Complexity and Expressivity of Branchingand Alternating-Time Temporal Logics with Finitely Many Variables^{*}

Mikhail Rybakov¹ and Dmitry Shkatov²

¹Tver State University and University of the Witwatersrand, Johannesburg ²University of the Witwatersrand, Johannesburg

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Abstract

We show that Branching-time temporal logics **CTL** and **CTL**^{*}, as well as Alternating-time temporal logics **ATL** and **ATL**^{*}, are as semantically expressive in the language with a single propositional variable as they are in the full language, i.e., with an unlimited supply of propositional variables. It follows that satisfiability for **CTL**, as well as for **ATL**, with a single variable is EXPTIME-complete, while satisfiability for **CTL**^{*}, as well as for **ATL**^{*}, with a single variable is 2EXPTIME-complete,—i.e., for these logics, the satisfiability for formulas with only one variable is as hard as satisfiability for arbitrary formulas.

KEYWORDS: branching-time temporal logics, alternating-time temporal logics, finite-variable fragments, computational complexity, semantic expressivity, satisfiability problem

1 Introduction

The propositional Branching-time temporal logics **CTL** [4, 7] and **CTL**^{*} [10, 7] have for a long time been used in formal specification and verification of (parallel) non-terminating computer programs [23, 7], such as (components of) operating systems, as well as in formal specification and verification of hardware. More recently, Alternating-time temporal logics **ATL** and **ATL**^{*} [1, 7] have been used for formal specification and verification of multi-agent [33] and,

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more broadly, so-called open systems, i.e., systems whose correctness depends on the actions of external entities, such as the environment or other agents making up a multi-agent system.

Logics CTL, CTL^{*}, ATL, and ATL^{*} have two main applications to computer system design, corresponding to two different stages in the system design process, traditionally conceived of as having specification, implementation, and verification phases. First, the task of verifying that an implemented system conforms to a specification can be carried out by checking that a formula expressing the specification is satisfied in the structure modelling the system,—for program verification, this structure usually models execution paths of the program; this task corresponds to the model checking problem [5] for the logic. Second, the task of verifying that a specification of a system is satisfiable—and, thus, can be implemented by some system—corresponds to the satisfiability problem for the logic. Being able to check that a specification is satisfiable has the obvious advantage of avoiding wasted effort in trying to implement unsatisfiable systems. Moreover, an algorithm that checks for satisfiability of a formula expressing a specification builds, explicitly or implicitly, a model for the formula, thus supplying a formal model of a system conforming to the specification; this model can subsequently be used in the implementation phase. There is hope that one day such models can be used as part of a "push-button" procedure producing an assuredly correct implementation from a specification model, avoiding the need for subsequent verification altogether. Tableaux-style satisfiability-checking algorithms developed for CTL in [9], for CTL^{*} in [28], for ATL in [17], and for \mathbf{ATL}^* in [6] all implicitly build a model for the formula whose satisfiability is being checked.

In this paper, we are concerned with the satisfiability problem for **CTL**, **CTL**^{*}, **ATL**, and **ATL**^{*}; clearly, the complexity of satisfiability for these logics is of crucial importance to their applications to formal specification. It is well-known that, for formulas that might contain contain an arbitrary number of propositional variables, the complexity of satisfiability for all of these logics is quite high: it is EXPTIME-complete for **CTL** [12, 9], 2EXPTIME-complete for **CTL**^{*} [37], EXPTIME-complete for **ATL** [19, 40], and 2EXPTIME-complete for **ATL**^{*} [32].

It has, however, been observed (see, for example, [8]) that, in practice, formulas expressing formal specifications, despite being quite long and containing deeply nested temporal operators, usually contain only a very small number of propositional variables,—typically, two or three. The question thus arises whether limiting the number of propositional variables allowed to be used in the construction of formulas we take as inputs can bring down the complexity of the satisfiability problem for **CTL**, **CTL**^{*}, **ATL**, and **ATL**^{*}. Such an effect is not, after all, unknown in logic: examples are known of logics whose satisfiability problem goes down from "intractable" to "tractable" once we place a limit on the number of propositional variables allowed in the language: thus, satisfiability for the classical propositional logic as well as the extensions of the modal logic **K5** [25], which include such logics as **K45**, **KD45**, and **S5** (see also [20]), goes down from NP-complete to polynomial-time decidable once we limit the number of propositional variables in the language to an (arbitrary) finite number.¹ Similarly, as follows from [26], satisfiability for the intuitionistic propositional logic goes down from PSPACE-complete to polynomial-time decidable if we allow only a single propositional variable in the language.

The question of whether the complexity of satisfiability for **CTL**, **CTL**^{*}, **ATL**, and **ATL**^{*} can be reduced by restricting the number of propositional variables allowed to be used in the formulas has not, however, been investigated in the literature. The present paper is mostly meant to fill that gap.

A similar question has been answered in the negative for Linear-time temporal logic LTL in [8], where it was shown, using a proof technique peculiar to LTL (in particular, [8] relies on the fact that for LTL with a finite number of propositional variables satisfiability reduces to model-checking), that a singlevariable fragment of **LTL** is PSPACE-complete, i.e., as computationally hard as the entire logic [34]. It should be noted that, in this respect, LTL behaves like most "natural" modal and temporal logics, for which the presence of even a single variable in the language is sufficient to generate a fragment whose satis fiability is as hard as satisfiability for the entire logic. The first results to this effect have been proven in [2] for logics for reasoning about linguistic structures and in [38] for provability logic. A general method of proving such results for PSPACE-complete logics has been proposed in [20]; even though [20] considers only a handful of logics, the method can be generalised to large classes of logics, often in the language without propositional variables [22, 3] (it is not, however, applicable to LTL, as it relies on unrestricted branching in the models of the logic, which runs contrary to the semantics of LTL,—hence the need for a different approach, as in [8]). In this paper, we use a suitable modification of the technique from [20] (see [29, 30]) to show that single-variable fragments of CTL, CTL^{*}, ATL, and ATL^{*} are as computationally hard as the entire logics; thus, for these logics, the complexity of satisfiability cannot be reduced by restricting the number of variables in the language.

Before doing so, a few words might be in order to explain why the technique from [20] is not directly applicable to the logics we are considering in this paper. The approach of [20] is to model propositional variables by (the so-called pp-like) formulas of a single variable; to establish the PSPACE-harness results presented in [20], a substitution is made of such pp-like formulas for propositional variables into formulas encoding a PSPACE-hard problem. In the case of logics containing modalities corresponding to transitive relations, such as the modal logic **S4**, for such a substitution to work, the formulas into which the substitution is made need to satisfy the property referred to in [20] as "evidence in a structure,"—a formula is evident in a structure if it has a model satisfying the following heredity condition: if a propositional variable is true at a state, it has to be true at all the states accessible from that state. In the case of

¹To avoid ambiguity, we emphasise that we use the standard complexity-theoretic convention of measuring the complexity of the input as its size; in our case, this is the length of the input formula. In other words, we do not measure the complexity of the input according to how many distinct variables it contains; limiting the number of variables simply provides a restriction on the languages we consider.

