}' = (V, \mathbf{E}', \tau')$  as follows. Intuitively, we slowly construct and subsequently deconstruct each layer  $E_i$  of  $\mathbf{G}$ . Formally, for each  $i \in [\tau]$  we write  $E_i := E_i(\mathbf{G}) =:$  $\{(e_{i,j}, i) \mid j \in [|E_i|]\}$  and define an auxiliary temporal graph  $\mathbf{G}_i := (V, \mathbf{E}_i, 2 \cdot |E_i|)$  where  $\mathbf{E}_i := \{(e_{i,j}, t) \mid j \in [|E_i|] \land ||E_i| - t| < j\}$ . In particular, we have  $|E_1(\mathbf{G}_i)| = 1, E_{2 \cdot |E_i|}(\mathbf{G}_i) = \emptyset$ , and  $E_{|E_i|}(\mathbf{G}_i) = E_i$ . Now we construct  $\mathbf{G}'$  as  $\mathbf{G}' := \mathbf{G}_1 \circ \mathbf{G}_2 \circ \ldots \circ \mathbf{G}_{[\tau]}$ . Observe that  $\mathbf{G}'$  is 1-steady. Furthermore, for any temporal (s, z)-path in  $\mathbf{G}'$ , there is a temporal (s, z)-path in  $\mathbf{G}$  that uses the same vertices and vice versa. Hence TEMPORAL (s, z)-SEPARATION is equivalent on inputs  $\mathbf{G}$  and  $\mathbf{G}'$ .

The reduction of Proposition 5.11 increases the maximum label by a factor depending on the input size. Indeed, from previous results [31] it follows that for any fixed  $\lambda$ , TEM-PORAL (s, z)-SEPARATION on  $\lambda$ -steady temporal graphs is fixed-parameter tractable when parameterized by  $\tau$ . **Corollary 5.12.** For any fixed  $\lambda$  we have that TEMPORAL (s, z)-SEPARATION on  $\lambda$ -steady temporal graphs is fixed-parameter tractable when parameterized by the maximum label  $\tau$ .

Proof. For a temporal graph  $G = (V, E, \tau)$ , the vertex set  $W := \{v \in V \mid \exists \{v, w\} \in (\bigcup_{i=1}^{\tau} E_i) \setminus (\bigcap_{i=1}^{\tau} E_i)\} \subseteq V$  is called the *temporal core* of G. Zschoche et al. [31] showed that TEMPORAL (s, z)-SEPARATION is fixed-parameter tractable when parameterized by the size of the temporal core.

The statement follows directly from the fact that the temporal core of a  $\lambda$ -steady temporal graph  $\mathbf{G} = (V, \mathbf{E}, \tau)$  is upper-bounded by  $2 \cdot \lambda \cdot \tau$ .

# 6 (Almost) Order-Preserving Temporal Unit Interval Graphs

In this section, we sort of combine aspects studied in Section 4 (restrictions of the underlying graph) and Section 5 (temporal restrictions). To this end, we focus on temporal graphs where each layer is a unit interval graph and we further restrict how much the intervals may change over time. This is a layer-wise restriction with, additionally, a temporal restriction. Recall from Proposition 3.3 that TEMPORAL (s, z)-SEPARATION remains NP-complete on temporal graphs where each layer is a unit interval graph, even if the maximum label  $\tau$  is a small constant.

Now we show in the following that if there is an ordering on the vertices that matches the relative positions of the intervals in all layers, then we can solve TEMPORAL (s, z)-SEP-ARATION in polynomial time. We then generalize this by introducing a parameter that, informally speaking, describes how much the interval orderings may change over time, and show fixed-parameter tractability with respect to the combination of this new parameter and the maximum label  $\tau$ .

We call a total ordering  $\langle V \rangle$  on a vertex set V compatible with a unit interval graph G = (V, E) if there are unit intervals  $[a_v, a_v + 1]$  with  $a_v \in \mathbb{R}$  for all vertices  $v \in V$  that induce the graph G and for all  $u, v \in V$  with  $u <_V v$  we have that  $a_u \leq a_v$ . Note that for every unit interval graph there is a total ordering on the vertices that is compatible with it.

**Definition 6.1.** A temporal graph  $G = (V, E, \tau)$  is an order-preserving temporal unit interval graph if G is a temporal unit interval graph and there is a total ordering  $\langle V \rangle$  on the vertex set V that is compatible with every layer  $G_i$ .

Given an order-preserving temporal unit interval graph  $G = (V, E, \tau)$ , we denote by  $\langle_V$  a compatible total ordering on V. Let n := |V|, and number the vertices in  $V =: \{v_1, v_2, \ldots, v_n\}$  such that  $v_i <_V v_j \Leftrightarrow i \leq j$ . Furthermore, we use the following notation:  $V_{\langle i} := \{v_j \mid 1 \leq j < i\}$  and  $V_{\geq i} := \{v_j \mid n \geq j > i\}$  and  $N^{\geq}_{G_t}(v_i) := N_{G_t}(v_i) \cap V_{\geq i}$ . If the ordering  $\langle_V$  is clear from the context, then we refer to vertices as smaller or larger than other vertices to express that they appear before or after, respectively, in the ordering  $\langle_V$ .

**Lemma 6.2.** Order-preserving temporal unit interval graphs can be recognized in polynomial time and a compatible vertex ordering for a given order-preserving temporal unit interval graph can be computed in polynomial time.

*Proof.* Let  $\mathbf{G} = (V, \mathbf{E}, \tau)$  be a temporal graph. Then, due to Looges and Olariu [24, Theorem 1], we know that  $\mathbf{G}$  is an order-preserving temporal unit interval graph with vertex ordering  $\langle V \rangle$  if and only if the vertices in of every closed neighborhood  $N_{G_i}[v] := N_{G_i}(v) \cup \{v\}$ 

with  $v \in V$  of every layer  $i \in [\tau]$  appear consecutively in the ordering  $\langle_V$ . Thus, the problem can be solved by searching a column ordering of the matrix  $M \in \{0,1\}^{|V| \cdot \tau \times |V|}$  defined by  $M[(i,t),j] = 1 \iff v_j \in N_{G_t}[v_i]$  that has the *consecutive ones property*, a task for which a linear-time algorithm is known [7].

We now state some useful properties of temporal paths and separators in order-preserving temporal unit interval graphs. Due to (iii) of the following lemma, we will henceforth assume without loss of generality that  $v_1 = s$  and  $v_n = z$ .

**Lemma 6.3.** Let  $G = (V, E, \tau)$  be an order-preserving temporal unit interval graph with ordering  $\leq_V$ .

- (i) For all  $1 \le a \le b \le \tau$  and for all  $S \subseteq V$  we have that  $G_{[a:b]} S$  is also an order-preserving temporal unit interval graph.
- (ii) If for some  $1 \leq i < j \leq n$  there is a temporal  $(v_i, v_j)$ -path P in G, then there is temporal  $(v_i, v_j)$ -path P' in G that visits its vertices in the order given by  $<_V$ .
- (iii) Let  $S \subseteq V$  be a temporal  $(v_i, v_j)$ -separator in G for some  $1 \leq i < j \leq n$ . Then  $S' := S \setminus (V_{\leq i} \cup V_{\geq j})$  is also a temporal  $(v_i, v_j)$ -separator in G.
- (iv) A temporal  $(v_i, v_j)$ -separator in G is also a temporal  $(v_{i'}, v_{j'})$ -separator in G for all  $1 \le i' \le i < j \le j' \le n$ .
- (v) Let  $S \subseteq V \setminus \{s, z\}$  such that  $v_i$  is the largest vertex reachable from s in  $\mathbf{G} S$ . Let t denote the first time  $v_i$  is reachable from s in  $\mathbf{G} S$ , and let  $t \leq t' \leq \tau$ . Then  $N_{G_{t'}}^{>}(v_i) \subseteq S$ .
- (vi) Let  $S_1 \subseteq V \setminus \{s, z\}$  such that  $v_i$  is the largest vertex reachable from s in  $\mathbf{G}_{[1:t]} S_1$  for some  $t \in [\tau - 1]$ . Let  $S_2 \subseteq V \setminus \{s, z\}$  such that  $v_j$  is the largest vertex reachable from sin  $\mathbf{G}_{[t+1:\tau]} - S_2$ . If  $i \leq j$ , then  $S := S_1 \cup S_2$  is a temporal (s, z)-separator in  $\mathbf{G}$  such that there is no vertex reachable from s in  $\mathbf{G} - S$  that is larger than  $v_j$ .
- (vii) Let  $S \subseteq V$  be an inclusion-wise minimal temporal (s, z)-separator in G with the property that a given  $v_i$  is the largest vertex that is reachable from s in G - S and let  $v_j$  be the smallest vertex that is not in S such that S is also a temporal  $(s, v_j)$ -separator in G. Then for all  $v_i <_V v <_V v_j$  with  $v_i \neq v \neq v_j$  we have that  $v \in S$ , and we have that  $S \cap V_{>j} = \emptyset$ .

*Proof.* (i): Obvious.

(ii): We prove that there is a temporal  $(v_i, v_j)$ -path P' in G that visits its vertices in the order given by  $<_V$  and  $t \leq t'$ , where t and t' denote the first time label in P and in P', respectively. We give an inductive proof over the number of edges in the temporal  $(v_i, v_j)$ -path P. For the base case, if P has only one edge, then  $E(P) = (\{v_i, v_j\}, t)$  for some  $t \in [\tau]$ . Hence, P' := P clearly is the sought temporal path. Now, assume that the statement holds for all temporal  $(v_i, v_j)$ -paths with at most  $\ell \in \mathbb{N}$  edges. For the inductive step, let P be a temporal  $(v_i, v_j)$ -path with exactly  $\ell + 1$  edges. Let  $v_{i'}$  be the last vertex on P such that  $i' \leq i$ , and let  $t \in [\tau]$  be the index of the layer where P contains the edge  $\{v_{i'}, v_x\}$ , where  $v_x$  is the successor of  $v_{i'}$  on P. Since  $G_t$  is a unit interval graph with order  $<_V$ , the edge  $\{v_i, v_x\}$  is present in  $G_t$ . Denote by  $P_x$  the temporal  $(v_x, v_j)$ -subpath of P, starting at vertex  $v_x$ . Observe that  $P_x$  has at most  $\ell$  edges, and hence there is a path  $P'_x$  visiting its vertices in the order given by  $<_V$  and starting at some time label  $t' \geq t$ . Thus, the path  $P' = (\{v_i\} \cup V(P'_x), \{(\{v_i, v_x\}, t)\} \cup E(P'_x)),$  that starts with edge  $\{v_i, v_x\}, t\}$  and then follows  $P'_x$ , visits its vertices in the order given by  $<_V$  and starts at time label t being at least the first time label appearing on the edges of P.

