

# The Queue-Number of Posets of Bounded Width or Height

Kolja Knauer<sup>1\*</sup>, Piotr Micek<sup>2\*\*</sup> and Torsten Ueckerdt<sup>3</sup>

<sup>1</sup> Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France  
kolja.knauer@lis-lab.fr

<sup>2</sup> Jagiellonian University, Faculty of Mathematics and Computer Science, Theoretical  
Computer Science Department, Poland piotr.micek@tcs.uj.edu.pl

<sup>3</sup> Karlsruhe Institute of Technology (KIT), Institute of Theoretical Informatics,  
Germany torsten.ueckerdt@kit.edu

**Abstract.** Heath and Pemmaraju [9] conjectured that the queue-number of a poset is bounded by its width and if the poset is planar then also by its height. We show that there are planar posets whose queue-number is larger than their height, refuting the second conjecture. On the other hand, we show that any poset of width 2 has queue-number at most 2, thus confirming the first conjecture in the first non-trivial case. Moreover, we improve the previously best known bounds and show that planar posets of width  $w$  have queue-number at most  $3w - 2$  while any planar poset with 0 and 1 has queue-number at most its width.

## 1 Introduction

A *queue layout* of a graph consists of a total ordering on its vertices and an assignment of its edges to *queues*, such that no two edges in a single queue are nested. The minimum number of queues needed in a queue layout of a graph  $G$  is called its *queue-number* and denoted by  $qn(G)$ .

To be more precise, let  $G$  be a graph and let  $L$  be a linear order on the vertices of  $G$ . We say that the edges  $uv, u'v' \in E(G)$  are *nested* with respect to  $L$  if  $u < u' < v' < v$  or  $u' < u < v < v'$  in  $L$ . Given a linear order  $L$  of the vertices of  $G$ , the edges  $u_1v_1, \dots, u_kv_k$  of  $G$  form a *rainbow* of size  $k$  if  $u_1 < \dots < u_k < v_k < \dots < v_1$  in  $L$ . Given  $G$  and  $L$ , the edges of  $G$  can be partitioned into  $k$  queues if and only if there is no rainbow of size  $k + 1$  in  $L$ , see [10].

The queue-number was introduced by Heath and Rosenberg in 1992 [10] as an analogy to book embeddings. Queue layouts were implicitly used before and have applications in fault-tolerant processing, sorting with parallel queues, matrix computations, scheduling parallel processes, and communication management in distributed algorithm (see [8,10,12]).

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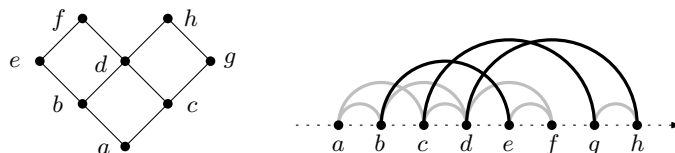
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Perhaps the most intriguing question concerning queue-numbers is whether planar graphs have bounded queue-number.

*Conjecture 1 (Heath and Rosenberg [10]).*

The queue-number of planar graphs is bounded by a constant.

In this paper we study queue-numbers of posets. The parameter was introduced in 1997 by Heath and Pemmaraju [9] and the main idea is that given a poset one should lay it out respecting its relation. Two elements  $a, b$  of a poset are called *comparable* if  $a < b$  or  $b < a$ , and *incomparable*, denoted by  $a \parallel b$ , otherwise. Posets are visualized by their *diagrams*: Elements are placed as points in the plane and whenever  $a < b$  in the poset, and there is no element  $c$  with  $a < c < b$ , there is a curve from  $a$  to  $b$  going upwards (that is  $y$ -monotone). We denote this case as  $a < b$ . The diagram represents those relations which are essential in the sense that they are not implied by transitivity, also known as *cover relations*. The undirected graph implicitly defined by such a diagram is the *cover graph* of the poset. Given a poset  $P$ , a *linear extension*  $L$  of  $P$  is a linear order on the elements of  $P$  such that  $x <_L y$ , whenever  $x <_P y$ . (Throughout the paper we use a subscript on the symbol  $<$ , if we want to emphasize which order it represents.) Finally, the *queue-number of a poset*  $P$ , denoted by  $\text{qn}(P)$ , is the smallest  $k$  such that there is a linear extension  $L$  of  $P$  for which the resulting linear layout of  $G_P$  contains no  $(k + 1)$ -rainbow. Clearly we have  $\text{qn}(G_P) \leq \text{qn}(P)$ , i.e., the queue-number of a poset is at least the queue-number of its cover graph. It is shown in [9] that even for *planar posets*, that is posets admitting crossing-free diagrams, there is no function  $f$  such that  $\text{qn}(P) \leq f(\text{qn}(G_P))$ .



**Fig. 1.** A poset and a layout with two queues (gray and black). Note that the order of the elements on the spine is a linear extension of the poset.

Heath and Pemmaraju [9] investigated the maximum queue-number of several classes of posets, in particular with respect to bounded width (the maximum number of pairwise incomparable elements) and height (the maximum number of pairwise comparable elements). A set with every two elements being comparable is a *chain*. A set with every two distinct elements being incomparable is an *antichain*. They proved that if  $\text{width}(P) \leq w$ , then  $\text{qn}(P) \leq w^2$ . The lower bound is attained by *weak orders*, i.e., chains of antichains and is conjectured to be the upper bound as well:

*Conjecture 2 (Heath and Pemmaraju [9]).*

Every poset of width  $w$  has queue-number at most  $w$ .

Furthermore, they made a step towards this conjecture for planar posets: if a planar poset  $P$  has  $\text{width}(P) \leq w$ , then  $\text{qn}(P) \leq 4w - 1$ . For the lower bound side they provided planar posets of width  $w$  and queue-number  $\lceil \sqrt{w} \rceil$ .