PSPACE-complete logics, formulas satisfying the evidence condition can always be found, as the intuitionistic logic, which is PSPACE-complete, has the heredity condition built into its semantics. The situation is drastically different for logics that are EXPTIME-hard, which is the case for all the logics considered in the present paper: to show that a logic is EXPTIME-hard, one uses formulas that require for their satisfiability chains of states of the length exponential in the size of the formula.—this cannot be achieved with formulas that are evident in a structure, as by varying the valuations of propositional variables that have to satisfy the heredity condition we can only describe chains whose length is linear in the size of the formula. Thus, the technique from [20] is not directly applicable to EXPTIME-hard logics with "transitive" modalities, as the formulas into which the substitution of pp-like formulas needs to be made do not satisfy the condition that has to be met for such a substitution to work. As all the logics considered in this paper do have a "transitive" modality—namely, the temporal connective "always in the future", which is interpreted by the reflexive, transitive closure of the relation corresponding to the temporal connective "at the next instance"—this limitation prevents the technique from [20] from being directly applied to them.

In the present paper, we modify the approach of [20] by coming up with substitutions of single-variable formulas for propositional variables that can be made into arbitrary formulas, rather than formulas satisfying a particular property, such as evidence in a structure. This allows us to break away from the class PSPACE and to deal with **CTL**, **CTL**^{*}, **ATL**, and **ATL**^{*}, all of which are at least EXPTIME-hard. A similar approach has recently been used in [29] and [30] for some other propositional modal logics.

A by-product of our approach, and another contribution of this paper, is that we establish that single-variable fragments of **CTL**, **CTL**^{*}, **ATL**, and **ATL**^{*} are as semantically expressive as the entire logic, i.e., all properties that can be specified with any formula of the logic can be specified with a formula containing only one variable—indeed, our complexity results follow from this. In this light, the observation cited above—that in practice most properties of interest are expressible in these logics using only a very small number of variables—is not at all surprising from a purely mathematical point of view, either.

The paper is structured as follows. In Section 2, we introduce the syntax and semantics of **CTL** and **CTL**^{*}. Then, in Section 3, we show that **CTL** and **CTL**^{*} can be polynomial-time embedded into their single-variable fragments. As a corollary, we obtain that satisfiability for the single variable fragment of **CTL** is EXPTIME-complete and satisfiability for the single variable of of **CTL**^{*} is 2EXPTIME-complete. In Section 4, we introduce the syntax and semantics of **ATL** and **ATL**^{*}. Then, in Section 5, we prove results for **ATL** and **ATL**^{*} that are analogous to those proven in Section 3 for **CTL** and **CTL**^{*}. We conclude in Section 6 by discussing other formalisms related to the logics considered in this paper to which our proof technique can be applied to obtain similar results.

2 Branching-time temporal logics

We start by briefly recalling the syntax and semantics of **CTL** and **CTL**^{*}.

The language of **CTL**^{*} contains a countable set $Var = \{p_1, p_2, ...\}$ of propositional variables, the propositional constant \perp ("falsehood"), the Boolean connective \rightarrow ("if ..., then ..."), the path quantifier \forall , and temporal connectives \bigcirc ("next") and \mathcal{U} ("until"). The language contains two kinds of formulas: state formulas and path formulas, so called because they are evaluated in the models at states and paths, respectively. State formulas φ and path formulas ϑ are simultaneously defined by the following BNF expressions:

$$\begin{split} \varphi &::= p \mid \bot \mid (\varphi \to \varphi) \mid \forall \vartheta, \\ \vartheta &::= \varphi \mid (\vartheta \to \vartheta) \mid (\vartheta \, \mathcal{U} \vartheta) \mid \bigcirc \vartheta, \end{split}$$

where p ranges over *Var*. Other Boolean connectives are defined as follows: $\neg A := (A \to \bot), (A \land B) := \neg (A \to \neg B), (A \lor B) := (\neg A \to B), \text{ and}$ $(A \leftrightarrow B) := (A \to B) \land (B \to A), \text{ where } A \text{ and } B \text{ can be either state or path}$ formulas. We also define $\top := \bot \to \bot, \diamond \vartheta := (\top \mathcal{U}\vartheta), \Box \vartheta := \neg \diamond \neg \vartheta, \text{ and}$ $\exists \vartheta := \neg \forall \neg \vartheta.$

Formulas are evaluated in Kripke models. A Kripke model is a tuple $\mathfrak{M} = (\mathcal{S}, \longmapsto, V)$, where \mathcal{S} is a non-empty set (of states), \longmapsto is a binary (transition) relation on \mathcal{S} that is serial (i.e., for every $s \in \mathcal{S}$, there exists $s' \in \mathcal{S}$ such that $s \longmapsto s'$), and V is a (valuation) function $V : Var \to 2^{\mathcal{S}}$.

An infinite sequence s_0, s_1, \ldots of states in \mathfrak{M} such that $s_i \mapsto s_{i+1}$, for every $i \ge 0$, is called a *path*. Given a path π and some $i \ge 0$, we denote by $\pi[i]$ the *i*th element of π and by $\pi[i, \infty]$ the suffix of π beginning at the *i*th element. If $s \in S$, we denote by $\Pi(s)$ the set of all paths π such that $\pi[0] = s$.

The satisfaction relation between models \mathfrak{M} , states s, and state formulas φ , as well as between models \mathfrak{M} , paths π , and path formulas ϑ , is defined as follows:

- $\mathfrak{M}, s \models p_i \leftrightarrows s \in V(p_i);$
- $\mathfrak{M}, s \models \bot$ never holds;
- $\mathfrak{M}, s \models \varphi_1 \rightarrow \varphi_2 \rightleftharpoons \mathfrak{M}, s \models \varphi_1 \text{ implies } \mathfrak{M}, s \models \varphi_2;$
- $\mathfrak{M}, s \models \forall \vartheta_1 \leftrightarrows \mathfrak{M}, \pi \models \vartheta_1 \text{ for every } \pi \in \Pi(s).$
- $\mathfrak{M}, \pi \models \varphi_1 \rightleftharpoons \mathfrak{M}, \pi[0] \models \varphi_1;$
- $\mathfrak{M}, \pi \models \vartheta_1 \rightarrow \vartheta_2 \rightleftharpoons \mathfrak{M}, \pi \models \vartheta_1 \text{ implies } \mathfrak{M}, \pi \models \vartheta_2;$
- $\mathfrak{M}, \pi \models \bigcirc \vartheta_1 \leftrightarrows \mathfrak{M}, \pi[1, \infty] \models \vartheta_1;$
- $\mathfrak{M}, \pi \models \vartheta_1 \mathcal{U} \vartheta_2 \rightleftharpoons \mathfrak{M}, \pi[i, \infty] \models \vartheta_2$ for some $i \ge 0$ and $\mathfrak{M}, \pi[j, \infty] \models \vartheta_1$ for every j such that $0 \le j < i$.