(iii): Follows directly from (ii).

(iv): Follows directly from (ii).

(v): Suppose not. Then there is a time step t'' with larger neighborhood and hence there is a vertex  $v_j \in N^{>}_{G_{t''}}(v_i) \setminus N^{>}_{G_{t'}}(v_i)$ . Hence,  $v_j$  with j > i is reachable from s in  $G_{[1:t'']} - S$ , contradicting the definition of  $v_i$ .

(vi): Follows directly from (ii).

(vii): Assume towards a contradiction that there is a vertex  $v \notin S$  with  $v_i <_V v <_V v_j$ . Then either v is reachable from s in  $\mathbf{G} - S$ , which would be a contradiction to  $v_i$  being the largest vertex reachable from s in  $\mathbf{G} - S$ , or v is not reachable from s in  $\mathbf{G} - S$ , a contradiction to the assumption that  $v_j$  is the smallest vertex such that S is also a temporal  $(s, v_j)$ -separator in  $\mathbf{G}$ . Furthermore,  $S \cap V_{>j} = \emptyset$  follows from the assumption that S is inclusion-wise minimal and Lemma 6.3(iii).

Now we have the necessary tools to prove that TEMPORAL (s, z)-SEPARATION can be solved in polynomial time on order-preserving temporal unit interval graphs.

**Theorem 6.4.** TEMPORAL (s, z)-SEPARATION on order-preserving temporal unit interval graphs is solvable in  $\mathcal{O}(|V|^2 \cdot \tau^2)$  time.

Proof. Let  $\mathbf{G} = (V, \mathbf{E}, \tau)$  be a given order-preserving temporal unit interval graph and k be a given upper bound on the temporal separator size. By Lemma 6.2 we can find a total vertex ordering  $\langle_V$  compatible with every layer. Assume that there is no layer with an edge between s and z. In order to solve the problem, we use the following dynamic programming table T of size  $\tau \times (n-1)$ . In the table entry T[t,i] we store a minimum temporal (s,z)-separator S for  $\mathbf{G}_{[1:t]}$  with the property that there is no vertex reachable from s in  $\mathbf{G}_{[1:t]} - S$  that is larger than  $v_i$ . Let

$$\mathcal{N}(v, t, t') := \begin{cases} \left\{ N^{>}_{G_{t''}}(v) \mid t \le t'' \le t' \right\}, & \text{if } \forall t \le t'' \le t' : (\{v, z\}, t'') \notin \mathbf{E}, \\ \{V \setminus \{s, z\}\}, & \text{otherwise.} \end{cases}$$

Let T be defined in the following way:

$$T[1,1] := N_{G_1}(s), \tag{1}$$

$$T[t,1] := \underset{S \in \mathcal{N}(s,1,t)}{\operatorname{arg\,max}} |S|, \tag{2}$$

$$T[1,i] := \underset{S \in Y_i}{\operatorname{arg\,min}} |S|, \text{ where } Y_i := \{T[1,i-1]\} \cup \mathcal{N}(v_i,1,1),$$
(3)

$$T[t,i] := \underset{S \in X_{t,i}}{\operatorname{arg\,min}} |S|, \text{ where}$$

$$\tag{4}$$

$$X_{t,i} := \left\{ T[t',i'] \cup \underset{S \in \mathcal{N}(v_i,t'+1,t)}{\operatorname{arg\,max}} |S| \left| i' \in [i-1] \land t' \in [t-1] \right\} \\ \cup \left\{ T[t,i-1] \right\} \cup \left\{ \underset{S \in \mathcal{N}(v_i,1,t)}{\operatorname{arg\,max}} |S| \right\}.$$

We decide whether we face a yes-instance by checking if there is an  $i \in [n-1]$  such that  $|T[\tau, i]| \le k$ .

It is easy to see that each table entry can be computed in  $\mathcal{O}(|V| \cdot \tau)$  time and the table has size  $|V| \cdot \tau$ . Hence, the algorithm has the claimed polynomial running time.

Correctness. We prove by induction on both dimensions of T that T[t, i] is a minimum temporal (s, z)-separator S for  $G_{[1:t]}$  with the property that there is no vertex reachable from s in  $G_{[1:t]}-S$  that is larger than  $v_i$  with respect to  $<_V$ . First, observe that Lemma 6.3(v) implies that T[1, 1] and T[t, 1] are correctly filled in Equations (1) and (2). Hence, the base for our induction is correct.

We proceed with the proof of the cases specified by Equations (3) and (4) in two steps. First we show that for all T[t, i] with  $t \ge 1$  and i > 1, we have that T[t, i] is a temporal (s, z)-separator S for  $G_{[1:t]}$  with the property that there is no vertex reachable from s in  $G_{[1:t]} - S$  that is larger than  $v_i$ . Then, in a second step, we show that said separator is minimum.

It is easy to check that if t = 1, then for all  $i \in [n - 1]$  we have that T[1, i] (as specified in Equation (3)) is a temporal (s, z)-separator with the desired properties. Next, we consider the case that t, i > 1. We show that every set in  $X_{t,i}$  is a temporal (s, z)-separator with the desired properties. By induction we know that this holds for T[t, i - 1]. It is also easy to check that it holds for  $S' := \arg \max_{S \in \mathcal{N}(v_i, 1, t)} |S|$ . For arbitrary  $i' \in [i - 1]$  and  $t' \in [t - 1]$ (Equation (4)) it is also straightforward to see that  $S' := T[t', i'] \cup \arg \max_{S \in \mathcal{N}(v_i, t'+1, t)} |S|$  has the desired properties. By induction, T[t', i'] contains a temporal (s, z)-separator for  $G_{[1:t']}$ with the property that there is no vertex reachable from s in  $G_{[1:t']} - T[t', i']$  that is larger than  $v_{i'}$ . The set  $S'' := \arg \max_{S \in \mathcal{N}(v_i, t'+1, t)} |S|$  either equals  $V \setminus \{s, z\}$ , in which case we clearly have a separator with the desired properties, or it forms a temporal (s, z)-separator for  $G_{[t'+1:t]}$  with the property that there is no vertex reachable from s in  $G_{[t'+1:t]} - S'$  that is larger than  $v_i$ . Then by Lemma 6.3(vi) we get that we have a separator with the desired properties.

Now we show that for all  $t \ge 1$  and i > 1, the separator contained in T[t, i] is of minimum size. Let  $S^* \subseteq V \setminus \{s, z\}$  be a minimum set of vertices such that in  $\mathbf{G}_{[1:t]} - S^*$  the vertex  $v_j$ ,  $j \le i$ , is the largest reachable vertex from s. If j < i, then by induction hypothesis (both for t = 1 and t > 1) we have that  $|S^*| \ge |T[t, i - 1]|$  and hence  $|T[t, i]| \le |S^*|$ .

We continue with the case that j = i. If  $v_i$  is reachable in  $G_{[1:1]} - S^*$  from s, then by Lemma 6.3(v) we know that  $N_{G_{t'}}^>(v_i) \subseteq S^*$  for all  $t' \in [t]$ . As  $S^*$  is minimum, it holds that  $|S^*| = \max_{S \in \mathcal{N}(v_i, 1, t)} |S|$ , and we have that  $\arg \max_{S \in \mathcal{N}(v_i, 1, t)} |S| \in X_{t,i}$  (if t = 1, then  $\arg \max_{S \in \mathcal{N}(v_i, 1, t)} |S| \in Y_i$ ) which implies that  $|T[t, i]| \leq |S^*|$ .

Now assume that t > 1 and  $v_i$  is not reachable from s in  $G_{[1:1]} - S^*$ . Let t' be the largest time-step in which  $v_i$  is not reachable from s in  $G_{[1:t']} - S^*$ , and let i' < i be the largest index such that  $v_{i'}$  is reachable from s in  $G_{[1:t']} - S^*$ . By Lemma 6.3(v), we know that  $S'' := N_{G_{t''}}^>(v_i)$ , where  $t' + 1 \le t'' \le t$  achieves the maximum cardinality, is contained in  $S^*$ . Let S' be the smallest subset of  $S^*$  such that in  $G_{[1:t']} - S'$  the vertex  $v_{i'}$  is the largest reachable vertex from s. By induction hypothesis, we have that  $|S'| \ge |T[t', i']|$ . From Lemma 6.3(vii) it follows that  $S' \cap S'' = \emptyset$ . Hence, because  $S^*$  is minimum, we can write  $S^* = S' \uplus S''$ . Hence, we have

$$|S| = |S'| + |S''| \ge |T[t', i']| + |N_{G_{t''}}^{>}(v_i)| \ge \min_{S \in X_{t,i}} |S| = |T[t, i]|,$$

where the second inequality follows from the fact that  $T[t', i'] \cup N^{>}_{G_{*'}}(v_i) \in X_{t,i}$ .

Next we show how to use the derived polynomial-time algorithm as a basis for a distanceto-triviality parameterization [10, 18]. For a temporal unit interval graph we introduce a parameter  $\kappa$  that bounds how much the compatible vertex orderings of two consecutive layers of a temporal unit interval graph differ. We use the *Kendall tau* distance [20] to measure the similarity of vertex orderings. The Kendall tau distance K is a metric that counts the number of pairwise disagreements between two total orderings; it is also known as "bubble sort distance". We call the parameter  $\kappa$  the *shuffle number* of a temporal unit interval graph and define it as follows.