We improve the bounds for planar posets and get the following:

**Theorem 1.** *Every planar poset of width  $w$  has queue-number at most  $3w - 2$ . Moreover, there are planar posets of width  $w$  and queue-number  $w$ .*

As an ingredient of the proof we show that posets without certain subdivided crowns satisfy Conjecture 2 (c.f. Theorem 5). This implies the conjecture for interval orders and planar posets with (unique minimum) 0 and (unique maximum) 1 (c.f. Corollary 2). Moreover, we confirm Conjecture 2 for the first non-trivial case  $w = 2$ :

**Theorem 2.** *Every poset of width 2 has queue-number at most 2.*

An easy corollary of this is that all posets of width  $w$  have queue-number at most  $w^2 - w + 1$  (c.f. Corollary 1).

Another conjecture of Heath and Pemmaraju concerns planar posets of bounded height:

*Conjecture 3 (Heath and Pemmaraju [9]).*

Every planar poset of height  $h$  has queue-number at most  $h$ .

We show that Conjecture 3 is false for the first non-trivial case  $h = 2$ :

**Theorem 3.** *There is a planar poset of height 2 with queue-number at least 4.*

Furthermore, we establish a link between a relaxed version of Conjecture 3 and Conjecture 1, namely we show that the latter is equivalent to planar posets of height 2 having bounded queue-number (c.f. Theorem 6). On the other hand, we show that Conjecture 3 holds for planar posets with 0 and 1:

**Theorem 4.** *Every planar poset of height  $h$  with 0 and 1 has queue-number at most  $h - 1$ .*

*Organization of the paper.* In Section 2 we consider general (not necessarily planar) posets and give upper bounds on their queue-number in terms of their width, such as Theorem 2. In Section 3 we consider planar posets and bound the queue-number in terms of the width, both from above and below, i.e., we prove Theorem 1. In Section 4 we give a counterexample to Conjecture 3 by constructing a planar poset with height 2 and queue-number at least 4. Here we also argue that proving *any* upper bound on the queue-number of such posets is equivalent to proving Conjecture 1. Finally, we show that Conjecture 3 holds for planar posets with 0 and 1 and that for every  $h$  there is a planar poset of height  $h$  and queue-number  $h - 1$  (c.f. Proposition 3).

## 2 General Posets of Bounded Width

By Dilworth's Theorem [3], the width of a poset  $P$  coincides with the smallest integer  $w$  such that  $P$  can be decomposed into  $w$  chains of  $P$ . Let us derive Proposition 1 of Heath and Pemmaraju [9] from such a chain partition.

**Proposition 1.** *For every poset  $P$ , if  $\text{width}(P) \leq w$  then  $\text{qn}(P) \leq w^2$ .*

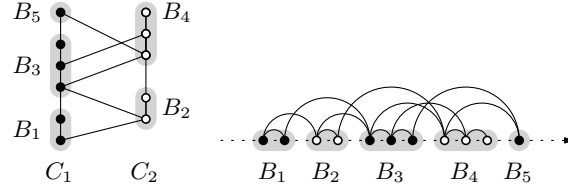
*Proof.* Let  $P$  be a poset of width  $w$  and  $C_1, \dots, C_w$  be a chain partition of  $P$ . Let  $L$  be any linear extension of  $P$  and  $a <_L b <_L c <_L d$  with  $a \prec d$  and  $b \prec c$ . Note that we must have either  $a \parallel b$  or  $c \parallel d$ . It follows that if  $a \in C_i$ ,  $b \in C_j$ ,  $c \in C_k$ , and  $d \in C_\ell$ , then  $(i, \ell) \neq (j, k)$ . As there are only  $w^2$  ordered pairs  $(x, y)$  with  $x, y \in [w]$ , we can conclude that every nesting set of covers has cardinality at most  $w^2$ .  $\square$

Note that in the above proof  $L$  is *any* linear extension and that without choosing the linear extension  $L$  carefully, upper bound  $w^2$  is best-possible. Namely, if  $P = \{a_1, \dots, a_k, b_1, \dots, b_k\}$  with comparabilities  $a_i < b_j$  for all  $1 \leq i, j \leq k$ , then  $P$  has width  $k$  and the linear extension  $a_1 < \dots < a_k < b_k < \dots < b_1$  creates a rainbow of size  $k^2$ .

We continue by showing that every poset of width 2 has queue-number at most 2, that is, we prove Theorem 2.

*Proof (Theorem 2).* Let  $P$  be a poset of width 2 and minimum element 0 and  $C_1, C_2$  be a chain partition of  $P$ . Note that the assumption of the minimum causes no loss of generality, since a 0 can be added without increasing the width nor decreasing the queue-number. Any linear extension  $L$  of  $P$  partitions the ground set  $X$  naturally into inclusion-maximal sets of elements, called *blocks*, from the same chain in  $\{C_1, C_2\}$  that appear consecutively along  $L$ , see Figure 2. We denote the blocks by  $B_1, \dots, B_k$  according to their appearance along  $L$ . We say that  $L$  is *lazy* if for each  $i = 2, \dots, k$ , each element  $x \in B_i$  has a relation to some element  $y \in B_{i-1}$ . A linear extension  $L$  can be obtained by picking any minimal element  $m \in P$ , put it into  $L$ , and recurse on  $P \setminus \{m\}$ . Lazy linear extensions (with respect to  $C_1, C_2$ ) can be constructed by the same process where additionally the next element is chosen from the same chain as the element before, if possible. Note that the existence of a 0 is needed in order to ensure the property of laziness with respect to  $B_2$ .

Now we shall prove that in a lazy linear extension no three covers are pairwise nesting. So assume that  $a \prec b$  is any cover and that  $a \in B_i$  and  $b \in B_j$ . As  $L$  is lazy,  $b$  is comparable to some element in  $B_{j-1}$  (if  $j \geq 2$ ) and all elements in  $B_1, \dots, B_{j-2}$  (if  $j \geq 3$ ). With  $a \prec b$  being a cover, it follows from  $L$  being lazy that  $i \in \{j-2, j-1, j\}$ . If  $i = j$ , then no cover is nested under  $a \prec b$ . If  $i = j-1$ , then no cover  $c \prec d$  is nested above  $a \prec b$ : either  $c \in B_i$  and  $d \in B_j$  and hence  $c \prec d$  is not a cover, or both endpoints would be inside the same chain, i.e.,  $c, d$  are the last and first element of  $B_{j-2}$  and  $B_j$  or  $B_i$  and  $B_{i+2}$ , respectively. This implies  $c <_L a <_L d <_L b$  or  $a <_L c <_L b <_L d$ , respectively, and  $c \prec d$  cannot nest above  $a \prec b$ . If  $i = j-2$ , then no cover is nested above  $a \prec b$ . Thus, either



**Fig. 2.** A poset of width 2 with a 0 and a chain partition  $C_1, C_2$  and the blocks  $B_1, \dots, B_5$  induced by a lazy linear extension with respect to  $C_1, C_2$ .