A **CTL***-formula is a state formula in this language. A **CTL***-formula is satisfiable if it is satisfied by some state of some model, and valid if it is satisfied by every state of every model. Formally, by **CTL*** we mean the set of valid **CTL***-formulas. Notice that this set is closed under uniform substitution.

Logic **CTL** can be thought of as a fragment of **CTL**^{*} containing only formulas where a path quantifier is always paired up with a temporal connective. This, in particular, disallows formulas whose main sign is a temporal connective and, thus, eliminates path-formulas. Such composite "modal" operators are $\forall \bigcirc$ (universal "next"), $\forall \mathcal{U}$ (universal "until"), and $\exists \mathcal{U}$ (existential "until"). Formulas are defined by the following BNF expression:

$$\varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid \forall \bigcirc \varphi \mid \forall (\varphi \, \mathcal{U}\varphi) \mid \exists (\varphi \, \mathcal{U}\varphi),$$

where p ranges over *Var*. We also define $\neg \varphi := (\varphi \to \bot), (\varphi \land \psi) := \neg(\varphi \to \neg \psi), (\varphi \lor \psi) := (\neg \varphi \to \psi), \top = \bot \to \bot, \exists \bigcirc \varphi := \neg \forall \bigcirc \neg \varphi, \exists \Diamond \varphi := \exists (\top \mathcal{U}\varphi), \text{ and } \forall \Box \varphi := \neg \exists \Diamond \neg \varphi.$

The satisfaction relation between models \mathfrak{M} , states s, and formulas φ is inductively defined as follows (we only list the cases for the "new" modal operators):

- $\mathfrak{M}, s \models \forall \bigcirc \varphi_1 \leftrightarrows \mathfrak{M}, s' \models \varphi_1$ whenever $s \longmapsto s'$;
- $\mathfrak{M}, s \models \forall (\varphi_1 \ \mathcal{U} \varphi_2) \rightleftharpoons \text{ for every path } s_0 \longmapsto s_1 \longmapsto \dots \text{ with } s_0 = s,$ $\mathfrak{M}, s_i \models \varphi_2, \text{ for some } i \ge 0, \text{ and } \mathfrak{M}, s_j \models \varphi_1, \text{ for every } 0 \le j < i;$
- $\mathfrak{M}, s \models \exists (\varphi_1 \ \mathcal{U} \varphi_2) \rightleftharpoons$ there exists a path $s_0 \mapsto s_1 \mapsto \ldots$ with $s_0 = s$, such that $\mathfrak{M}, s_i \models \varphi_2$, for some $i \ge 0$, and $\mathfrak{M}, s_j \models \varphi_1$, for every $0 \le j < i$.

Satisfiable and valid formulas are defined as for **CTL**^{*}. Formally, by **CTL** we mean the set of valid **CTL**-formulas; this set is closed under substitution.

For each of the logics described above, by a variable-free fragment we mean the subset of the logic containing only formulas without any propositional variables. Given formulas φ , ψ and a propositional variable p, we denote by $\varphi[p/\psi]$ the result of uniformly substituting ψ for p in φ .

3 Finite-variable fragments of CTL^{*} and CTL

In this section, we consider the complexity of satisfiability for finite-variable fragments of \mathbf{CTL} and \mathbf{CTL}^* , as well as semantic expressivity of those fragments.

We start by noticing that for both **CTL** and **CTL**^{*} satisfiability of the variable-free fragment is polynomial-time decidable. Indeed, it is easy to check that, for these logics, every variable-free formula is equivalent to either \perp or \top . Thus, to check for satisfiability of a variable-free formula φ , all we need to do is to recursively replace each subformula of φ by either \perp or \top , which gives us an algorithm that runs in time linear in the size of φ . Since both **CTL** and

CTL^{*} are at least EXPTIME-hard and $P \neq EXPTIME$, variable-free fragments of these logics cannot be as expressive as the entire logic.

We next prove that the situation changes once we allow just one variable to be used in the construction of formulas. Then, we can express everything we can express in the full languages of **CTL** and **CTL**^{*}; as a consequence, the complexity of satisfiability becomes as hard as satisfiability for the full languages. In what follows, we first present the proof for **CTL**^{*}, and then point out how that work carries over to **CTL**.

Let φ be an arbitrary **CTL**^{*}-formula. Without a loss of generality we may assume that φ contains propositional variables $p_1, \ldots p_n$. Let p_{n+1} be a variable not occurring in φ . First, inductively define the translation \cdot' as follows:

$$\begin{array}{lll} p_i' & = & p_i, & \text{where } i \in \{1, \dots, n\}; \\ \bot' & = & \bot; \\ (\phi \to \psi)' & = & \phi' \to \psi'; \\ (\forall \alpha)' & = & \forall (\Box p_{n+1} \to \alpha'); \\ (\bigcirc \alpha)' & = & \bigcirc \alpha'; \\ (\alpha \, \mathcal{U}\beta)' & = & \alpha' \, \mathcal{U}\beta'. \end{array}$$

Next, let

$$\Theta = p_{n+1} \land \forall \Box (\exists \bigcirc p_{n+1} \leftrightarrow p_{n+1}),$$

and define

$$\widehat{\varphi} = \Theta \wedge \varphi'.$$

Intuitively, the translation \cdot' restricts evaluation of formulas to the paths where every state makes the variable p_{n+1} true, while Θ acts as a guard making sure that all paths in a model satisfy this property. Notice that φ is equivalent to $\widehat{\varphi}[p_{n+1}/\top]$.

Lemma 3.1 Formula φ is satisfiable if, and only if, formula $\hat{\varphi}$ is satisfiable.

Proof. Suppose that $\widehat{\varphi}$ is not satisfiable. Then, $\neg \widehat{\varphi} \in \mathbf{CTL}^*$ and, since \mathbf{CTL}^* is closed under substitution, $\neg \widehat{\varphi}[p_{n+1}/\top] \in \mathbf{CTL}^*$. As $\widehat{\varphi}[p_{n+1}/\top] \leftrightarrow \varphi \in \mathbf{CTL}^*$, so $\neg \varphi \in \mathbf{CTL}^*$; thus, φ is not satisfiable.