**Definition 6.5** (Shuffle Number). Given a temporal unit interval graph  $\boldsymbol{G} = (V, \boldsymbol{E}, \tau)$ , its shuffle number  $\kappa$  is the smallest integer such that there are vertex orderings  $\langle_V^1, \langle_V^2, \ldots, \langle_V^\tau \rangle$  with the property that  $\langle_V^t$  is compatible with layer  $G_t$  for all  $t \in [\tau]$ , and the orderings of any two consecutive layers have Kendall tau distance at most  $\kappa$ , that is, for all  $t \in [\tau - 1]$  we have that  $K(\langle_V^t, \langle_V^{t+1}) \leq \kappa$ . We say that the vertex orderings  $\langle_V^1, \langle_V^2, \ldots, \langle_V^\tau \rangle$  witness the shuffle number of  $\boldsymbol{G}$ .

Clearly for order-preserving temporal unit interval graphs we have that  $\kappa = 0$  and it is easy to observe (with the help of Lemma 3.1) that we get NP-completeness for  $\kappa = 1$ . However, if we consider the parameter combination ( $\kappa + \tau$ ) the problem becomes fixed-parameter tractable.

**Theorem 6.6.** Given the a temporal unit interval graph and a vertex orderings that witness its shuffle number  $\kappa$ , TEMPORAL (s, z)-SEPARATION is fixed-parameter tractable when parameterized by  $\kappa + \tau$ , where  $\tau$  is the maximum label.

*Proof.* Let  $G = (V, E, \tau)$  be a temporal unit interval graph given as input together with vertex orderings  $<_V^1, <_V^2, \ldots, <_V^{\tau}$ , and let k be the size bound on the separator. The algorithm proceeds as follows. We first "mark" all vertices u, v with the property that for some  $t \in [\tau - 1]$  we have that  $u <_V^t v$  and  $v <_V^{t+1} u$ , that is, their relative order is flipped at some point in time. We also always mark s and z. Let M be the set of marked vertices. More formally, let M be the largest subset of V that contains s and z with the property that for all  $u \in M \setminus \{s, z\}$  there is a  $v \in M$  and a  $t \in [\tau - 1]$  such that either  $u <_V^t v$  and  $v <_V^{t+1} u$ , or  $v <_V^t u$  and  $u <_V^{t+1} v$ .

Note that we can compute M in polynomial time when given the vertex orderings using bubble sort and we have that  $|M| \leq 2 \cdot \kappa \cdot \tau + 2$ . If M = V, then we can solve the problem in the desired running time by trying out every possible separator. From now on we assume that  $M \neq V$ .

Next, we define two partitions, one for the vertex set M and one for the vertex set  $V' := V \setminus M$ . Intuitively, the partition of V' describes which parts of the orderings stay the same over the whole lifetime of the temporal graph, or in other words, which parts of the graph are order-preserving. The partition of M describes which vertices lie between parts of the temporal graphs that are order-preserving.

We define a partition of the vertices in  $M = M_1 \uplus M_2 \uplus \ldots \uplus M_p$  as follows: Let  $V = \{v_1, v_2, \ldots, v_n\}$  be the vertex ordering given by  $<_V^1$  (that is,  $v_i <_V^1 v_j$  if and only if i < j).

- We have that  $s \in M_1$  and  $z \in M_p$ .
- If  $v_i \in M$  and  $v_{i+1} \in M$  for some  $i \in [n-1]$ , then  $v_i \in M_j$  and  $v_{i+1} \in M_j$  for some  $j \in [p]$ .
- If  $v_i \in M_j$  and  $v_{i'} \in M_j$  with i < i' for some  $j \in [p]$ , then for all  $i < i^* < i'$  we have that  $v_{i^*} \in M_j$ .
- For all  $j \in [p]$  we have that  $M_j \neq \emptyset$ , and if  $v_i$  in  $M_j$  and  $v_{i'}$  in  $M_{j+1}$  for some  $j \in [p-1]$ , then we have that i < i'.

Analogously, we define a partition of the remaining vertices  $V' = V'_1 \uplus V'_2 \uplus \ldots \uplus V'_q$  in the following way:

- If  $v_i \in V'$  and  $v_{i+1} \in V'$  for some  $i \in [n-1]$ , then  $v_i \in V'_j$  and  $v_{i+1} \in V'_j$  for some  $j \in [q]$ . If  $v_i \in V'_j$  and  $v_{i'} \in V'_j$  with i < i' for some  $j \in [q]$ , then for all  $i < i^* < i'$  we have that  $v_{i^\star} \in V'_i$ .
- For all  $j \in [q]$  we have that  $V'_{j} \neq \emptyset$ , and if  $v_{i}$  in  $V'_{j}$  and  $v_{i'}$  in  $V'_{j+1}$  for some  $j \in [q-1]$ , then we have that i < i'.

We can easily compute both partitions by iterating over all vertices in V in the order given by  $<_V^1$  and checking whether a vertex is contained in M. It is also easy to check that  $q \leq$  $p+1 \leq \kappa \cdot \tau + 3 \leq n$ , since for all 1 < j < p we have that  $|M_j| \geq 2$ .

Note that any vertex ordering  $<_V^t$  with  $t \in [\tau]$  defines the same partitions.

Now we are ready to construct a separator S. First we guess the set  $M_S := S \cap M$ . Then for each  $1 < j \le p$  we guess the earliest time  $a_j$  a temporal path starting from s should be able to reach a vertex in the set  $M_j$  in G-S or we set  $a_j := \tau + 1$  if no temporal path from s should be able to reach a vertex in  $M_j$  in G - S. For each  $1 \leq j < p$  we guess the earliest time  $d_j > a_j$  a temporal path from s should be able to reach a vertex in  $V'_j$  in G - S or, in other words, leave the set  $M_j$ , or we set  $d_j := \tau + 1$  if no temporal path from s should be able to reach a vertex in  $V'_i$  in G - S.

Now we create the following instances of TEMPORAL (s, z)-SEPARATION on order-preserving temporal unit interval graphs: For each  $j \in [q]$  we do the following: If  $d_i < a_{i+1}$ , then we create an order-preserving temporal unit interval graph by taking the graph  $G_{[d_j:a_{j+1}-1]}[V'_j]$ and adding two new vertices  $s_j$  and  $z_j$ . We further add the time-edge  $(\{s_j, u\}, t)$  to the temporal graph if  $d_j \leq t \leq a_{j+1} - 1$  and  $(\{u', u\}, t) \in E$  for some  $u' \in M_j \setminus M_S$ . We add the edge  $(\{z_j, u\}, t)$  to the graph if  $d_j \leq t \leq a_{j+1} - 1$  and  $(\{u', u\}, t) \in \mathbf{E}$  for some  $u' \in M_{j+1} \setminus M_S$ . We call the constructed graph  $G_j$ . Intuitively, we merge all vertices in  $M_j \setminus M_S$  to a vertex  $s_j$ and all vertices in  $M_{j+1} \setminus M_S$  to a vertex  $z_j$ . It is easy to check that  $G_j$  is an order-preserving temporal unit interval graph. Now we solve the optimization variant of TEMPORAL (s, z)-SEPARATION on  $(G_j, s_j, z_j)$  using Theorem 6.4<sup>4</sup>. Let  $S_j$  be the solution, that is, a minimum temporal  $(s_j, z_j)$ -separator for  $G_j$ . If there is no valid solution or if  $d_j \geq a_{j+1}$ , then we set  $S_j = \emptyset$ .

Finally, we set  $S = \bigcup_{i \in [a]} S_i \cup M_S$ . If  $|S| \leq k$  and there is no temporal (s, z)-path in G-S, then we output yes. Otherwise, we output no.

It is easy to check that the algorithm runs in FPT-time with respect to parameter  $(\kappa + \tau)$ . We next prove the correctness of the algorithm.

 $(\Rightarrow)$ : If the algorithm outputs yes, then we face a yes-instance. This is trivially true since the algorithm does a sanity check as a last step.

 $(\Leftarrow)$ : If we face a yes-instance, then there is a temporal (s, z)-separator  $S^*$  with  $|S^*| \leq k$ for G. We claim that in this case, our algorithm outputs yes. Since we try out all possible sets  $M_S$  we can assume that there is a branch of our algorithm where we have that  $M_S = M \cap S^*$ . Similarly, we can assume that we are in a branch where all values  $a_j$  and  $d_j$  for  $j \in [q]$  are "correct", that is, they are the largest numbers with the property that no vertex  $v \in M_j$  is reachable from s in  $G - S^*$  earlier than  $a_j$  and no vertex  $u \in V'_j$  is reachable from s in  $G - S^*$ earlier than  $d_i$ .

Then we can show that  $S = \bigcup_{j \in [q]} S_j \cup M_S$  is a temporal (s, z)-separator and  $|S| \leq |S^*|$ : We first check that S is a temporal (s, z)-separator. Since  $M \cap S^* = M \cap S$  we know that for each part  $M_j$  with 1 < j < p we have that a temporal path from s that arrives at a vertex in  $M_j$  no earlier than  $a_j$  cannot arrive at a vertex in  $V'_j$  earlier than  $d_j$  in G - S. Furthermore,

<sup>&</sup>lt;sup>4</sup>Note that the corresponding algorithm can easily be modified to output a solution.

no temporal path from s can arrive at a vertex in  $V'_1$  earlier than  $d_1$  in  $\mathbf{G} - S$  and no temporal path from s that arrives at a vertex in  $M_p$  at time  $a_p$  or later can reach z in  $\mathbf{G} - S$ . The sets  $S_j$  are chosen in a way that ensures that a temporal path from s that does not arrive at any vertex in  $V'_j$  earlier than  $d_j$  cannot reach a vertex in  $M_{j+1}$  earlier than  $a_{j+1}$  in  $\mathbf{G} - S_j$  and hence also in  $\mathbf{G} - S$ . We can conclude that S is a temporal (s, z)-separator for  $\mathbf{G}$ . Now assume for contradiction that  $|S| > |S^*|$ . Then there is a set  $S_j$  such that  $|S_j| > |V'_j \cap S^*|$ . This is a contradiction to the fact that  $S_j$  is a minimum temporal  $(s_j, z_j)$ -separator for  $(\mathbf{G}_j, s_j, z_j)$ since  $V'_j \cap S^*$  is also a temporal  $(s_j, z_j)$ -separator for  $(\mathbf{G}_j, s_j, z_j)$  since otherwise there would be a temporal path from s that arrives at a vertex in  $M_{j+1}$  earlier than  $a_{j+1}$  in  $\mathbf{G} - S^*$ . This completes the correctness proof.