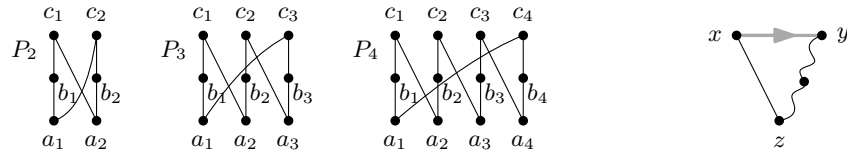
no cover is nested below  $a \prec b$ , or no cover is nested above  $a \prec b$ , or both. In particular, there is no three nesting covers and  $\text{qn}(P) \leq 2$ .  $\square$

**Corollary 1.** *Every poset of width  $w$  has queue-number at most  $w^2 - 2\lfloor w/2 \rfloor$ .*

*Proof.* We take any chain partition of size  $w$  and pair up chains to obtain a set  $S$  of  $\lfloor w/2 \rfloor$  disjoint pairs. Each pair from  $S$  induces a poset of width at most 2, which by Theorem 2 admits a linear order with at most two nesting covers. Let  $L$  be a linear extension of  $P$  respecting all these partial linear extensions.

Now, following the proof of Proposition 1 any cover can be labeled by a pair  $(i, j)$  corresponding to the chains containing its endpoint. Thus, in a set of nesting covers any pair appears at most once, but for each  $i, j$  such that  $(i, j) \in S$  only two of the four possible pairs can appear simultaneously in a nesting. This yields the upper bound.  $\square$

For an integer  $k \geq 2$  we define a *subdivided  $k$ -crown* as the poset  $P_k$  as follows. The elements of  $P_k$  are  $\{a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_k\}$  and the cover relations are given by  $a_i \prec b_i$  and  $b_i \prec c_i$  for  $i = 2, \dots, k$ ,  $a_i \prec c_{i-1}$  for  $i = 1, \dots, k-1$ , and  $a_1 \prec c_k$ ; see the left of Figure 3. We refer to the covers of the form  $a_i \prec c_j$  as the *diagonal covers* and we say that a poset  $P$  has an *embedded  $P_k$*  if  $P$  contains  $3k$  elements that induce a copy of  $P_k$  in  $P$  with all diagonal covers of that copy being covers of  $P$ .

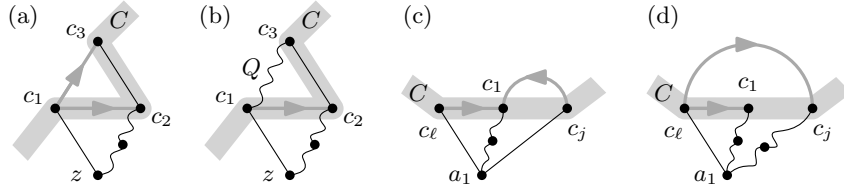


**Fig. 3.** Left: The posets  $P_2, P_3$ , and  $P_4$ . Right: The existence of an element  $z$  with cover relation  $z \prec x$  and non-cover relation  $z \prec y$  gives rise to a gray edge from  $x$  to  $y$ .

**Theorem 5.** *If  $P$  is a poset that for no  $k \geq 2$  has an embedded  $P_k$ , then the queue-number of  $P$  is at most the width of  $P$ .*

*Proof.* Let  $P$  be any poset. For this proof we consider the cover graph  $G_P$  of  $P$  as a directed graph with each edge  $xy$  directed from  $x$  to  $y$  if  $x \prec y$  in  $P$ . We call these edges the *cover edges*. Now we augment  $G_P$  to a directed graph  $G$  by introducing for some incomparable pairs  $x \parallel y$  a directed edge. Specifically, we add a directed edge from  $x$  to  $y$  if there exists a  $z$  with  $z < x, y$  in  $P$  where  $z \prec x$  is a cover relation and  $z < y$  is not a cover relation; see the right of Figure 3. We call these edges the *gray edges* of  $G$ .

Now we claim that if  $G$  has a directed cycle, then  $P$  has an embedded subdivided crown. Clearly, every directed cycle in  $G$  has at least one gray edge. We consider the directed cycles with the fewest gray edges and among those let  $C = [c_1, \dots, c_\ell]$  be one with the fewest cover edges. First assume that  $C$  has a cover edge (hence  $\ell \geq 3$ ), say  $c_1 c_2$  is a gray edge followed by a cover edge  $c_2 c_3$ . Consider the element  $z$  with cover relation  $z \prec c_1$  and non-cover relation  $z < c_2$  in  $P$ . By  $z < c_2 \prec c_3$  we have a non-cover relation  $z < c_3$  in  $P$ . Now if  $c_1 \parallel c_3$  in  $P$ , then  $G$  contains the gray edge  $c_1 c_3$  (see Figure 4(a)) and  $[c_1, c_3, \dots, c_\ell]$  is a directed cycle with the same number of gray edges as  $C$  but fewer cover edges, a contradiction. On the other hand, if  $c_1 < c_3$  in  $P$  (note that  $c_3 < c_1$  is impossible as  $z \prec c_1$  is a cover), then there is a directed path  $Q$  of cover edges from  $c_1$  to  $c_3$  (see Figure 4(b)) and  $C + Q - \{c_1 c_2, c_2 c_3\}$  contains a directed cycle with fewer gray edges than  $C$ , again a contradiction.