Suppose that $\widehat{\varphi}$ is satisfiable. In particular, let $\mathfrak{M}, s_0 \models \widehat{\varphi}$ for some model \mathfrak{M} and some s_0 in \mathfrak{M} . Define \mathfrak{M}' to be the smallest submodel of \mathfrak{M} such that

- s_0 is in \mathfrak{M}' ;
- if x is in $\mathfrak{M}', x \mapsto y$, and $\mathfrak{M}, y \models p_{n+1}$, then y is also in \mathfrak{M}' .

Notice that, since $\mathfrak{M}, s_0 \models p_{n+1} \land \forall \Box (\exists \bigcirc p_{n+1} \leftrightarrow p_{n+1})$, the model \mathfrak{M}' is serial, as required, and that p_{n+1} is true at every state of \mathfrak{M}' .

We now show that $\mathfrak{M}', s_0 \models \varphi$. Since $\mathfrak{M}, s_0 \models \varphi'$, it suffices to prove that, for every state x in \mathfrak{M}' and every state subformula ψ of φ , we have $\mathfrak{M}, x \models \psi'$ if, and only if, $\mathfrak{M}', x \models \psi$; and that, for every path π in \mathfrak{M}' and every path subformula α of φ , we have $\mathfrak{M}, \pi \models \alpha'$ if, and only if, $\mathfrak{M}', \pi \models \alpha$. This can be done by simultaneous induction on ψ and α . The base case as well as Boolean cases are straightforward.

Let $\psi = \forall \alpha$, so $\psi' = \forall (\Box p_{n+1} \to \alpha')$. Assume that $\mathfrak{M}, x \not\models \forall (\Box p_{n+1} \to \alpha')$. Then, $\mathfrak{M}, \pi \not\models \alpha'$, for some $\pi \in \Pi(x)$ such that $\mathfrak{M}, \pi[i] \models p_{n+1}$, for every $i \ge 0$. By construction of \mathfrak{M}', π is a path is \mathfrak{M}' ; thus, we can apply the inductive hypothesis to conclude that $\mathfrak{M}', \pi \not\models \alpha$. Therefore, $\mathfrak{M}', x \not\models \forall \alpha$, as required. Conversely, assume that $\mathfrak{M}', x \not\models \forall \alpha$. Then, $\mathfrak{M}', \pi \not\models \alpha$, for some $\pi \in \Pi(x)$. Clearly, π is a path in \mathfrak{M} . Since p_{n+1} is true at every state in \mathfrak{M}' , and thus, at every state in π , using the inductive hypothesis, we conclude that $\mathfrak{M}, x \not\models \forall (\Box p_{n+1} \to \alpha')$.

The cases for the temporal connectives are straightforward.

Lemma 3.2 If $\hat{\varphi}$ is satisfiable, then it is satisfied in a model where p_{n+1} is true at every state.

Proof. If $\widehat{\varphi}$ is satisfiable, then, as has been shown in the proof of Lemma 3.1, φ is satisfied in a model where p_{n+1} is true at every state; i.e., $\mathfrak{M}, s \models \varphi$ for some $\mathfrak{M} = (\mathcal{S}, \longmapsto, V)$ such that p_{n+1} is true at every state in \mathcal{S} and some $s \in S$. Since φ is equivalent to $\widehat{\varphi}[p_{n+1}/\top]$, clearly $\mathfrak{M}, s \models \widehat{\varphi}$.

Next, we model all the variables of $\hat{\varphi}$ by single-variable formulas A_1, \ldots, A_m . This is done in the following way. Consider the class M of models that, for each $m \in \{1, \ldots, n+1\}$, contains a model $\mathfrak{M}_m = (\mathcal{S}_m, \longmapsto, V_m)$ defined as follows:

- $S_m = \{r_m, b^m, a_1^m, a_2^m, \dots, a_{2m}^m\};$
- $\mapsto = \{\langle r_m, b^m \rangle, \langle r_m, a_1^m \rangle\} \cup \{\langle a_i^m, a_{i+1}^m \rangle : 1 \le m \le 2m 1\} \cup \{\langle s, s \rangle : s \in \mathcal{S}_m\};$
- $s \in V_m(p)$ if, and only if, $s = r_m$ or $s = a_{2k}^m$, for some $k \in \{1, \ldots, m\}$.

The model \mathfrak{M}_m is depicted in Figure 1, where circles represent states with loops. With every such \mathfrak{M}_m , we associate a formula A_m , in the following way. First, inductively define the sequence of formulas

$$\begin{array}{lll} \chi_0 & = & \forall \,\Box \, p; \\ \chi_{k+1} & = & p \land \exists \bigcirc (\neg p \land \exists \bigcirc \chi_k). \end{array}$$

Next, for every $m \in \{1, \ldots, n+1\}$, let

$$A_m = \chi_m \land \exists \bigcirc \forall \Box \neg p.$$

Lemma 3.3 Let $\mathfrak{M}_k \in \mathsf{M}$ and let x be a state in \mathfrak{M}_k . Then, $\mathfrak{M}_k, x \models A_m$ if, and only if, k = m and $x = r_m$.

Proof. Straightforward.

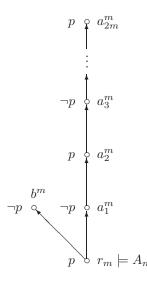


Figure 1: Model \mathfrak{M}_m

Now, for every $m \in \{1, \ldots, n+1\}$, define

$$B_m = \exists \bigcirc A_m.$$

Finally, let σ be a (substitution) function that, for every $i \in \{1 \dots n + 1\}$, replaces p_i by B_i , and let

$$\varphi^* = \sigma(\widehat{\varphi}).$$

Notice that the formula φ^* contains only a single variable, p.

Lemma 3.4 Formula φ is satisfiable if, and only if, formula φ^* is satisfiable.

Proof. Suppose that φ is not satisfiable. Then, in view of Lemma 3.1, $\widehat{\varphi}$ is not satisfiable. Then, $\neg \widehat{\varphi} \in \mathbf{CTL}^*$ and, since \mathbf{CTL}^* is closed under substitution, $\neg \varphi^* \in \mathbf{CTL}^*$. Thus, φ^* is not satisfiable.

Suppose that φ is satisfiable. Then, in view of Lemmas 3.1 and 3.2, $\widehat{\varphi}$ is satisfiable in a model $\mathfrak{M} = (\mathcal{S}, \longmapsto, V)$ where p_{n+1} is true at every state. We can assume without a loss of generality that every $x \in \mathcal{S}$ is connected by some path to s. Define model \mathfrak{M}' as follows. Append to \mathfrak{M} all the models from M (i.e., take their disjoint union), and for every $x \in \mathcal{S}$, make r_m , the root of \mathfrak{M}_m , accessible from x in \mathfrak{M}' exactly when $\mathfrak{M}, x \models p_m$. The evaluation of p is defined as follows: for states from each $\mathfrak{M}_m \in \mathsf{M}$, the evaluation is the same as in \mathfrak{M}_m , and for every $x \in \mathcal{S}$, let $x \notin V'(p)$.