Running time. There are  $2^{|M|}$  possible guesses for  $M_S$  and then a total of  $\tau^{2(p-1)}$  possible guesses for the  $a_i$  and  $d_i$  values. The polynomial part of the running time is  $q \cdot \mathcal{O}(|V|^2 \cdot \tau^2)$ . Together with the bounds we know for q, p, and |M| we get a running time upper bound of  $\mathcal{O}((4\tau)^{\tau \cdot \kappa} \cdot (\kappa + \tau) \cdot |V|^2 \cdot \tau^2)$ .

We remark that is remains an open question how to compute the shuffle number of a given temporal unit interval graph and a set of vertex orderings that witness the shuffle number. We conjecture that deciding whether a temporal unit interval graph has shuffle number  $\kappa = 1$  is already NP-hard.

# 7 Conclusion

We studied TEMPORAL (s, z)-SEPARATION on different temporal graph classes—with structural and temporal restrictions on temporal graph models. We proved TEMPORAL (s, z)-SEPARATION to remain NP-complete on the majority of the considered classes of restricted temporal graphs. Polynomial-time solvability was achieved for temporal graphs where the underlying graph has bounded treewidth, on grounded temporal graphs, temporal graphs with many periods, and temporal graphs where each layer is a unit interval graph with respect to the same vertex ordering.

Our results exemplify that many notions of temporal graph classes that are currently considered in the literature do not impose useful restrictions on temporal graphs when dealing with separation problems. We introduced the class of order-preserving temporal unit interval graphs which is more restrictive than just requiring the layers to fall into a specific graphs class. However, also this notion does not capture temporal aspects (that is, it is invariant under reordering of layers). We defined a distance measure for temporal unit interval graph to order-preserving temporal unit interval graph, the *shuffle number* of a temporal unit interval graph, and showed that this is a useful restriction for TEMPORAL (s, z)-SEPARATION. Exploring further, more sophisticated structural restrictions of temporal graphs, whose definitions may rely on global properties and on temporal aspects, is of particular interest when asking for computationally tractable cases of TEMPORAL (s, z)-SEPARATION.

We further briefly discuss STRICT TEMPORAL (s, z)-SEPARATION, the main difference to TEMPORAL (s, z)-SEPARATION being that we are looking for a *strict* temporal (s, z)-separator that removes all *strict* temporal (s, z)-paths from the input graph. A temporal path is *strict* if the time-edges of the path have strictly increasing time labels. In certain circumstances STRICT TEMPORAL (s, z)-SEPARATION and TEMPORAL (s, z)-SEPARATION can behave very differently with respect to their computational complexity [31], nevertheless we believe that most of our results can be adapted to the strict case. More specifically, we believe that the results presented in Section 3 and Section 4 all carry over, however the algorithms of course need suitable adjustments. For our results on temporal restrictions (Section 5) it is easy to show that most of the polynomial-time solvable cases become NP-hard in the strict case. This follows from the fact that STRICT TEMPORAL (s, z)-SEPARATION is NP-complete even if all layers are the same, or in other words, all edges appear either in all time steps or never [31]. We also believe that the algorithm of Section 6 concerning temporal unit interval graph can be adapted to the strict case.

# References

- E. C. Akrida, L. Gąsieniec, G. B. Mertzios, and P. G. Spirakis. The complexity of optimal design of temporally connected graphs. *Theory of Computing Systems*, 61(3):907–944, Oct 2017. 2
- [2] E. C. Akrida, J. Czyzowicz, L. Gąsieniec, Ł. Kuszner, and P. G. Spirakis. Temporal flows in temporal networks. *Journal of Computer and System Sciences*, 103:46–60, 2019.
- K. Axiotis and D. Fotakis. On the size and the approximability of minimum temporally connected subgraphs. In Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming (ICALP '16), volume 55 of LIPIcs, pages 149:1–149:14. Schloss Dagstuhl Leibniz-Zentrum fuer Informatik, 2016. 2, 3
- [4] A. Barrat and J. Fournet. Contact patterns among high school students. *PLoS ONE*, 9 (9):e107878, 2014.
- [5] L. W. Beineke. Characterizations of derived graphs. Journal of Combinatorial Theory, 9 (2):129–135, 1970.
- [6] H. L. Bodlaender, P. G. Drange, M. S. Dregi, F. V. Fomin, D. Lokshtanov, and M. Pilipczuk. A c<sup>k</sup>n 5-approximation algorithm for treewidth. SIAM Journal on Computing, 45(2):317–378, 2016. 27
- [7] K. S. Booth and G. S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences*, 13(3):335–379, 1976. 18
- [8] A. Brandstadt, J. P. Spinrad, et al. Graph classes: a survey, volume 3. Siam, 1999. 9
- B. Bui-Xuan, A. Ferreira, and A. Jarry. Computing shortest, fastest, and foremost journeys in dynamic networks. *International Journal of Foundations of Computer Science* (*IJFCS '03*), 14(2):267–285, 2003.
- [10] L. Cai. Parameterized complexity of vertex colouring. Discrete Applied Mathematics, 127 (3):415–429, 2003. 20
- [11] A. Casteigts, P. Flocchini, W. Quattrociocchi, and N. Santoro. Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems*, 27(5):387–408, 2012. 2, 3, 13, 14, 15

- [12] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015. 5, 12, 27
- [13] R. Diestel. Graph Theory, 5th Edition, volume 173 of Graduate Texts in Mathematics. Springer, 2016. 4
- [14] R. G. Downey and M. R. Fellows. Fundamentals of Parameterized Complexity. Springer, 2013. 5, 12
- [15] T. Erlebach, M. Hoffmann, and F. Kammer. On temporal graph exploration. In Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP '15), volume 9134 of LNCS, pages 444–455. Springer, 2015. 2, 3
- [16] P. Flocchini, B. Mans, and N. Santoro. On the exploration of time-varying networks. *Theoretical Computer Science*, 469:53–68, 2013. 2, 3, 15
- [17] J. Flum and M. Grohe. Parameterized Complexity Theory, volume XIV of Texts in Theoretical Computer Science. An EATCS Series. Springer Verlag, Berlin, 2006. 5, 12
- [18] J. Guo, F. Hüffner, and R. Niedermeier. A structural view on parameterizing problems: Distance from triviality. In Proceedings of the 1st International Workshop on Parameterized and Exact Computation (IWPEC '04), pages 162–173. Springer, 2004. 20
- [19] D. Kempe, J. Kleinberg, and A. Kumar. Connectivity and inference problems for temporal networks. Journal of Computer and System Sciences, 64(4):820–842, 2002. 2, 3, 6
- [20] M. G. Kendall. A new measure of rank correlation. *Biometrika*, 30(1/2):81–93, 1938. 21
- [21] A. Khodaverdian, B. Weitz, J. Wu, and N. Yosef. Steiner network problems on temporal graphs. CoRR, abs/1609.04918v2, 2016. 14
- [22] F. Kuhn, N. A. Lynch, and R. Oshman. Distributed computation in dynamic networks. In Proceedings of the 42th Annual ACM Symposium on Theory of Computing (STOC '10), pages 513–522. ACM, 2010. 2, 3, 15, 16
- [23] C. Liu and J. Wu. Scalable routing in cyclic mobile networks. *IEEE Transactions on Parallel and Distributed Systems*, 20(9):1325–1338, 2009. 15
- [24] P. J. Looges and S. Olariu. Optimal greedy algorithms for indifference graphs. Computers & Mathematics with Applications, 25(7):15-25, 1993.
- [25] G. B. Mertzios, O. Michail, I. Chatzigiannakis, and P. G. Spirakis. Temporal network optimization subject to connectivity constraints. In *Proceedings of the 40th International Colloquium on Automata, Languages, and Programming (ICALP '13)*, volume 7966 of *LNCS*, pages 657–668. Springer, 2013. 2
- [26] O. Michail. An introduction to temporal graphs: An algorithmic perspective. Internet Mathematics, 12(4):239–280, 2016. 2
- [27] O. Michail and P. G. Spirakis. Traveling salesman problems in temporal graphs. Theoretical Computer Science, 634:1–23, 2016. 2, 3

- [28] J. Nešetřil and P. O. de Mendez. Sparsity: Graphs, Structures, and Algorithms. Springer, 2012. 12
- [29] R. Niedermeier. Invitation to Fixed-Parameter Algorithms. Oxford University Press, 2006. 5, 12
- [30] H. Wu, J. Cheng, Y. Ke, S. Huang, Y. Huang, and H. Wu. Efficient algorithms for temporal path computation. *IEEE Transactions on Knowledge and Data Engineering*, 28(11):2927-2942, 2016. 4
- [31] P. Zschoche, T. Fluschnik, H. Molter, and R. Niedermeier. The complexity of finding separators in temporal graphs. *Journal of Computer and System Sciences*, 107:72–92, 2020. 2, 3, 5, 6, 7, 9, 14, 15, 16, 17, 23, 24, 32

# A Proof of Theorem 4.7

**Theorem 4.7.** For a given tree decomposition of the underlying graph, one can solve TEMPO-RAL (s, z)-SEPARATION in  $\mathcal{O}((\tau+2)^{\operatorname{tw}(\mathbf{G}_{\downarrow})+2} \cdot \operatorname{tw}(\mathbf{G}_{\downarrow}) \cdot |V| \cdot |\mathbf{E}|)$  time, where  $\tau$  is the maximum time label.