**Fig. 4.** Illustrations for the proof of Theorem 5.

Hence  $C = [c_1, \dots, c_\ell]$  is a directed cycle consisting solely of gray edges. Note that by the first paragraph  $\{c_1, \dots, c_\ell\}$  is an antichain in  $P$ . For  $i = 2, \dots, \ell$  let  $a_i$  be the element of  $P$  with cover relation  $a_i \prec c_{i-1}$  and non-cover relation  $a_i < c_i$ , as well as  $a_1$  with cover relation  $a_1 \prec c_\ell$  and non-cover relation  $a_1 < c_1$ . As  $\{c_1, \dots, c_\ell\}$  is an antichain and  $a_i < c_i$  holds for  $i = 1, \dots, \ell$ , we have  $\{c_1, \dots, c_\ell\} \cap \{a_1, \dots, a_\ell\} = \emptyset$ . Let us assume that  $a_1 < c_j$  in  $P$  for some  $j \neq 1, \ell$ . If  $a_1 \prec c_j$  is a cover relation, then there is a gray edge  $c_j c_1$  in  $G$  (see Figure 4(c)) and the cycle  $[c_1, \dots, c_j]$  is shorter than  $C$ , a contradiction. If  $a_1 < c_j$  is a non-cover relation, then there is a gray edge  $c_\ell c_j$  in  $G$  (see Figure 4(d)) and the cycle  $[c_j, \dots, c_\ell]$  is shorter than  $C$ , again a contradiction.

Hence, the only relations between  $a_1, \dots, a_\ell$  and  $c_1, \dots, c_\ell$  are cover relations  $a_1 \prec c_\ell$  and  $a_i \prec c_{i-1}$  for  $i = 2, \dots, \ell$  and the non-cover relations  $a_i < c_i$  for  $i = 1, \dots, \ell$ . Hence  $a_1, \dots, a_\ell$  are pairwise distinct. Moreover,  $\{a_1, \dots, a_\ell\}$  is an antichain in  $P$  since the only possible relations among these elements are of the

form  $a_1 < a_\ell$  or  $a_i < a_{i-1}$ , which would contradict that  $a_1 \prec c_\ell$  and  $a_i \prec c_{i-1}$  are cover relations. Finally, we pick for every  $i = 1, \dots, \ell$  an element  $b_i$  with  $a_i < b_i < c_i$ , which exists as  $a_i < c_i$  is a non-cover relation. Together with the above relations between  $a_1, \dots, a_\ell$  and  $c_1, \dots, c_\ell$  we conclude that  $b_1, \dots, b_\ell$  are pairwise distinct and these  $3\ell$  elements induce a copy of  $P_\ell$  in  $P$  with all diagonal covers in that copy being covers of  $P$ .

Thus, if  $P$  has no embedded  $P_k$ , then the graph  $G$  we constructed has no directed cycles, and we can pick  $L$  to be any topological ordering of  $G$ . As  $G_P \subseteq G$ ,  $L$  is a linear extension of  $P$ . For any two nesting covers  $x_2 <_L x_1 <_L y_1 <_L y_2$  we have  $x_1 \parallel x_2$  or  $y_1 \parallel y_2$  or both, since  $x_2 \prec y_2$  is a cover. However, if  $x_2 < x_1$  in  $P$ , then there would be a gray edge from  $y_2$  to  $y_1$  in  $G$ , contradicting  $y_1 <_L y_2$  and  $L$  being a topological ordering of  $G$ . We conclude that  $x_1 \parallel x_2$  and the left endpoints of any rainbow form an antichain, proving  $\text{qn}(P) \leq \text{width}(P)$ .  $\square$

Let us remark that several classes of posets have no embedded subdivided crowns, e.g., graded posets, interval orders (since these are 2+2-free, see [6]), or (quasi-)series-parallel orders (since these are N-free, see [7]). Here, 2+2 and N are the four-element posets defined by  $a < b, c < d$  and  $a < b, c < d, c < b$ , respectively. Also note that while subdivided crowns are planar posets, no planar poset with 0 and 1 has an embedded  $k$ -crown. Indeed, already looking at the subposet induced by the  $k$ -crown and the 0 and the 1, it is easy to see that there must be a crossing in any diagram. Thus, we obtain:

**Corollary 2.** *For any interval order, series-parallel order, and planar poset with 0 and 1,  $P$  we have  $\text{qn}(P) \leq \text{width}(P)$ .*

### 3 Planar Posets of Bounded Width

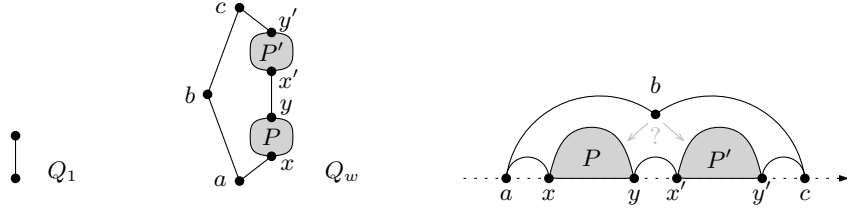
Heath and Pemmaraju [9] show that the largest queue-number among planar posets of width  $w$  lies between  $\lceil \sqrt{w} \rceil$  and  $4w - 1$ . Here we improve the lower bound to  $w$  and the upper bound to  $3w - 2$ .

**Proposition 2.** *For each  $w$  there exists a planar poset  $Q_w$  with 0 and 1 of width  $w$  and queue-number  $w$ .*

*Proof.* We shall define  $Q_w$  recursively, starting with  $Q_1$  being any chain. For  $w \geq 2$ ,  $Q_w$  consists of a *lower copy*  $P$  and a disjoint *upper copy*  $P'$  of  $Q_{w-1}$ , three additional elements  $a, b, c$ , and the following cover relations in between:

- $a \prec x$ , where  $x$  is the 0 of  $P$
- $y \prec x'$ , where  $y$  is the 1 of  $P$  and  $x'$  is the 0 of  $P'$
- $y' \prec c$ , where  $y'$  is the 1 of  $P'$
- $a \prec b \prec c$

It is easily seen that all cover relations of  $P$  and  $P'$  remain cover relations in  $Q_w$ , and that  $Q_w$  is planar, has width  $w$ ,  $a$  is the 0 of  $Q_w$ , and  $c$  is the 1 of  $Q_w$ . See Figure 5 for an illustration.