We now show that $\mathfrak{M}', s \models \varphi^*$. It is easy to check that $\mathfrak{M}', s \models \sigma(\Theta)$. It thus remains to show that $\mathfrak{M}', s \models \sigma(\varphi')$. Since $\mathfrak{M}, s \models \varphi'$, it suffices to prove that $\mathfrak{M}, x \models \psi'$ if, and only if, $\mathfrak{M}', x \models \sigma(\psi')$, for every state x in \mathfrak{M} and every

state subformula ψ of φ ; and that $\mathfrak{M}, \pi \models \alpha'$ if, and only if, $\mathfrak{M}', \pi \models \sigma(\alpha')$, for every path π in \mathfrak{M} and every path subformula α of φ . This can be done by simultaneous induction on ψ and α .

Let $\psi = p_i$, so $\psi' = p_i$ and $\sigma(\psi') = B_i$. Assume that $\mathfrak{M}, x \models p_i$. Then, by construction of \mathfrak{M}' , we have $\mathfrak{M}', x \models B_i$. Conversely, assume that $\mathfrak{M}', x \models B_i$. As $\mathfrak{M}', x \models B_i$ implies $\mathfrak{M}', x \models \exists \bigcirc p$ and since $\mathfrak{M}, y \not\models p$, for every $y \in S$, this can only happen if $x \mapsto \mathfrak{M}' r_m$, for some $m \in \{1, \ldots, n+1\}$. Since, then, $r_m \models A_i$, in view of Lemma 3.3, m = i, and thus, by construction of \mathfrak{M}' , we have $\mathfrak{M}, x \models p_i$.

The Boolean cases are straightforward.

Let $\psi = \forall \alpha$, so $\psi' = \forall (\Box p_{n+1} \to \alpha')$ and $\sigma(\psi') = \forall (\Box B_{n+1} \to \sigma(\alpha'))$. Assume that $\mathfrak{M}, x \not\models \forall (\Box p_{n+1} \to \alpha')$. Then, for some $\pi \in \Pi(x)$ such that $\mathfrak{M}, \pi[i] \models p_{n+1}$ for every $i \ge 0$, we have $\mathfrak{M}, \pi \not\models \alpha'$. Clearly, π is a path in \mathfrak{M}' , and thus, by inductive hypothesis, $\mathfrak{M}', \pi[i] \models B_{n+1}$, for every $i \ge 0$, and $\mathfrak{M}', \pi \not\models \sigma(\alpha')$. Hence, $\mathfrak{M}', x \not\models \forall (\Box B_{n+1} \to \sigma(\alpha'))$, as required. Conversely, assume that $\mathfrak{M}', x \not\models \forall (\Box B_{n+1} \to \sigma(\alpha'))$. Then, for some $\pi \in \Pi(x)$ such that $\mathfrak{M}', \pi[i] \models B_{n+1}$ for every $i \ge 0$, we have $\mathfrak{M}', \pi \not\models \sigma(\alpha')$. Since by construction of \mathfrak{M}' , no state outside of \mathcal{S} satisfies B_{n+1} , we know that π is a path in \mathfrak{M} . Thus, we can use the inductive hypothesis to conclude that $\mathfrak{M}, x \not\models \forall (\Box p_{n+1} \to \alpha')$.

The cases for the temporal connectives are straightforward.

Lemma 3.4, together with the observation that the formula φ^* is polynomialtime computable from φ , give us the following:

Theorem 3.5 There exists a polynomial-time computable function e assigning to every \mathbf{CTL}^* -formula φ a single-variable formula $e(\varphi)$ such that $e(\varphi)$ is satisfiable if, and only if, φ is satisfiable.

Theorem 3.6 The satisfiability problem for the single-variable fragment of CTL^{*} is 2EXPTIME-complete.

Proof. The lower bound immediately follows from Theorem 3.5 and 2EXPTIMEhardness of satisfiability for \mathbf{CTL}^* [37]. The upper bound follows from the 2EXPTIME upper bound for satisfiability for \mathbf{CTL}^* [37]. \Box

We now show how the argument presented above for \mathbf{CTL}^* can be adapted to \mathbf{CTL} . First, we notice that if our sole purpose were to prove that satisfiability for the single-variable fragment of \mathbf{CTL} is EXPTIME-complete, we would not need to work with the entire set of connectives present in the language of \mathbf{CTL} ,—it would suffice to work with a relatively simple fragment of \mathbf{CTL} containing the modal operators $\forall \bigcirc$ and $\forall \square$, whose satisfiability, as follows from [12], is EXPTIME-hard. We do, however, also want to establish that the single-variable fragment of \mathbf{CTL} is as expressive the entire logic; therefore, we embed the entire \mathbf{CTL} into its single-variable fragment. To that end, we can carry out an argument similar to the one presented above for \mathbf{CTL}^* .

First, we define the translation \cdot' as follows:

$$\begin{array}{lll} p_i' & = & p_i \text{ where } i \in \{1, \dots, n\}; \\ (\bot)' & = & \bot; \\ (\phi \to \psi)' & = & \phi' \to \psi'; \\ (\forall \bigcirc \phi)' & = & \forall \bigcirc (p_{n+1} \to \phi'); \\ (\forall (\phi \mathcal{U}\psi))' & = & \forall (\phi' \mathcal{U}(p_{n+1} \land \psi')); \\ (\exists (\phi \mathcal{U}\psi))' & = & \exists (\phi' \mathcal{U}(p_{n+1} \land \psi')). \end{array}$$

Next, let

$$\Theta = p_{n+1} \land \forall \Box (\exists \bigcirc p_{n+1} \leftrightarrow p_{n+1}).$$

and define

$$\widehat{\varphi} = \Theta \wedge \varphi'.$$

Intuitively, the translation \cdot' restricts the evaluation of formulas to the states where p_{n+1} is true. Formula Θ acts as a guard making sure that all states in a model satisfy this property. We can then prove the analogues of Lemmas 3.1 and 3.2.

Lemma 3.7 Formula φ is satisfiable if, and only if, formula $\hat{\varphi}$ is satisfiable.

Proof. Analogous to the proof of Lemma 3.1. In the right-to-left direction, inductive steps for modal connectives rely on the fact that in a submodel we constructed every state makes the variable p_{n+1} true.

Lemma 3.8 If $\hat{\varphi}$ is satisfiable, then it is satisfied in a model where p_{n+1} is true at every state.

Proof. Analogous to the proof of Lemma 3.2.