We prove Theorem 4.7 by introducing a dynamic program which is executed on a nice tree decomposition.

**Definition A.1.** A tree decomposition  $\mathcal{T} := (T, (B_i)_{i \in V(T)})$  of a graph G is a nice tree decomposition if T is rooted, every node of the tree T has at most two children nodes, and for each node  $i \in V(T)$  the following conditions are satisfied:

- (i) If i has two children nodes  $k, j \in V(T)$  in T, then  $B_i = B_k = B_j$ . Node i is called a *join node*.
- (ii) If i has one child node j, then one of the following conditions must hold:
  - (a)  $B_i = B_i \cup \{v\}$ . Node *i* is called an *introduce node* of *v*.
  - (b)  $B_i = B_i \setminus \{v\}$ . Node *i* is called a *forget node* of *v*.
- (iii) If i is a leaf in T, then  $|B_i| = 1$ . Node i is called a *leaf node*.

For the node  $i \in V(T)$ , the tree  $T_i$  denotes the subtree of T rooted at i. The set  $B(T_i) := \bigcup_{i \in V(T_i)} B_i$  is the union of all bags of  $T_i$ .

Note that a tree decomposition of width  $\mathcal{O}(\operatorname{tw}(G))$  for a given graph G with n vertices can be computed in  $2^{\mathcal{O}(\operatorname{tw}(G))} \cdot n$  time [6] and can be turned into a nice tree decomposition in polynomial-time [12, Lemma 7.4].

We are going to color V with  $\tau + 2$  colors  $\langle A_{[1:\tau]}, S, Z \rangle$ . If a vertex  $v \in V$  has color  $Y \in \{A_{[1:\tau]}, S, Z\}$ , then we denote this by  $v \in Y$ . Thus, formally each color forms a subset of the vertices. The meaning of colors is that if  $v \in S$ , then v is in the temporal (s, z)-separator; if  $v \in Z$ , then v is not reachable from s in  $\mathbf{G} - S$ ; and if  $v \in A_i$ , then v cannot be reached before time point i from s.

**Definition A.2.** We say that  $\langle A_{[1:\tau]}, S, Z \rangle$  is a coloring of  $X \subseteq V(\mathbf{G})$  if  $X = A_1 \uplus A_2 \uplus \cdots \uplus A_\tau \amalg S \amalg Z$ . A coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $X \subseteq V(\mathbf{G})$  is valid if (i)  $s \in A_1$ , (ii)  $z \in Z$ , and (iii) for all  $a \in A_i, a' \in A_j$ , and  $b \in Z$ 

- there is no temporal (a, b)-path with departure time at least i in G[X] S, and
- there is no temporal (a, a')-path with departure time at least i and arrival time at most j 1 in G[X] S.

For  $Y \supseteq X$ , a valid coloring  $\langle A'_{[1:\tau]}, S', Z' \rangle$  of Y is called an *extension* of  $\langle A_{[1:\tau]}, S, Z \rangle$ if  $S \subseteq S', Z \subseteq Z'$ , and  $A_i \subseteq A'_i$  for all  $i \in [\tau]$ . If such an extension exists,  $\langle A_{[1:\tau]}, S, Z \rangle$  is said to be *extendable* to Y.

**Lemma A.3.** Let  $G = (V, E, \tau)$  be a temporal graph, and  $s, z \in V$ . There is a valid coloring  $\langle A_{[1;\tau]}, S, Z \rangle$  of V if and only if S is a temporal (s, z)-separator in G.

*Proof.*  $\Rightarrow$ : Let  $\langle A_{[1:\tau]}, S, Z \rangle$  be a valid coloring of V such that |S| = k. Vertex s has color  $A_1$  and vertex z has color Z. We know that there is no temporal (s, z)-path in G[V] - S = G - S, otherwise condition (iii) of the definition of a valid coloring is violated. Hence, S is a temporal (s, z)-separator of size k in G.

 $\Leftarrow$ : Let S be a given temporal (s, z)-separator of size k in G. Let  $A \subseteq V(G)$  contain all vertices in G - S that are reachable from s. We construct a valid coloring as follows. Assign color Z to all vertices in  $V(\mathbf{G}) \setminus (A \cup S)$ . Note that  $z \in Z$ . For each  $v \in A$  we set  $v \in A_t$  where  $t \in [\tau]$  is the earliest point of time at which v can be reached from s. In particular  $s \in A_1$ . As a consequence, there is no  $w \in A_{t'}$  such that there is a temporal (w, v)-path with departure time at least t' and arrival time at most t - 1, as otherwise there is a temporal (s, v)-path with arrival time at most t - 1 contradicting that t is the earliest time point in which v is reachable from s. Finally, we can observe that there are no  $a \in A_i$  and  $b \in Z$  such that there is a temporal (a, b)-path with departure time at least i, because a can be reached at time point i from s and all vertices of color Z are not reachable in  $\mathbf{G} - S$ . Hence,  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of V.

Let  $G = (V, E, \tau)$  be a temporal graph,  $s, z \in V$ , and  $\mathcal{T} = (T, (B_i)_{i \in V(T)})$  be a nice tree decomposition of  $G_{\downarrow}$  of width tw $(G_{\downarrow})$ . We add s and z to every bag of  $\mathcal{T}$ . Thus,  $\mathcal{T}$  is of width at most tw $(G_{\downarrow}) + 2$ .

In the following, we give a dynamic program on  $\mathcal{T}$ . For each node x in T we compute a table  $D_x$  which stores for each coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  the minimum size of S' over all extensions  $\langle A'_{[1:\tau]}, S', Z' \rangle$  of  $\langle A_{[1:\tau]}, S, Z \rangle$  to  $B(T_x)$ :

$$D_x[A_{[1:\tau]}, S, Z] := \min\left\{\infty, \left|S'\right| \left| \begin{array}{cc} \langle A'_{[1:\tau]}, S', Z' \rangle & \text{is an extension} \\ \text{of } \langle A_{[1:\tau]}, S, Z \rangle & \text{to } B(T_x) \end{array} \right. \right\}$$
(5)

Let  $r \in V(T)$  be the root of T. If  $D_r[A_{[1:\tau]}, S, Z] = k' < \infty$ , then the coloring  $\langle A_{[1:\tau]}, S, Z \rangle$ of  $B_r$  is extendable to  $B(T_r) = V(\mathbf{G})$  and there is a temporal (s, z)-separator of size k' in  $\mathbf{G}$ . Hence, the input instance  $(\mathbf{G}, s, z, k)$  is a yes-instance if and only if  $k' \leq k$ .

The dynamic program first computes the tables for all leaf nodes of T and then, in a "bottom-up" manner, all tables of nodes of which all child nodes are already computed. The computation of  $D_x$ ,  $x \in V(T)$ , depends on the type of x, that is, whether x is a leaf, introduce, forget, or join node.

**Leaf node.** Let  $x \in V(T)$  be a leaf node of  $\mathcal{T}$ . Thus,  $B_x = \{s, v, z\}$ . We test each coloring of  $B_x$  and set  $D_x[A_{[1:\tau]}, S, Z] = \infty$  if  $s \notin A_1$  or  $z \notin Z$ , because the coloring cannot be valid. Assume  $s \in A_1$  or  $z \in Z$ . We distinguish three cases.

**Case 1:** If  $v \in S$ , then this is a valid coloring. We set  $D_x[A_{[1;\tau]}, S, Z] := 1$ .

- **Case 2:** If  $v \in Z$ , then we set  $D_x[A_{[1:\tau]}, S, Z] := \infty$  if there is a  $(\{s, v\}, t) \in E(G[B_x])$ , and  $D_x[A_{[1:\tau]}, S, Z] := 0$  otherwise.
- **Case 3:** If  $v \in A_i$ ,  $i \in [\tau]$ , then we set  $D_x[A_{[1:\tau]}, S, Z] := \infty$  if there is a  $(\{s, v\}, t) \in E(G[B_x])$  with t < i or if there is a  $(\{v, z\}, t) \in E(G[B_x])$  with  $i \le t$ , and  $D_x[A_{[1:\tau]}, S, Z] := 0$  otherwise.

**Lemma A.4.** Let G be a temporal graph and  $\mathcal{T}$  be a tree decomposition of G as described above,  $x \in V(T)$  be a leaf node, and  $\langle A_{[1:\tau]}, S, Z \rangle$  be a coloring of  $B_x$ . Then the following holds:

- (i)  $D_x[A_{[1:\tau]}, S, Z] < \infty$  if and only if  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$ .
- (ii) If  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$ , then  $D_x[A_{[1:\tau]}, S, Z] = |S|$ .
- (iii) The table entry  $D_x[A_{[1:\tau]}, S, Z]$  can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time.

*Proof.* We first prove (i).

 $\Leftarrow$ : Let  $D_x[A_{[1:\tau]}, S, Z] = \infty$  There are five cases in which  $D_x[A_{[1:\tau]}, S, Z]$  is set to  $\infty$ . Either  $s \notin A_1, z \notin Z, v \in Z$  and there is a time-edge  $(\{s, v\}, t) \in E(\boldsymbol{G}[B(T_x)])$ , or  $v \in A_i$  and

there is a time-edge  $(\{s, v\}, t) \in E(\mathbf{G}[B(T_x)])$  with t < i or there is a time-edge  $(\{v, z\}, t) \in E(\mathbf{G}[B(T_x)])$  with  $i \leq t$ , where  $i \in [\tau]$ . It follows that  $\langle A_{[1:\tau]}, S, Z \rangle$  is no valid coloring of  $B_x$ .  $\Rightarrow$ : Let  $D_x[A_{[1:\tau]}, S, Z] < \infty$ . Note that s must be of color  $A_1$  and z must be of color Z. Observe that  $D_x[A_{[1:\tau]}, S, Z] = 0$  or  $D_x[A_{[1:\tau]}, S, Z] = 1$ . Consider the case of  $D_x[A_{[1:\tau]}, S, Z] = 1$ . Thus,  $v \in S$ . This implies that  $\mathbf{G}[B(T_x)] - S$  is time-edgeless and therefore  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$ . Next, consider the case of  $D_x[A_{[1:\tau]}, S, Z] = 0$ . If  $v \in Z$ , then there is no time-edge from s to v which means  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$ . If  $v \in A_i$ , then there is no time-edge  $(\{s, v\}, t)$  with t < i and there is no time-edge from  $(\{z, v\}, t)$  with  $i \leq t$ . In both cases  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$ .