**Fig. 5.** Recursively constructing planar posets  $Q_w$  of width  $w$  and queue-number  $w$ . Left:  $Q_1$  is a two-element chain. Middle:  $Q_w$  is defined from two copies  $P, P'$  of  $Q_{w-1}$ . Right: The general situation for a linear extension of  $Q_w$ .

To prove that  $\text{qn}(Q_w) = w$  we argue by induction on  $w$ , with the case  $w = 1$  being immediate. Let  $L$  be any linear extension of  $Q_w$ . Then  $a$  is the first element in  $L$  and  $c$  is the last. Since  $y \prec x'$ , all elements in  $P$  come before all elements of  $P'$ . Now if in  $L$  the element  $b$  comes after all elements of  $P$ , then  $P$  is nested under cover  $a \prec b$ , and if  $b$  comes before all elements of  $P'$ , then  $P'$  is nested under cover  $b \prec c$ . We obtain  $w$  nesting covers by induction on  $P$  in the former case, and by induction on  $P'$  in the latter case. This concludes the proof.  $\square$

Next we prove Theorem 1, namely that the maximum queue-number of planar posets of width  $w$  lies between  $w$  and  $3w - 2$ .

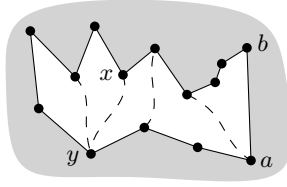
*Proof (Theorem 1).* By Proposition 2 some planar posets of width  $w$  have queue-number  $w$ . So it remains to consider an arbitrary planar poset  $P$  of width  $w$  and show that  $P$  has queue-number at most  $3w - 2$ . To this end, we shall add some relations to  $P$ , obtaining another planar poset  $Q$  of width  $w$  that has a 0 and 1, with the property that  $\text{qn}(P) \leq \text{qn}(Q) + 2w - 2$ . Note that this will conclude the proof, as by Corollary 2 we have  $\text{qn}(Q) \leq w$ .

Given a planar poset  $P$  of width  $w$ , there are at most  $w$  minima and at most  $w$  maxima. Hence there are at most  $2w - 2$  extrema that are not on the outer face. For each such extremum  $x$ —say  $x$  is a minimum—consider the unique face  $f$  with an obtuse angle at  $x$ . We introduce a new relation  $y < x$ , where  $y$  is a smallest element at face  $f$ , see Figure 6. Note that this way we introduce at most  $2w - 2$  new relations, and that these can be drawn  $y$ -monotone and crossing-free by carefully choosing the other element in each new relation. Furthermore, every inner face has a unique source and unique sink.

Now consider a cover relation  $a \prec_P b$  that is not a cover relation in the new poset  $Q$ . For the corresponding edge  $e$  from  $a$  to  $b$  in  $Q$  there is one face  $f$  with unique source  $a$  and unique sink  $b$ . Now either way the other edge in  $f$  incident to  $a$  or to  $b$  must be one of the  $2w - 2$  newly inserted edges, see again Figure 6. This way we assign  $a \prec b$  to one of  $2w - 2$  queues, one for each newly inserted edge. Every such queue contains either at most one edge or two incident edges, i.e., a nesting is impossible, no matter what linear ordering is chosen later.

We create at most  $2w - 2$  queues to deal with the cover relations of  $P$  that are not cover relations of  $Q$  and spend another  $w$  queues for  $Q$  dealing with the remaining cover relations of  $P$ . Thus,  $\text{qn}(P) \leq \text{qn}(Q) + 2w - 2 \leq 3w - 2$ .  $\square$



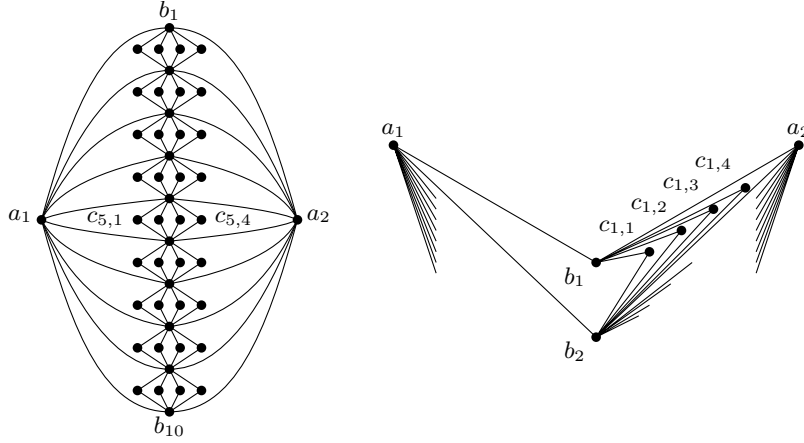


**Fig. 6.** Inserting new relations (dashed) into a face of a plane diagram. Note that relation  $a < b$  is a cover relation in  $P$  but not in  $Q$ .

## 4 Planar Posets of Bounded Height

Recall Conjecture 3, which states that every planar poset of height  $h$  has queue-number at most  $h$ . In the following, we give a counterexample to this conjecture:

*Proof (Theorem 3).* Consider the graph  $G$  that is constructed as follows: Start with  $K_{2,10}$  with bipartition classes  $\{a_1, a_2\}$  and  $\{b_1, \dots, b_{10}\}$ . For every  $i = 1, \dots, 9$  add four new vertices  $c_{i,1}, \dots, c_{i,4}$ , each connected to  $b_i$  and  $b_{i+1}$ . The resulting graph  $G$  has 46 vertices, is planar and bipartite with bipartition classes  $X = \{b_1, \dots, b_{10}\}$  and  $Y = \{a_1, a_2\} \cup \{c_{i,j} \mid 1 \leq i \leq 9, 1 \leq j \leq 4\}$ . See Figure 7.