Next, we model propositional variables p_1, \ldots, p_{n+1} in the formula $\hat{\varphi}$ exactly as in the argument for \mathbf{CTL}^* , i.e., we use formulas A_m and their associated models \mathfrak{M}_m , where $m \in \{1, \ldots, n+1\}$. This can be done since formulas A_m are, in fact, \mathbf{CTL} -formulas. Lemma 3.3 can, thus, be reused for \mathbf{CTL} , as well.

We then define a single-variable **CTL**-formula φ^* analogously to the way it had been done for **CTL**^{*}:

$$\varphi^* = \sigma(\widehat{\varphi}),$$

where σ is a (substitution) function that, for every $i \in \{1 \dots n+1\}$, replaces p_i by $B_i = \exists \bigcirc A_i$. We can then prove the analogue of Lemma 3.4.

Lemma 3.9 Formula φ is satisfiable if, and only if, formula φ^* is satisfiable.

Proof. Analogous to the proof of Lemma 3.4. In the left-to-right direction, the inductive steps for the modal connectives rely on the fact that the formula B_{n+1} is true precisely at the states of the model that satisfies φ .

We, thus, obtain the following:

Theorem 3.10 There exists a polynomial-time computable function e assigning to every **CTL**-formula φ a single-variable formula $e(\varphi)$ such that $e(\varphi)$ is satisfiable if, and only if, φ is satisfiable.

Theorem 3.11 The satisfiability problem for the single-variable fragment of CTL is EXPTIME-complete.

Proof. The lower bound immediately follows from Theorem 3.10 and EXPTIMEhardness of satisfiability for **CTL** [12]. The upper bound follows from the EX-PTIME upper bound for satisfiability for **CTL** [9]. \Box

4 Alternating-time temporal logics

Alternating-time temporal logics \mathbf{ATL}^* and \mathbf{ATL} can be conceived of as generalisations of \mathbf{CTL}^* and \mathbf{CTL} , respectively. Their models incorporate transitions occasioned by simultaneous actions of the agents in the system rather than abstract transitions, as in \mathbf{CTL}^* and \mathbf{CTL} , and we now reason about paths that can be forced by cooperative actions of coalitions of agents, rather than just about all (\forall) and some (\exists) paths. We do not lose the ability to reason about all and some paths in \mathbf{ATL}^* and \mathbf{ATL} , however, so these logics are generalisations of \mathbf{CTL}^* and \mathbf{CTL}^* and \mathbf{CTL}^* and \mathbf{CTL}^* .

The language of **ATL**^{*} contains a non-empty, finite set \mathbb{AG} of names of agents (subsets of \mathbb{AG} are called coalitions); a countable set $Var = \{p_1, p_2, \ldots\}$ of propositional variables; the propositional constant \bot ; the Boolean connective \rightarrow ; coalition quantifiers $\langle\!\langle C \rangle\!\rangle$, for every $C \subseteq \mathbb{AG}$; and temporal connectives \bigcirc ("next"), \mathcal{U} ("until"), and \Box ("always in the future"). The language contains two kinds of formulas: state formulas and path formulas. State formulas φ and path formulas α are simultaneously defined by the following BNF expressions:

$$\begin{split} \varphi &::= p \mid \perp \mid (\varphi \to \varphi) \mid \langle\!\langle C \rangle\!\rangle \vartheta, \\ \vartheta &::= \varphi \mid (\vartheta \to \vartheta) \mid (\vartheta \, \mathcal{U} \vartheta) \mid \bigcirc \vartheta \mid \Box \vartheta \end{split}$$

where C ranges over subsets of AG and p ranges over *Var*. Other Boolean and temporal connectives are defined as for \mathbf{CTL}^* .

Formulas are evaluated in concurrent game models. A concurrent game model is a tuple $\mathfrak{M} = (\mathbb{AG}, \mathcal{S}, Act, act, \delta, V)$, where

- $\mathbb{AG} = \{1, \dots, k\}$ is a finite, non-empty set of agents;
- S is a non-empty set of states;
- *Act* is a non-empty set of actions;
- $act : \mathbb{AG} \times S \mapsto 2^{Act}$ is an action manager function assigning a non-empty set of "available" actions to an agent at a state;

- δ is a transition function assigning to every state $s \in S$ and every action profile $\alpha = (\alpha_1, \ldots, \alpha_k)$, where $\alpha_a \in act(a, s)$, for every $a \in \mathbb{AG}$, an outcome state $\delta(s, \alpha)$;
- V is a (valuation) function $V: Var \to 2^{\mathcal{S}}$.

A few auxiliary notions need to be introduced for the definition of the satisfaction relation.

A path is an infinite sequence s_0, s_1, \ldots of states in \mathfrak{M} such that, for every $i \ge 0$, the following holds: $s_{i+1} \in \delta(s_i, \alpha)$, for some action profile α . The set of all such sequences is denoted by \mathcal{S}^{ω} . The notation $\pi[i]$ and $\pi[i, \infty]$ is used as for **CTL**^{*}. Initial segments $\pi[0, i]$ of paths are called *histories*; a typical history is denoted by h, and its last state, $\pi[i]$, is denoted by last(h). Note that histories are non-empty sequences of states in \mathcal{S} ; we denote the set of all such sequences by \mathcal{S}^+ .

Given $s \in S$ and $C \subseteq \mathbb{AG}$, a *C*-action at *s* is a tuple α_C such that $\alpha_C(a) \in act(a, s)$, for every $a \in C$, and $\alpha_C(a')$, for every $a' \notin C$, is an unspecified action of agent a' at *s* (technically, a *C*-action might be thought of as an equivalence class on action profiles determined by a vector of chosen actions for every $a \in C$); we denote by act(C, s) the set of *C*-actions at *s*. An action profile α extends a *C*-action α_C , symbolically $\alpha_C \subseteq \alpha$, if $\alpha(a) = \alpha_C(a)$, for every $a \in C$. The outcome set of the *C*-action α_C at *s* is the set of states $out(s, \alpha_C) = \{\delta(s, \alpha) \mid \alpha \in act(\mathbb{AG}, s) \text{ and } \alpha_C \subseteq \alpha\}.$

A strategy for an agent a is a function $str_a(h) : S^+ \mapsto act(a, last(h))$ assigning to every history an action available to a at the last state of the history. A C-strategy is a tuple of strategies for every $a \in C$. The function $out(s, \alpha_C)$ can be naturally extended to the functions $out(s, str_C)$ and $out(h, str_C)$ assigning to a given state s, or more generally a given history h, and a given C-strategy the set of states that can result from applying str_C at s or h, respectively. The set of all paths that can result when the agents in C follow the strategy str_C from a given state s is denoted by $\Pi(s, str_C)$ and defined as $\{\pi \in S^{\omega} \mid \pi[0] = s \text{ and } \pi[j+1] \in out(\pi[0, j], str_C), \text{ for every } j \ge 0\}.$