If  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$ , then  $D_x[A_{[1:\tau]}, S, Z] = |S|$  as we set  $D_x[A_{[1:\tau]}, S, Z] = 1$  if and only if  $v \in S$ . This proves (ii). Furthermore, we can check by iterating over all timeedges whether  $\langle A_{[1:\tau]}, S, Z \rangle$  is a valid coloring of  $B_x$  This proves (iii), and hence (i)–(iii) hold true.

**Introduce node.** Let  $x \in V(T)$  be an introduce node of  $\mathcal{T}, y \in V(T)$  denote its child node, and  $B_x \setminus B_y = \{v\}$ . We distinguish three cases.

- **Case 1:** If  $v \in S$ , then we set  $D_x[A_{[1:\tau]}, S, Z] := D_y[A_{[1:\tau]}, S \setminus \{v\}, Z] + 1$ .
- **Case 2:** If  $v \in Z$ , then we set  $D_x[A_{[1:\tau]}, S, Z] := D_y[A_{[1:\tau]}, S, Z \setminus \{v\}]$  if for all  $w \in V$  with  $(\{w, v\}, t) \in E(G[B(T_x)])$  it holds that  $w \in A_i \Rightarrow t < i$ . Otherwise, we set  $D_x[A_{[1:\tau]}, S, Z] := \infty$ .
- **Case 3:** If  $v \in A_i$ ,  $i \in [\tau]$ , then we set  $D_x[A_{[1:\tau]}, S, Z] := D_y[A_{[1:t-1]}, A_i \setminus \{v\}, A_{[i+1:\tau]}, S, Z]$ , if for all  $(\{v, w\}, t) \in E(G[B(T_x)])$  it holds that  $t \ge i \Rightarrow w \in \bigcup_{j=1}^t A_j \cup S$  and  $t < i \Rightarrow w \in \bigcup_{j=t+1}^\tau A_j \cup S \cup Z$ . Otherwise, we set  $D_x[A_{[1:\tau]}, S, Z] := \infty$ .

We prove the correctness for each case separately. We start with the first case.

**Lemma A.5.** Let G and  $\mathcal{T}$  be as described above,  $x \in V(T)$  be an introduce node of  $v, y \in V(T)$  be the child node of x,  $\langle A_{[1:\tau]}, S, Z \rangle$  be a coloring of  $B_x$  and  $v \in S$ . Then the following holds:

- 1. Coloring  $\langle A_{[1:\tau]}, S \setminus \{v\}, Z \rangle$  of  $B_y$  is extendable to  $B(T_y)$  if and only if coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  is extendable to  $B(T_x)$ .
- 2. The value of  $D_x[A_{[1:\tau]}, S, Z]$  agrees with Equation (5) and can be computed in  $\mathcal{O}(1)$  time.

Proof.  $\Rightarrow$ : Let  $\langle A_{[1:\tau]}, S \setminus \{v\}, Z \rangle$  be a coloring of  $B_y$  and  $\langle A'_{[1:\tau]}, S', Z' \rangle$  be an extension to  $B(T_y)$ , where  $|S'| = D_y[A_{[1:\tau]}, S \setminus \{v\}, Z]$ . Note that  $v \notin S'$ , because  $v \notin B(T_y)$  since xis the introduce node for v. Since  $B(T_x) \setminus B(T_y) = \{v\}$ , we know that  $G[B(T_y)] - S'$  is the same temporal graph as  $G[B(T_x)] - (S' \cup \{v\})$ . Hence, the coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  is extendable to  $B(T_x)$  and  $|S' \cup \{v\}| = |S'| + 1$  implies that the table entry  $D_x[A_{[1:\tau]}, S, Z] = D_y[A_{[1:\tau]}, S \setminus \{v\}, Z] + 1$ .

 $\begin{array}{ll} \leftarrow: \ \operatorname{Let} \ \langle A_{[1:\tau]}, S \setminus \{v\}, Z \rangle \ \text{be not extendable to} \ B(T_y), \ \text{then} \ \langle A_{[1:\tau]}, S, Z \rangle \ \text{is not extendable to} \ B(T_x) \ \text{because} \ \boldsymbol{G}[B(T_y)] \ \text{is a temporal subgraph of} \ \boldsymbol{G}[B(T_x)], \ \text{where} \ v \notin B(T_y). \\ \text{Hence,} \ D_x[A_{[1:\tau]}, S, Z] = D_y[A_{[1:\tau]}, S \setminus \{v\}, Z] + 1 = \infty + 1 = \infty. \end{array}$ 

Note that  $D_x[A_{[1:\tau]}, S, Z]$  can be computed in  $\mathcal{O}(1)$  time because we just have to look up the value of  $D_y[A_{[1:\tau]}, S, Z]$ .

Next, we move to the correctness of the second case.

**Lemma A.6.** Let G and  $\mathcal{T}$  be as described above,  $x \in V(T)$  be an introduce node of  $v, y \in V(T)$  be the child node of x,  $\langle A_{[1:\tau]}, S, Z \rangle$  be a coloring of  $B_x$  and  $v \in Z$ . Then the following holds:

- 1. The coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  is extendable to  $B(T_x)$  if and only if the coloring  $\langle A_{[1:\tau]}, S, Z \rangle \{v\} \rangle$  of  $B_y$  is extendable to  $B(T_y)$  and for all  $(\{w, v\}, t) \in E(\mathbf{G}[B(T_x)])$  it holds that  $w \in A_i \Rightarrow t < i$ .
- 2. The value of  $D_x[A_{[1:\tau]}, S, Z]$  agrees with Equation (5) and can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time.

Proof.  $\Rightarrow$ : Let the coloring  $\langle A'_{[1:\tau]}, S', Z' \rangle$  be an extension of the coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  to  $B(T_x)$ . Since  $B(T_y) = B(T_x) \setminus \{v\}$  and  $(Z \setminus \{v\}) \subseteq Z \subseteq Z'$ , the coloring  $\langle A_{[1:\tau]}, S, Z \setminus \{v\} \rangle$  of  $B_y$  is extendable to  $B(T_y)$ . Furthermore,  $v \in Z$  implies that for all time-edges  $(\{w, v\}, t) \in E(\boldsymbol{G}[B(T_x)])$  it holds that  $w \in A_i \Rightarrow t < i$ .

 $\Leftarrow$ : First, if coloring  $\langle A_{[1:\tau]}, S, Z \setminus \{v\} \rangle$  of  $B_y$  is not extendable to  $B(T_y)$  then coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  cannot be extendable to  $B(T_x)$  because  $G[B(T_y)]$  is a temporal subgraph of  $G[B(T_x)]$ . Hence,  $D_x[A_{[1:\tau]}, S, Z] = \infty$ .

Let  $\langle A_{[1:\tau]}, S, Z \setminus \{v\} \rangle$  be a coloring of  $B_y$  which is extendable to  $B(T_y)$  and for all  $(\{w, v\}, t) \in E(\mathbf{G}[B(T_x)])$  it holds that  $w \in A_i \Rightarrow t < i$ . Then we know that there is an extension  $\langle A'_{[1:\tau]}, S', Z' \rangle$  of  $\langle A_{[1:\tau]}, S, Z \setminus \{v\} \rangle$  to  $B(T_y)$ . We claim that  $\langle A'_{[1:\tau]}, S', Z' \cup \{v\} \rangle$  is a valid coloring of  $B(T_x)$ . Since  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a valid coloring of  $B(T_y)$ , we have that  $s \in A'_1, z \in Z'$ , and for all  $i, j \in [\tau]$ ,  $a \in A'_i$  and  $a' \in A'_j$  there is no temporal (a, a')-path with departure time at least i and arrival time at most j - 1 in  $\mathbf{G}[B(T_y)] - S$ . Suppose there exist  $a \in A'_i$  and  $b \in Z'$  such that there is a temporal (a, b)-path P in  $\mathbf{G}[B(T_x)] - S$  with departure time at least i, for some  $i \in [\tau]$ . Since  $B(T_x) \setminus B(T_y) = \{v\}$  and  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a valid coloring of  $B(T_y)$ , vertex v is the first vertex of color Z which is visited by P. Hence, there is a time-edge  $(\{w, v\}, t) \in E(\mathbf{G}[B(T_x)])$  such that  $w \in A_i$  and  $i \leq t$ , contradicting  $w \in A_i \Rightarrow t < i$ . It follows that  $\langle A'_{[1:\tau]}, S', Z' \cup \{v\} \rangle$  is a valid coloring of  $B(T_x)$ . Since  $v \in Z$ , we have  $D_x[A_{[1:\tau]}, S, Z] = D_y[A_{[1:\tau]}, S, Z \setminus \{v\}]$ .

Note that  $D_x[A_{[1:\tau]}, S, Z]$  can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time, since we can decide whether for all  $(\{w, v\}, t) \in E(\mathbf{G}[B(T_x)])$  it holds that  $w \in A_i \Rightarrow t < i$  by iterating once over the time-edges in  $\mathbf{E}$ .

Last, we show the correctness of the third case.

**Lemma A.7.** Let G and  $\mathcal{T}$  be as described above,  $x \in V(T)$  be an introduce node of  $v, y \in V(T)$  be the child node of x,  $\langle A_{[1:\tau]}, S, Z \rangle$  be a coloring of  $B_x$  and  $v \in A_i$ , where  $i \in [\tau]$ . Then the following holds:

- 1. Coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  is extendable to  $B(T_x)$  if and only if coloring  $\langle A_{[1:i-1]}, A_i \setminus \{v\}, A_{[i+1:\tau]}, S, Z \rangle$  of  $B_y$  is extendable to  $B(T_y)$  and for each  $(\{v, w\}, t) \in E(\boldsymbol{G}[B(T_x)])$  it holds that:  $t \ge i \Rightarrow w \in \bigcup_{j=1}^t A_j \cup S$  and  $t < i \Rightarrow w \in \bigcup_{j=t+1}^\tau A_j \cup S \cup Z$ .
- 2. The value of  $D_x[A_{[1:\tau]}, S, Z]$  agrees with Equation (5) and can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time.