**Fig. 7.** A planar poset  $P$  of height 2 and queue-number at least 4. Left: The cover graph  $G_P$  of  $P$ . Right: A part of a planar diagram of  $P$ .

Let  $P$  be the poset arising from  $G$  by introducing the relation  $x < y$  for every edge  $xy$  in  $G$  with  $x \in X$  and  $y \in Y$ . Clearly,  $P$  has height 2 and hence the cover relations of  $P$  are exactly the edges of  $G$ . Moreover, by a result of Moore [11] (see also [2])  $P$  is planar because  $G$  is planar, also see the right of Figure 7.

We shall argue that  $\text{qn}(P) \geq 4$ . To this end, let  $L$  be any linear extension of  $P$ . Without loss of generality we have  $a_1 <_L a_2$ . Note that since in  $P$  one bipartition

class of  $G$  is entirely below the other, any 4-cycle in  $G$  gives a 2-rainbow. Let  $b_{i_1}, b_{i_2}$  be the first two elements of  $X$  in  $L$ ,  $b_{j_1}, b_{j_2}$  be the last two such elements. As  $|X| = 10$  there exists  $1 \leq i \leq 9$  such that  $\{i, i+1\} \cap \{i_1, i_2, j_1, j_2\} = \emptyset$ , i.e., we have  $b_{i_1}, b_{i_2} <_L b_i, b_{i+1} <_L b_{j_1}, b_{j_2} <_L a_1 <_L a_2$ , where we use that  $a_1$  and  $a_2$  are above all elements of  $X$  in  $P$ .

Now consider the elements  $C = \{c_{i,1}, \dots, c_{i,4}\}$  that are above  $b_i$  and  $b_{i+1}$  in  $P$ . As  $|C| \geq 4$ , there are two elements  $c_1, c_2$  of  $C$  that are both below  $a_1, a_2$  in  $L$ , or both between  $a_1$  and  $a_2$  in  $L$ , or both above  $a_1, a_2$  in  $L$ . Consider the 2-rainbow  $R$  in the 4-cycle  $[c_1, b_i, c_2, b_{i+1}]$ . In the first case  $R$  is nested below the 4-cycle  $[a_1, b_{i_1}, a_2, b_{i_2}]$ , in the second case the cover  $b_{j_1} \prec a_1$  is nested below  $R$  and  $R$  is nested below the cover  $b_{i_1} \prec a_2$ , and in the third case 4-cycle  $[a_1, b_{j_1}, a_2, b_{j_2}]$  is nested below  $R$ . As each case results in a 4-rainbow, we have  $\text{qn}(P) \geq 4$ .  $\square$

Even though Conjecture 3 has to be refuted in its strongest meaning, it might hold that planar posets of height  $h$  have queue-number  $O(h)$ , or at least bounded by some function  $f(h)$  in terms of  $h$ , or at least that planar posets of height 2 have bounded queue-number. As it turns out, all these statements are equivalent, and in turn equivalent to Conjecture 1.

**Theorem 6.** *The following statements are equivalent:*

- (i) *Planar graphs have queue-number  $O(1)$  (Conjecture 1).*
- (ii) *Planar posets of height  $h$  have queue-number  $O(h)$ .*
- (iii) *Planar posets of height  $h$  have queue-number at most  $f(h)$  for a function  $f$ .*
- (iv) *Planar posets of height 2 have queue-number  $O(1)$ .*
- (v) *Planar bipartite graphs have queue-number  $O(1)$ .*

*Proof.* (i) $\Rightarrow$ (ii) Pemmaraju proves in his thesis [13] (see also [4]) that if  $G$  is a graph,  $\pi$  is a vertex ordering of  $G$  with no  $(k+1)$ -rainbow,  $V_1, \dots, V_m$  are color classes of any proper  $m$ -coloring of  $G$ , and  $\pi'$  is the vertex ordering with  $V_1 <_{\pi'} \dots <_{\pi'} V_m$ , where within each  $V_i$  the ordering of  $\pi$  is inherited, then  $\pi'$  has no  $(2(m-1)k+1)$ -rainbow. So if  $P$  is any poset of height  $h$ , its cover graph  $G_P$  has  $\text{qn}(G_P) \leq c$  by (i) for some global constant  $c > 0$ . Splitting  $P$  into  $h$  antichains  $A_1, \dots, A_h$  by iteratively removing all minimal elements induces a proper  $h$ -coloring of  $G_P$  with color classes  $A_1, \dots, A_h$ . As every vertex ordering  $\pi'$  of  $G$  with  $A_1 <_{\pi'} \dots <_{\pi'} A_h$  is a linear extension of  $P$ , it follows by Pemmaraju's result that  $\text{qn}(P) \leq 2(h-1) \text{qn}(G_P) \leq 2ch$ , i.e.,  $\text{qn}(P) \in O(h)$ .

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) These implications are immediate.

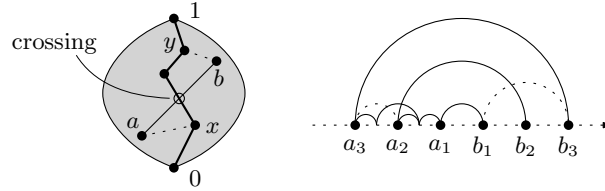
(iv) $\Rightarrow$ (v) Moore proves in his thesis [11] (see also [2]) that if  $G$  is a planar and bipartite graph with bipartition classes  $A$  and  $B$ , and  $P_G$  is the poset on element set  $A \cup B = V(G)$  where  $x < y$  if and only if  $x \in A, y \in B, xy \in E(G)$ , then  $P_G$  is a planar poset of height 2. As  $G$  is the cover graph of  $P_G$ , we have  $\text{qn}(G) \leq \text{qn}(P_G) \leq c$  for some constant  $c > 0$  by (iv), i.e.,  $\text{qn}(G) \in O(1)$ .

(v) $\Rightarrow$ (i) This is a result of Dujmović and Wood [5].  $\square$

Finally, we show that Conjecture 3 holds for planar posets with 0 and 1.