The satisfaction relation between models \mathfrak{M} , states s, and state formulas φ , as well as between models \mathfrak{M} , paths π , and path formulas ϑ , is defined as follows:

- $\mathfrak{M}, s \models p_i \leftrightarrows s \in V(p_i);$
- $\mathfrak{M}, s \models \bot$ never holds;
- $\mathfrak{M}, s \models \varphi_1 \rightarrow \varphi_2 \rightleftharpoons \mathfrak{M}, s \models \varphi_1$ implies $\mathfrak{M}, s \models \varphi_2$;
- $\mathfrak{M}, s \models \langle\!\langle C \rangle\!\rangle \vartheta_1 \rightleftharpoons$ there exists a *C*-strategy str_C such that $\mathfrak{M}, \pi \models \vartheta_1$ holds for every $\pi \in \Pi(s, str_C)$;
- $\mathfrak{M}, \pi \models \varphi_1 \rightleftharpoons \mathfrak{M}, \pi[0] \models \varphi_1;$
- $\mathfrak{M}, \pi \models \vartheta_1 \rightarrow \vartheta_2 \rightleftharpoons \mathfrak{M}, \pi \models \vartheta_1$ implies $\mathfrak{M}, \pi \models \vartheta_2$;

- $\mathfrak{M}, \pi \models \bigcirc \vartheta_1 \leftrightarrows \mathfrak{M}, \pi[1, \infty] \models \vartheta_1;$
- $\mathfrak{M}, \pi \models \Box \vartheta_1 \rightleftharpoons \mathfrak{M}, \pi[i, \infty] \models \vartheta_1$, for every $i \ge 0$;
- $\mathfrak{M}, \pi \models \vartheta_1 \mathcal{U} \vartheta_2 \rightleftharpoons \mathfrak{M}, \pi[i, \infty] \models \vartheta_2$ for some $i \ge 0$ and $\mathfrak{M}, \pi[j, \infty] \models \vartheta_1$ for every j such that $0 \le j < i$.

An **ATL***-formula is a state formula in this language. An **ATL***-formula is satisfiable if it is satisfied by some state of some model, and valid if it is satisfied by every state of every model. Formally, by **ATL*** we mean the set of all valid **ATL***-formulas; notice that this set is closed under uniform substitution.

Logic **ATL** can be thought of as a fragment of **ATL**^{*} containing only formulas where a coalition quantifier is always paired up with a temporal connective. This, as in the case of **CTL**, eliminates path-formulas. Such composite "modal" operators are $\langle\!\langle C \rangle\!\rangle \bigcirc$, $\langle\!\langle C \rangle\!\rangle \square$, and $\langle\!\langle C \rangle\!\rangle \mathcal{U}$. Formulas are defined by the following BNF expression:

$$\varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid \langle\!\langle C \rangle\!\rangle \bigcirc \varphi \mid \langle\!\langle C \rangle\!\rangle \Box \varphi \mid \langle\!\langle C \rangle\!\rangle (\varphi \,\mathcal{U}\varphi),$$

where C ranges over subsets of AG and p ranges over *Var*. The other Boolean connectives and the constant \top are defined as for **CTL**.

The satisfaction relation between concurrent game models \mathfrak{M} , states s, and formulas φ is inductively defined as follows (we only list the cases for the "new" modal operators):

- $\mathfrak{M}, s \models \langle\!\langle C \rangle\!\rangle \bigcirc \varphi_1 \rightleftharpoons$ there exists a *C*-action α_C such that $\mathfrak{M}, s' \models \varphi_1$ whenever $s' \in out(s, act_C)$;
- $\mathfrak{M}, s \models \langle\!\langle C \rangle\!\rangle \Box \varphi_1 \rightleftharpoons$ there exists a *C*-strategy str_C such that $\mathfrak{M}, \pi[i] \models \varphi_1$ holds for all $\pi \in out(s, str_C)$ and all $i \ge 0$;
- $\mathfrak{M}, s \models \langle\!\!\langle C \rangle\!\!\rangle (\varphi_1 \, \mathcal{U} \varphi_2) \rightleftharpoons$ there exists a *C*-strategy str_C such that, for all $\pi \in out(s, str_C)$, there exists $i \ge 0$ with $\mathfrak{M}, \pi[i] \models \varphi$ and $\mathfrak{M}, \pi[j] \models \varphi$ holds for every j such that $0 \le j < i$.

Satisfiable and valid formulas are defined as for **ATL**^{*}. Formally, by **ATL** we mean the set of all valid **ATL**^{*}-formulas; this set is closed under substitution.

Remark 4.1 We have given definitions of satisfiability and validity for ATL^* and ATL that assume that the set of all agents AG present in the language is "fixed in advance". At least two other notions of satisfiability (and, thus, validity) for these logics have been discussed in the literature (see, e.g., [40]) i.e., satisfiability of a formula in a model where the set of all agents coincides with the set of agents named in the formula and satisfiability of a formula in a model where the set of agents is any set including the agents named in the formula (in this case, it suffices to consider all the agents named in the formula plus one extra agent). In what follows, we explicitly consider only the notion of satisfiability for a fixed set of agents; other notions of satisfiability can be handled in a similar way.

5 Finite-variable fragments of ATL^{*} and ATL

We start by noticing that satisfiability for variable-free fragments of both **ATL**^{*} and **ATL** is polynomial-time decidable, using the algorithm similar to the one outlined for **CTL**^{*} and **CTL**. It follows that variable-free fragments of **ATL**^{*} and **ATL** cannot be as expressive as entire logics.

We also notice that, as is well-known, satisfiability for \mathbf{CTL}^* is polynomialtime reducible to satisfiability for \mathbf{ATL}^* and satisfiability for \mathbf{CTL} is polynomialtime reducible to satisfiability for \mathbf{ATL} , using the translation that replaces all occurrences of \forall by $\langle\!\langle \mathcal{O} \rangle\!\rangle$ and all occurrences of \exists by $\langle\!\langle \mathbb{AG} \rangle\!\rangle$. Thus, Theorems 3.6 and 3.11, together with the known upper bounds [19, 35, 32], immediately give us the following:

Theorem 5.1 The satisfiability problem for the single-variable fragment of ATL^{*} is 2EXPTIME-complete.

Theorem 5.2 The satisfiability problem for the single-variable fragment of **ATL** is EXPTIME-complete.

In the rest of this section, we show that single-variable fragments of ATL^* and ATL are as expressive as the entire logics by embedding both ATL^* and ATL into their single-variable fragments. The arguments closely resemble the ones for CTL^* and CTL, so we only provide enough detail for the reader to be able to easily fill in the rest.