*Proof.*  $\Rightarrow$ : Let  $\langle A_{[1:\tau]}, S, Z \rangle$  be a valid coloring of  $B_x$  and  $\langle A'_{[1:\tau]}, S', Z' \rangle$  be an extension to  $B(T_x)$ . Since  $B(T_y) = B(T_x) \setminus \{v\}$  and  $(A_i \setminus \{v\}) \subseteq A_i \subseteq A'_i$ , the coloring  $\langle A_1, A_2, \ldots, A_i \setminus \{v\}, \ldots, A_{\tau}, S, Z \rangle$  of  $B_y$  is extendable to  $B(T_y)$ . Let  $(\{v, w\}, t) \in E(\boldsymbol{G}[B(T_x)])$ . We distinguish two cases.

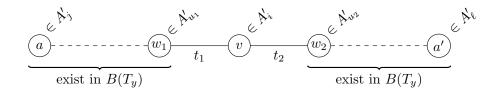


Figure 5: The temporal (a, a')-path P from the proof of Lemma A.7.

First, let  $t \ge i$ . Note that  $w \in B_y$  since x is an introduce node for v. Since  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a valid coloring of  $B(T_x), w \notin Z$  since there is no temporal (v, w)-path with departure time tin  $\mathbf{G}[B(T_x)] - S'$ . Assume towards a contradiction that  $w \in A_j$ , where  $j \in [t + 1 : \tau]$ . Then the time-edge  $(\{v, w\}, t)$  is a temporal (v, w)-path with departure time at least i and arrival time at most j - 1, contradicting the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a valid coloring of  $B(T_x)$ . Hence,  $w \in \bigcup_{i=1}^t A_j \cup S$ .

Second, let t < i. Again,  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a valid coloring of  $B(T_x)$  and therefore  $w \notin \bigcup_{j=1}^t A_j$  because otherwise there would be a temporal (w, v)-path in  $G[B(T_x)] - S'$  with departure time at least t and arrival time t < i, contradicting the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a valid coloring. Hence  $w \in \bigcup_{i=t+1}^{\tau} A_j \cup S \cup Z$ .

 $\Leftarrow$ : First, if coloring  $\langle A_1, A_2, \ldots, A_i \setminus \{v\}, \ldots, A_{\tau}, S, Z \rangle$  of  $B_y$  is not extendable to  $B(T_y)$  then coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  cannot be extendable to  $B(T_x)$  because  $\boldsymbol{G}[B(T_y)]$  is a temporal subgraph of  $\boldsymbol{G}[B(T_x)]$ . Hence,  $D_x[A_{[1:\tau]}, S, Z] = \infty$ .

Let coloring  $\langle A_1, A_2, \ldots, A_i \setminus \{v\}, \ldots, A_{\tau}, S, Z \rangle$  of  $B_y$  be extendable to  $B(T_y)$  and for each  $(\{v, w\}, t) \in E(\mathbf{G}[B(T_x)])$  it holds that:  $t \geq i \Rightarrow w \in \bigcup_{j=1}^t A_j \cup S$  and  $t < i \Rightarrow w \in \bigcup_{j=t+1}^t A_j \cup S \cup Z$ . Then let  $\langle A'_{[1:\tau]}, S', Z' \rangle$  be an extension of  $\langle A_1, A_2, \ldots, A_i \setminus \{v\}, \ldots, A_{\tau}, S, Z \rangle$ to  $B(T_y)$ . We claim that  $\langle A'_1, A'_2, \ldots, A'_i \cup \{v\}, \ldots, S', Z' \rangle$  is a valid coloring for  $B(T_x)$ . We know  $s \in A'_1$  and  $z \in Z'$ .

Suppose towards a contradiction that there exist  $a \in A'_j$  and  $a' \in A'_{\ell}$ ,  $j, \ell \in [\tau]$ , such that there is a temporal (a, a')-path P with departure time at least j and arrival time at most  $\ell - 1$ . Since coloring  $\langle A'_{[1:\tau]}, S', Z' \rangle$  of  $B(T_y)$  is valid, we know that v appears in P. Thus, there are time-edges  $(\{w_1, v\}, t_1), (\{v, w_2\}, t_2) \in E(\mathbf{G}[B(T_x)])$  in P such that  $t_1 \leq t_2$  and  $w_1$  appears before v and v appears before  $w_2$  in P, where  $w_1 \in A'_{u_1}, w_2 \in A'_{u_2}$ . Note that  $w_1 \in A_{u_1}$ and  $w_2 \in A_{u_2}$  as x is an introduce node of v. Refer to Figure 5 for an illustration.

We know the following:

- $u_1 \leq t_1$ , otherwise there is a temporal  $(a, w_1)$ -path with departure time at least j and arrival time at most  $u_1 1$  in  $G[B(T_y)]$ , contradicting the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is valid.
- $i \leq t_1$ , otherwise either  $w_1 \notin \bigcup_{j=t_1+1}^{\tau} A_j \cup S \cup Z$  contradicting the fact that for each  $(\{v, w\}, t) \in E(\mathbf{G}[B(T_x)])$  it holds that  $t < i \Rightarrow w \in \bigcup_{j=t+1}^{\tau} A_j \cup S \cup Z$ , or  $w_1 \in \bigcup_{j=t_1+1}^{\tau} A_j \cup S \cup Z$  and  $w \in A_{u_1}$ , contradicting the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a coloring of  $B(T_y)$ .
- $i \leq t_2$ , otherwise  $i > t_1$  since  $t_1 \leq t_2$ .
- $u_2 \leq t_2$ , otherwise  $i \leq t_2$  and  $w_2 \notin \bigcup_{j'=1}^{t_2} A_{j'}$ , contradicting the fact that for each  $(\{v, w\}, t) \in E(\boldsymbol{G}[B(T_x)])$  it holds that  $t < i \Rightarrow w \in \bigcup_{j=t+1}^{\tau} A_j \cup S \cup Z$ , or  $w_2 \in \bigcup_{j'=1}^{t_2} A_{j'}$  and  $w_2 \in A_{u_2}$ , contradicting the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$  is a coloring of  $B(T_y)$ .

It follows that P contains the temporal  $(w_2, a')$ -path as temporal subpath with departure time at least  $u_2 \leq t_2$  and arrival time  $\ell - 1$ . As this temporal subpath also exists in  $B(T_y)$ , this contradicts the fact that coloring  $\langle A'_{[1:\tau]}, S', Z' \rangle$  of  $B(T_y)$  is valid. We conclude that P does not exist.

Next, suppose towards a contradiction that there exist  $a \in A'_j$ ,  $j \in [\tau]$ , and  $b \in Z$  such that there is a temporal (a, b)-path P' with departure time at least j. The vertex  $v \in A_i$  is the last vertex visited by P' which is not colored by Z, otherwise we would be able to find a subsequence of P' similar to P. Thus, there are time-edges  $(\{w_1, v\}, t_1), (\{v, b\}, t_2) \in E(\mathbf{G}[B(T_x)])$  which are in P' such that  $w_1$  is visited before v and v is visited before b, where  $w_1 \in A'_{u_1}$ . We conclude analogously to the case of P that  $u_1 \leq t_1$ ,  $i \leq t_1$ ,  $i \leq t_2$ . Since  $i \leq t_2$ , we have that either  $b \notin \bigcup_{j=1}^t A_j \cup S$ , contradicting the fact that for each  $(\{v, w\}, t\} \in E(\mathbf{G}[B(T_x)])$  it holds that  $t \geq i \Rightarrow w \in \bigcup_{j=1}^t A_j \cup S$ , or  $b \in \bigcup_{j=1}^t A_j \cup S$  and  $b \in Z$ , contradicting the fact that  $\langle A_1, A_2, \ldots, A_i \setminus \{v\}, \ldots, A_{\tau}, S, Z\rangle$  is a coloring of  $B_y$ . Hence, P' does not exist.

Clearly,  $D_x[A_{[1:\tau]}, S, Z] = D_y[A_1, A_2, \dots, A_i \setminus \{v\}, \dots, A_{\tau}, S, Z]$  because  $v \notin S$ .

Note that  $D_x[A_{[1:\tau]}, S, Z]$  can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time because we can iterate once over the time-edge set  $\mathbf{E}$  and decide if for all  $(\{w, v\}, t) \in E(\mathbf{G}[B(T_x)])$  it holds that if  $t \geq i$ then  $w \in \bigcup_{j=1}^t A_j \cup S$  and if t < i then  $w \in \bigcup_{j=t+1}^\tau A_j \cup S \cup Z$ .  $\Box$ 

Forget node. Let  $x \in V(T)$  be a forget node of  $\mathcal{T}, y \in V(T)$  its child, and  $B_y \setminus B_x = \{v\}$ . We set

$$D_x[A_{[1:\tau]}, S, Z] = \min \left\{ \begin{array}{l} \min_{i \in [\tau]} D_y[A_{[1:i-1]}, A_i \cup \{v\}, A_{[i+1:\tau]}, S, Z], \\ D_y[A_{[1:\tau]}, S \cup \{v\}, Z], \\ D_y[A_{[1:\tau]}, S, Z \cup \{v\}] \end{array} \right\}$$

**Lemma A.8.** Let G and  $\mathcal{T}$  be as described above,  $x \in V(T)$  be a forget node of  $v, y \in V(T)$  be the child node of x, and  $\langle A_{[1:\tau]}, S, Z \rangle$  be a coloring of  $B_x$ . Then the following holds:

- 1. The coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  is extendable to  $B(T_x)$  if and only if it has an extension to  $B_y$  which is itself extendable to  $B(T_y)$ .
- 2. The value of  $D_x[A_{[1:\tau]}, S, Z]$  agrees with Equation (5) and can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time.