*Proof (Theorem 4).* Let  $P$  be a planar poset with 0 and 1. Then  $P$  has dimension at most two [1], i.e., it can be written as the intersection of two linear extensions of  $P$ . A particular consequence of this is, that there is a well-defined dual poset  $P^*$  in which two distinct elements  $x, y$  are comparable in  $P$  if and only if they are incomparable in  $P^*$ . Poset  $P^*$  reflects a “left of”-relation for each incomparable pair  $x \parallel y$  in  $P$  in the following sense: Any maximal chain  $C$  in  $P$  corresponds to a 0-1-path  $Q$  in  $G_P$ , which splits the elements of  $P \setminus C$  into those left of  $Q$  and those right of  $Q$ . Now  $x <_{P^*} y$  if and only if  $x$  is left of the path for every maximal chain containing  $y$  (equivalently  $y$  is right of the path for every maximal chain containing  $x$ ). Due to planarity, if  $a \prec b$  is a cover in  $P$  and  $C$  is a maximal chain containing neither  $a$  nor  $b$ , then  $a$  and  $b$  are on the same side of the path  $Q$  corresponding to  $C$ . In particular, if for  $x, y \in C$  we have  $a <_{P^*} x$  and  $b \parallel y$ , then  $b$  and  $y$  are comparable in  $P^*$ , but if  $y <_{P^*} b$  we would get a crossing of  $C$  and  $a \prec b$ . Also see the left of Figure 8. We summarize:

( $\star$ ) If  $a \prec b$ ,  $a <_{P^*} x$  for some  $x \in C$  and  $b \parallel y$  for some  $y \in C$ , then  $b <_{P^*} y$ .



**Fig. 8.** Left: Illustration of ( $\star$ ): If  $a <_{P^*} x$ ,  $b \parallel y$ ,  $x < y$ , and  $a \prec b$  is a cover, then  $b <_{P^*} y$  due to planarity. Right: If  $a_3 <_L a_2 <_L a_1 <_L b_1 <_L b_2 <_L b_3$  is a 3-rainbow with  $a_2, a_3 < a_1$ , then  $a_3 < a_2$ .

Now let  $L$  be the *leftmost* linear extension of  $P$ , i.e., the unique linear extension  $L$  with the property that for any  $x \parallel y$  in  $P$  we have  $x <_L y$  if and only if  $x < y$  in  $P^*$ . Assume that  $a_2 <_L a_1 <_L b_1 <_L b_2$  is a pair of nesting covers  $a_1 \prec b_1$  below  $a_2 \prec b_2$ . Then  $a_1 \parallel a_2$  (hence  $a_2 <_{P^*} a_1$ ) or  $b_1 \parallel b_2$  (hence  $b_1 <_{P^*} b_2$ ) or both. Observe that the latter case is impossible, as for any maximal chain  $C$  containing  $a_1 \prec b_1$  we would have  $a_2 <_{P^*} a_1$  with  $a_1 \in C$  and  $b_1 <_{P^*} b_2$  with  $b_1 \in C$ , contradicting ( $\star$ ). So the nesting of  $a_1 \prec b_1$  below  $a_2 \prec b_2$  is either of type A with  $a_2 < a_1$ , or of type B with  $b_1 < b_2$ . See Figure 9.

Now consider the case that cover  $a_2 \prec b_2$  is nested below another cover  $a_3 \prec b_3$ , see the right of Figure 8. Then also  $a_1 \prec b_1$  is nested below  $a_3 \prec b_3$  and we claim that if both, the nesting of  $a_1 \prec b_1$  below  $a_2 \prec b_2$  as well as the nesting of  $a_1 \prec b_1$  below  $a_3 \prec b_3$ , are of type A (respectively type B), then also the nesting of  $a_2 \prec b_2$  below  $a_3 \prec b_3$  is of type A (respectively type B). Indeed, assuming type B, we would get  $a_3 <_{P^*} a_2$  and  $b_1 <_{P^*} b_3$ , which together with any maximal chain  $C$  containing  $a_2 < a_1 < b_1$  contradicts ( $\star$ ).



**Fig. 9.** A nesting of  $a_1 \prec b_1$  below  $a_2 \prec b_2$  of type A (left) and type B (right).

Finally, let  $a_k <_L \dots <_L a_1 <_L b_1 <_L \dots <_L b_k$  be any  $k$ -rainbow and let  $I = \{i \in [k] \mid a_i < a_1\}$ , i.e., for each  $i \in I$  the nesting of  $a_1 \prec b_1$  below  $a_i \prec b_i$  is of type A. Then we have just shown that the nesting of  $a_j \prec b_j$  below  $a_i \prec b_i$  is of type A whenever  $i, j \in I$  and of type B whenever  $i, j \notin I$ . Hence, the set  $\{a_i \mid i \in I\} \cup \{a_1, b_1\} \cup \{b_i \mid i \notin I\}$  is a chain in  $P$  of size  $k+1$ , and thus  $k \leq h-1$ . It follows that  $P$  has queue-number at most  $h-1$ , as desired.  $\square$

The proof of the following can be found in the appendix.

**Proposition 3.** *For each  $h$  there exists a planar poset  $Q_h$  of height  $h$  and queue-number  $h-1$ .*

## 5 Conclusions

We studied the queue-number of (planar) posets of bounded height and width. Two main problems remain open: bounding the queue-number by the width and bounding it by a function of the height in the planar case, where the latter is equivalent to the central conjecture in the area of queue-numbers of graphs. For the first problem the biggest class known to satisfy it are posets without the embedded subdivided  $k$ -crowns for  $k \geq 2$  as defined in Section 2. Note, that proving it for  $k \geq 3$  would imply that Conjecture 2 holds for all 2-dimensional posets, which seems to be a natural next step.

Let us close the paper by recalling another interesting conjecture from [9], which we would like to see progress in:

*Conjecture 4 (Heath and Pemmaraju [9]).*

Every planar poset on  $n$  elements has queue-number at most  $\lceil \sqrt{n} \rceil$ .