First, consider \mathbf{ATL}^* . The translation \cdot' is defined as for \mathbf{CTL}^* , except that the clause for \forall is replaced by the following:

$$(\langle\!\langle C \rangle\!\rangle \alpha)' = \langle\!\langle C \rangle\!\rangle (\Box p_{n+1} \wedge \alpha').$$

Next, we define

$$\Theta = p_{n+1} \land \langle\!\langle \emptyset \rangle\!\rangle \Box (\langle\!\langle \mathbb{AG} \rangle\!\rangle \bigcirc p_{n+1} \leftrightarrow p_{n+1})$$

and

$$\widehat{\varphi} = \Theta \wedge \varphi'$$

Then, we can prove the analogues of Lemmas 3.1 and 3.2.

We next model all the variables of $\widehat{\varphi}$ by single-variable formulas A'_1, \ldots, A'_m . To that end, we use the class of concurrent game models $\mathsf{M} = \{\mathfrak{M}'_1, \ldots, \mathfrak{M}'_m\}$ that closely resemble models $\mathfrak{M}_1, \ldots, \mathfrak{M}_m$ used in the argument for \mathbf{CTL}^* . For every \mathfrak{M}'_i , with $i \in \{1, \ldots, m\}$, the set of states and the valuation V are the same as for \mathfrak{M}_i ; in addition, whenever $s \longmapsto s'$ holds in \mathfrak{M}_i , we set $\delta(s, \alpha) = s'$, for every action profile α . The actions available to an agent a at each state of \mathfrak{M}_i are all the actions available to a at any of the states of the model \mathfrak{M} to which we are going to attach models \mathfrak{M}'_i when proving the analogue of Lemma 3.4, as well as an extra action d_a that we need to set up transitions from the states of \mathfrak{M} to the roots of \mathfrak{M}'_i s.

With every \mathfrak{M}'_i we associate the formula A'_i . First, inductively define the sequence of formulas

$$\begin{array}{lll} \chi'_0 & = & \langle\!\!\langle \varnothing \rangle\!\!\rangle \Box \, p; \\ \chi'_{k+1} & = & p \wedge \langle\!\!\langle \mathbb{AG} \rangle\!\!\rangle \bigcirc (\neg p \wedge \langle\!\!\langle \mathbb{AG} \rangle\!\!\rangle \bigcirc \chi_k). \end{array}$$

Next, for every $m \in \{1, \ldots, n+1\}$, let

$$A'_m = \chi'_m \land \langle\!\!\langle \mathbb{AG} \rangle\!\!\rangle \bigcirc \langle\!\!\langle \mathcal{O} \rangle\!\!\rangle \Box \, \neg p$$

Lemma 5.3 Let $\mathfrak{M}'_k \in \mathsf{M}$ and let x be a state in \mathfrak{M}'_k . Then, $\mathfrak{M}'_k, x \models A'_m$ if, and only if, k = m and $x = r_m$.

Proof. Straightforward.

Now, for every $m \in \{1, \ldots, n+1\}$, define

$$B'_m = \langle\!\langle \mathbb{AG} \rangle\!\rangle \bigcirc A'_m.$$

Finally, let σ be a (substitution) function that, for every $i \in \{1, \ldots, n+1\}$, replaces p_i by B'_i , and let

 $\varphi^* = \sigma(\widehat{\varphi}).$

This allows us to prove the analogue of Lemma 3.4.

Lemma 5.4 Formula φ is satisfiable if, and only if, formula φ^* is satisfiable.

Proof. Analogous to the proof of Lemma 3.4. When constructing the model \mathfrak{M}' , whenever we need to connect a state s in \mathfrak{M} to the root r_i of \mathfrak{M}'_i , we make an extra action, d_a , available to every agent a, and define $\delta(s, \langle d_a \rangle_{a \in \mathbb{AG}}) = r_i$. \Box

Thus, we have the following:

Theorem 5.5 There exists a polynomial-time computable function e assigning to every \mathbf{ATL}^* -formula φ a single-variable formula $e(\varphi)$ such that $e(\varphi)$ is satisfiable if, and only if, φ is satisfiable.

We then can adapt the argument for \mathbf{ATL} form the one just presented in the same way we adapted the argument for \mathbf{CTL} from the one for \mathbf{CTL}^* , obtaining the following:

Theorem 5.6 There exists a polynomial-time computable function e assigning to every **ATL**-formula φ a single-variable formula $e(\varphi)$ such that $e(\varphi)$ is satisfiable if, and only if, φ is satisfiable.

6 Discussion

We have shown that logics **CTL**^{*}, **CTL**, **ATL**^{*}, and **ATL** can be polynomialtime embedded into their single-variable fragments; i.e., their single-variable fragments are as expressive as the entire logics. Consequently, for these logics, satisfiability is as computationally hard when one considers only formulas of one variable as when one considers arbitrary formulas. Thus, the complexity

of satisfiability for these logics cannot be reduced by restricting the number of variables allowed in the construction of formulas.

The technique presented in this paper can be applied to many other modal and temporal logics of computation considered in the literature. We will not here attempt a comprehensive list, but rather mention a few examples.

The proofs presented in this paper can be extended in a rather straightforward way to Branching- and Alternating-time temporal-epistemic logics [21, 35, 39, 16], i.e., logics that enrich the logics considered in this paper with the epistemic operators of individual, distributed, and common knowledge for the agents. Our approach can be used to show that single-variable fragments of those logics are as expressive as the entire logics and that, consequently, the complexity of satisfiability for them is as hard (EXPTIME-hard or 2EXPTIMEhard) as for the entire logics. Clearly, the same approach can be applied to epistemic logics [11, 14, 18], i.e., logics containing epistemic, but not temporal, operators—such logics are widely used for reasoning about distributed computation. Our argument also applies to logics with the so-called universal modality [13] to obtain EXPTIME-completeness of their variable-free fragments. The technique presented here has also been recently used [29] to show that propositional dynamic logics are as expressive in the language without propositional variables as in the language with an infinite supply of propositional variables. Since our method is modular in the way it tackles modalities present in the language, it naturally lends itself to modal languages combining various modalities—a trend that has been gaining prominence for some time now.

The technique presented in this paper can also be lifted to first-order languages to prove undecidability results about fragments of first-order modal and related logics,—see [31].

We conclude by noticing that, while we have been able to overcome the limitations of the technique from [20] described in the introduction, our modification thereof has limitations of its own. It is not applicable to logics whose semantics forbids branching, such as **LTL** or temporal-epistemic logics of linear time [21, 15]. Our technique cannot be used, either, to show that finite-variable fragments of logical systems that are not closed under uniform substitution—such as public announcement logic **PAL** [27, 36]—have the same expressive power as the entire system. This does not preclude it from being used in establishing complexity results for finite-variable fragments of such systems provided they contain fragments, as is the case with **PAL** [24], that are closed under substitution and have the same complexity as the entire system.

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