Proof.  $\Rightarrow$ : Let  $\langle A''_{[1:\tau]}, S'', Z'' \rangle$  be an extension of  $\langle A_{[1:\tau]}, S, Z \rangle$  to  $B(T_x)$ . Since y is a child of x and  $B_x \subseteq B_y$ , we know that  $B(T_x) = B(T_y)$  and therefore there is a coloring  $\langle A'_{[1:\tau]}, S', Z' \rangle$ of  $B_y$  which is extendable to  $B(T_y)$ , where  $S' \subseteq S'', Z' \subseteq Z''$ , and  $A'_i \subseteq A''_i$ , for all  $i \in [\tau]$ . It follows from  $B_x \subseteq B_y$ , that  $S \subseteq S', Z \subseteq Z'$ , and  $A_i \subseteq A'_i$ , for all  $i \in [\tau]$ .

 $\Leftarrow$ : It is easy to see that coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  is extendable to  $B(T_x)$  if there is a coloring  $\langle A'_{[1:\tau]}, S', Z' \rangle$  of  $B_y$  which is extendable to  $B(T_y)$  and is itself an extension of  $\langle A_{[1:\tau]}, S, Z \rangle$ , because  $\boldsymbol{G}[B(T_x)]$  is a temporal subgraph of  $\boldsymbol{G}[B(T_y)]$ . Since we want to extend the coloring of  $B_x$  such that we have a minimum size S, we select the minimum over all possible extensions of  $\langle A_{[1:\tau]}, S, Z \rangle$  to  $B_y$ .

Note that we can compute the table entry  $D_x[A_{[1:\tau]}, S, Z]$  in  $\mathcal{O}(|\mathbf{E}|)$  time, because we have to look up  $\tau + 2$  entries in  $D_y$  and  $\tau \leq |\mathbf{E}|$ , see [31].

**Join node.** Let  $x \in V(T)$  be a join node of  $\mathcal{T}, y_1, y_2 \in V(T)$  be children of x, and hence  $B_x = B_{y_1} = B_{y_2}$ . We set  $D_x[A_{[1:\tau]}, S, Z] := D_{y_1}[A_{[1:\tau]}, S, Z] + D_{y_1}[A_{[1:\tau]}, S, Z] - |S|$ .

**Lemma A.9.** Let G be a temporal graph and  $\mathcal{T}$  be a tree decomposition of G as described above,  $x \in V(T)$  be a join node of v,  $y_1, y_2 \in V(T)$  be the child nodes of x, and  $\langle A_{[1:\tau]}, S, Z \rangle$ be a coloring of  $B_x$ . Then the following holds:

- 1. The coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x = B_{y_1} = B_{y_2}$  is extendable to  $B(T_x)$  if and only if it is extendable to  $B(T_{y_1})$  and  $B(T_{y_2})$ .
- 2. The value of  $D_x[A_{[1:\tau]}, S, Z]$  agrees with Equation (5) and can be computed in  $\mathcal{O}(1)$  time.

*Proof.*  $\Rightarrow$ : Let  $\langle A_{[1:\tau]}, S, Z \rangle$  be a coloring of  $B_x = B_{y_1} = B_{y_2}$  and let  $\langle A'_{[1:\tau]}, S', Z' \rangle$  be an extension to  $B(T_x)$ . Since  $B(T_{y_1}), B(T_{y_2}) \subseteq B(T_x)$  and  $B_x = B_{y_1} = B_{y_2}$ , we know that  $\langle A_{[1:\tau]}, S, Z \rangle$  is extendable to  $B(T_{y_1})$  and  $B(T_{y_2})$ .

 $\Leftarrow$ : Let coloring  $\langle A_{[1:\tau]}, S, Z \rangle$  of  $B_x$  be extendable to  $B(T_{y_1})$  and  $B(T_{y_2})$ . Take  $\langle A'_{[1:\tau]}, S', Z' \rangle$ and  $\langle A''_{[1:\tau]}, S'', Z'' \rangle$  to be extensions to  $B(T_{y_1})$  respectively to  $B(T_{y_2})$ . We claim that  $\langle A'_1 \cup A''_1, A'_2 \cup A''_2, \ldots, A'_{\tau} \cup A''_{\tau}, S' \cup S'', Z' \cup Z'' \rangle$  is a valid coloring of  $B(T_x)$ . Suppose not, that is,  $\langle A'_1 \cup A''_1, A'_2 \cup A''_2, \ldots, A'_{\tau} \cup A''_{\tau}, S' \cup S'', Z' \cup Z'' \rangle$  is a coloring but not valid, or it forms no coloring.

In the first case, each  $s \notin A'_1 \cup A''_1$  or  $z \notin Z' \cup Z''$  contradicts the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$ and  $\langle A''_{[1:\tau]}, S'', Z'' \rangle$  are valid colorings. Next, suppose there are  $a \in A'_i \cup A''_i$ ,  $i \in \tau$ , and  $b \in Z' \cup$ Z'' such that there is a temporal (a, b)-path P with departure time at least i in  $G[B(T_x)] - (S' \cup$ S''). Then, either P exists in  $G[B(T_{y_1})]$  or in  $G[B(T_{y_2})]$ , contradicting the fact that  $\langle A'_{[1:\tau]}, S', Z' \rangle$ and  $\langle A''_{[1:\tau]}, S'', Z'' \rangle$  are valid colorings, or P contains an edge  $(\{v, w\}, t)$  that is neither in  $G[B(T_{y_1})]$  nor in  $G[B(T_{y_2})]$ . It follows that  $\{v, w\} \notin B_{y_1} \cap B_{y,2}$  but  $\{v, w\} \subseteq B_x$ , contradicting the fact that  $\mathcal{T}$  is a nice tree decomposition. It is not difficult to see that the case of  $a \in A'_i \cup A''_i$  and  $a \in A'_j \cup A''_j$ ,  $i, j \in \tau$ , such that there is a temporal (a, a')-path P with departure time at least i at arrival time at most j - 1 in  $G[B(T_x)] - (S' \cup S'')$ , follows the same argumentation.

In the second case, that is,  $\langle A'_1 \cup A''_1, A'_2 \cup A''_2, \ldots, A'_{\tau} \cup A''_{\tau}, S' \cup S'', Z' \cup Z'' \rangle$  forms no coloring, there is a vertex  $v \in B(T_{y_2}) \cap B(T_{y_1})$  which has different colors in  $\langle A'_{[1:\tau]}, S', Z' \rangle$  and  $\langle A''_{[1:\tau]}, S'', Z'' \rangle$ . If  $v \notin B_x = B_{y_1} = B_{y_2}$ , then  $B^{-1}(v)$  is not a connected subtree of T, contradicting the fact that  $\mathcal{T}$  is a nice tree decomposition. If  $v \in B_x$ , then v has different colors in  $\langle A_{[1:\tau]}, S, Z \rangle$ , contradicting the fact that  $\langle A_{[1:\tau]}, S, Z \rangle$  is a coloring of  $B_x$ . Altogether, it follows that  $\langle A'_1 \cup A''_1, A'_2 \cup A''_2, \ldots, A'_{\tau} \cup A''_{\tau}, S' \cup S'', Z' \cup Z'' \rangle$  is a valid coloring of  $B(T_x)$ .

Furthermore, this implies that for all vertices  $w \in B(T_x)$  it holds that  $w \in S' \cap S''$ implies  $w \in S$ . Hence,  $|S'| + |S''| - |S| = |S'| + |S''| - |S' \cap S''| = |S' \cup S''|$ .

Note that by a look up one table entry of  $D_{y_1}$  and one in  $D_{y_2}$ , we can compute the table entry  $D_x[A_{[1:\tau]}, S, Z]$  in  $\mathcal{O}(1)$  time.

Having Lemmata A.3, A.4, A.5, A.6, A.7, A.8 and A.9 at hand, we now prove Theorem 4.7.

Proof of Theorem 4.7. The algorithm works as follows on input instance  $\mathcal{I} = (\mathbf{G} = (V, \mathbf{E}, \tau), s, z, k)$  of TEMPORAL (s, z)-SEPARATION. Let  $\mathcal{T}$  be a nice tree decomposition for the underlying graph  $\mathbf{G}_{\downarrow}$  of width at most tw $(\mathbf{G}_{\downarrow})$ .

- 1. Add s and z to every bag in  $\mathcal{O}(\operatorname{tw}(\mathbf{G}_{\downarrow}) \cdot |V|)$  time. Note that  $|V(T)| \in \mathcal{O}(\operatorname{tw}(\mathbf{G}_{\downarrow}) \cdot |V|)$ and that each bag is of size at most tw $(\mathbf{G}_{\downarrow}) + 2$ .
- 2. Compute the dynamic program of Equation (5) on  $\mathcal{T}$ . This can be done in  $\mathcal{O}((\tau + 2)^{\operatorname{tw}(\mathbf{G}_{\downarrow})+2} \cdot \operatorname{tw}(\mathbf{G}_{\downarrow}) \cdot |V| \cdot |\mathbf{E}|)$  time because there are at most  $(\tau + 2)^{\operatorname{tw}(\mathbf{G}_{\downarrow})+2}$  possible colorings for each bag, there are at most  $\mathcal{O}(\operatorname{tw}(\mathbf{G}_{\downarrow}) \cdot |V|)$  many bags, and table entry

for one coloring can be computed in  $\mathcal{O}(|\mathbf{E}|)$  time, see Lemmata A.4, A.5, A.6, A.7, A.8 and A.9.

3. Iterate over the root table  $D_r$ . If there is an entry of size at most k, then output yes,

otherwise output no. The correctness of this step follows from Lemma A.3. Alltogether, the input instance  $\mathcal{I}$  can be decided in  $\mathcal{O}((\tau+2)^{\operatorname{tw}(\boldsymbol{G}_{\downarrow})+2} \cdot \operatorname{tw}(\boldsymbol{G}_{\downarrow}) \cdot |V| \cdot |\boldsymbol{E}|)$  time.