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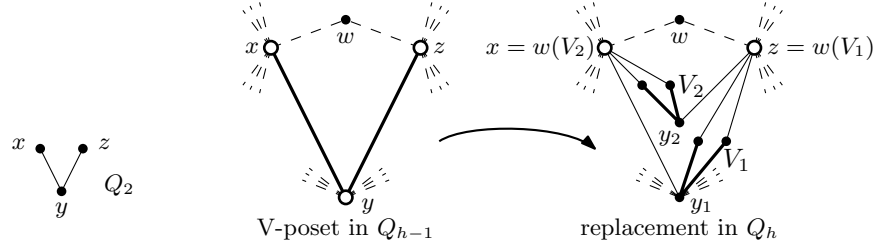
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## 6 Appendix

*Proof (Proposition 3).* We shall recursively define a planar poset  $Q_h$  of height  $h$  and queue-number  $h - 1$ , together with a certain set of marked subposets in  $Q_h$ . Each marked subposet consists of three elements  $x, y, z$  forming a *V-subposet* in  $Q_h$ , i.e.,  $y < x, z$  but  $x \parallel z$ , with both relations  $y < x$  and  $y < z$  being cover relations of  $Q_h$ , and  $y$  being a minimal element of  $Q_h$ . We call such a marked subposet in  $Q_h$  a V-poset. Finally, we ensure that the V-posets are pairwise incomparable, namely that any two elements in distinct V-posets are incomparable in  $Q_h$ .

For  $h = 2$  let  $Q_2$  be the three-element poset as shown in left of Figure 10, which also forms the only V-poset of  $Q_2$ . Clearly  $Q_2$  has height 2 and queue-number 1. For  $h \geq 3$  assume that we already constructed  $Q_{h-1}$  with a number of V-posets in it. Then  $Q_h$  is obtained from  $Q_{h-1}$  by replacing each V-poset by the eight-element poset shown in the right of Figure 10, which introduces (for each V-poset) five new elements. Moreover, two new V-posets are identified in  $Q_h$  as illustrated in Figure 10.

It is easy to check that  $Q_h$  is planar and has height  $h$ , since  $Q_{h-1}$  has height  $h - 1$  and the V-posets in  $Q_{h-1}$  are pairwise incomparable. Moreover, every V-poset in  $Q_h$  contains a minimal element of  $Q_h$  and all V-posets in  $Q_h$  are pairwise incomparable. Finally, observe that, as long as  $h \geq 3$ , for every V-poset  $V$  in  $Q_h$  there is a unique smallest element  $w = w(V)$  that is larger than all elements in  $V$ , see the right of Figure 10.



**Fig. 10.** Constructing planar posets of height  $h$  and queue-number  $h - 1$ . Left:  $Q_2$  is a three-element poset and its only V-poset. Right:  $Q_h$  is recursively defined from  $Q_{h-1}$  by replacing each V-poset by an eight-element poset and identifying two new V-posets.

In order to show that  $\text{qn}(Q_h) \geq h - 1$ , we shall show by induction on  $h$  that for every linear extension  $L$  of  $Q_h$  there exists a  $(h - 1)$ -rainbow in  $Q_h$  with respect to  $L$  whose innermost cover is contained in a V-poset  $V$  of  $Q_h$ , and, if  $h \geq 3$ , whose second innermost cover has the element  $w(V)$  as its upper end. This clearly holds for  $h = 2$ . For  $h \geq 3$ , consider any linear extension  $L$  of  $Q_h$ . This induces a linear extension  $L'$  of  $Q_{h-1}$  as follows: The set  $X$  of elements in  $Q_h$  not contained in any V-poset is also a subset of the elements in  $Q_{h-1}$ . The remaining elements of  $Q_{h-1}$  are the minimal elements of the V-posets in  $Q_{h-1}$ . For each minimal element  $y$  of  $Q_{h-1}$  consider the two corresponding V-posets in  $Q_h$  with its two corresponding minimal elements  $y_1, y_2$ . Let  $\hat{y} \in \{y_1, y_2\}$  be the element that comes first in  $L$ , i.e.,  $\hat{y} = y_1$  if and only if  $y_1 <_L y_2$ . Then we define  $L'$  to be the ordering of  $Q_{h-1}$  induced by the ordering of  $X \cup \{\hat{y} \mid y \in Q_{h-1} - X\}$  in  $L$ . Note that  $L'$  is a linear extension of  $Q_{h-1}$ , even though  $X \cup \{\hat{y} \mid y \in Q_{h-1} - X\}$  does not necessarily induce a copy of  $Q_{h-1}$  in  $Q_h$ .

By induction on  $Q_{h-1}$  there exists a  $(h - 2)$ -rainbow  $R$  with respect to  $L'$  whose innermost cover is contained in a V-poset  $V$  and, provided that  $h - 1 \geq 3$ , its second innermost cover has  $w = w(V)$  as its upper end. Consider the elements  $x, y, z$  of  $V$  with  $y$  being the minimal element, and the two corresponding V-posets  $V_1, V_2$  with minimal elements  $y_1, y_2$  of  $Q_h$ , where  $y_1x$  and  $y_2z$  are covers; see Figure 10. By definition of  $\hat{y}$  and  $L'$ , all elements of  $\{x, y\} \cup V_1 \cup V_2$  lie between  $\hat{y}$  (included) and  $w$  (excluded, if  $h - 1 \geq 3$ ) with respect to  $L$ .

Assume without loss of generality that  $x <_L z$ . If  $y_2 <_L y_1$  ( $\hat{y} = y_2$ ), then the V-poset with  $y_1$  is nested completely under the cover  $y_2z$  and replacing in  $R$  the innermost cover by the cover  $y_2z$  and any cover with  $y_1$  gives a  $(h - 1)$ -rainbow with the desired properties. If  $y_1 <_L y_2$  ( $\hat{y} = y_1$ ), then the V-poset with  $y_2$  is nested completely under the cover  $y_1x$  and replacing in  $R$  the innermost cover by the cover  $y_1x$  and any cover with  $y_2$  gives a  $(h - 1)$ -rainbow with the desired properties, which concludes the proof.  $\square$